Gen. Math. Notes, Vol. 6, No. 1, September 2011, pp. 15-24
ISSN 2219-7184; Copyright © ICSRS Publication, 2011
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Solution for Boundary Value Problem of Non-Integer Order in L^2-Space

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(Received: 12-4-11/Accepted: 15-9-11)

Abstract

In this paper, we shall prove the existence and uniqueness of a square - integrable solution in L^2-space, for the boundary value problem of non- integer order which has the form:.

$$^{c}D_x^{\alpha} y(x) = f(x, y(x)) \qquad , \quad 1 < \alpha \le 2$$
$$y(a) = y_a, \qquad y(b) = y_b$$

Where $^{c}D^{\alpha}$ is the Caputo fractional derivative and a, b are positive constants with $b \neq a$. The contraction mapping principle has been used in establishing our main results.

Keywords: *Fractional differential equation, Caputo's fractional derivative, L^2-Space, Boundary Value Problem.*

1 Introduction

Fractional differential equations have received considerable attention in the recent years, due to their wide applications in engineering, economy and other fields. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, and in many other fields see ([3],[4]).

Bai and Lu [1] discussed the existence and the multiplicity of positive solutions for boundary value problem of nonlinear fractional differential equation

$$\begin{aligned} &D_{0^+}^{\alpha} u(t) = f(t,u(t)) \qquad , \; 0 < t < 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{1}$$

Where $1 < \alpha \le 2$ is a real number, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differentiation, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

Hadid [2] studied local and global existence theorems of the nonlinear differential equation

$$\begin{aligned} &D^{\alpha} y(t) = f(t,y(t)) \qquad 0 < \alpha < 1 \\ &y(t_0) = t_0 \end{aligned} \tag{2}$$

by Shauder and Tyconove fixed point theorems.

Zhang [8] considered the existence of solution of nonlinear fractional differential equation boundary value problems involving Caputo's derivative.

$$\begin{aligned} &D_{0^+}^{\alpha} u(t) + f(t,u(t)) = 0 \qquad , \; 0 < t < 1 , \quad 1 < \alpha \le 2 \\ &u(0) = \alpha \ne 0 , \qquad u(1) = \beta \ne 0 \end{aligned} \tag{3}$$

Momani [5] studied local and global uniqueness theorems of the fractional differential equation (1) by using Biharie's and Gronwell's inequalities. In this paper, we consider the existence and uniqueness of a square - integrable solution of the following boundary value problem of fractional order:

$$\begin{aligned} &{}^{c}D_{x}^{\alpha} y(x) = f(x,y(x)) \qquad , \; 1 < \alpha \le 2 \\ &y(a) = y_a , \qquad y(b) = y_b \end{aligned} \tag{4}$$

Where ${}^{c}D_{a}^{\alpha}$ the Caputo's fractional derivative is $f(x,y)$ is a function on $(a,b) \times E$ in to E and E being an Euclidean – space.

Consider the space $L^2(a,b)$, the space of all measurable functions f such that $|f|^2$ is Lebesgue integrable on $[a,b]$. For any $f \in L^2(a,b)$ we define the norm as

$$\|f\|_2^2 = \int_a^b |f(x)|^2 dx \tag{5}$$

Under this norm, it is known that the space $L^2(a,b)$, is a Banach space.

2 Preliminaries

In this section, we mainly demonstrate and study the definitions, lemmas and some fundamental facts on Caputo's derivatives on fractional order, which can been founded in [6].

Definition 2.1. *Let f be a function which is defined almost everywhere (a.e) on [a,b] .for $\alpha > 0$,we define*

$${}_a^b I^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)dt$$

Provided that this integral (lebsegue) exists, where Γ is gamma function.

Definition 2.2. *For a function f defined on the interval $[a,b]$, the Caputo's fractional derivative of f is defined by*

$$\left(D_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s, y(s))ds$$

Where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α .

Definition 2.3. *If $\alpha \in R$ and a function $f(x)$ defined a.e on $a \le x \le b$, we define $f^{(\alpha)}(x) = {}_a^x I^{-\alpha} f$ for all $x \in [a,b]$, provided that ${}_a^x I^{-\alpha} f$ exists.*

Lemma 2.1. *If $\alpha > 0$ and $f(x)$ is continuous on $[a,b]$, then $D^{-\alpha} f$ exists and it is continuous with respect to* $\boldsymbol{x}$ *on $[a,b]$.*

Lemma 2.2. *If $\alpha > 0$ and $f(x)$ belongs to $L(a,b)$, then ${}_a^x D^{-\alpha} f$ exists for all $x \in L(a,b)$ if $\alpha \ge 1$ and a.e. if $\alpha < 1$.*

Lemma 2.3. *Let* $\alpha > 0$ *and* $n = [\alpha] + 1$ *. If* $y \in AC^n[a,b]$ *or* $y \in C^n[a,b]$, *then*

$$\left(I^{\alpha}\ {}^{c}D^{\alpha}\ y\right)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k .$$

Lemma 2.4. *If* $\alpha, \beta > 0$ *and* $f(x) \in L(a,b)$*,then on* $a \le x \le b$ *, we have:*

a. $\int_a^x {}_a^sD^{-\alpha} f \ \ ds = {}_a^xD^{-(\alpha+1)} f$, *or* ${}_a^xD^{-1}\ {}_a^sD^{-\alpha} f = {}_a^xD^{-(\alpha+1)} f$.

b. *If* $\alpha \ge 1$*,then* ${}_a^xD^{-\alpha} f$ *is absolutely continuous in* $x \in [a,b]$.

c. $\frac{d}{dx}\ {}_a^xD^{-(\alpha+1)} f = {}_a^xD^{-\alpha} f$, *everywhere if* $\alpha \ge 1$ *and a.e. if* $\alpha < 1$.

d. ${}_a^xD^{-(\alpha+\beta)} f = {}_a^xD^{-\alpha}\ {}_a^sD^{-\beta} f$, *a.e. if* $\alpha + \beta \le 1$.

Lemma 2.4. (Hölder's inequality)

Let X *be a measurable space, let* p *and* q *satisfy* $1 < p < \infty$, $1 < q < \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$.

If $f \in L^p(X)$ *and* $g \in L^q(X)$, *then* $(f\ g)$ *belongs to* $L(X)$ *and satisfies*

$$\int_X |f\ g|\, dx \le \left[\int_X |f|^{\,p} dx\right]^{\frac{1}{p}} \left[\int_X |g|^{\,q} dx\right]^{\frac{1}{q}}$$

3 The Main Results

In this section we shall prove existence and uniqueness of a square-integrable solution in $L^2(a,b)$ space for the boundary value problem of non-integer order of the form (4).

Lemma 3.1

Let $1 < \alpha \le 2$ *and let* $h : [a,b] \to R$ *be continuous. A function* $\boldsymbol{y}$ *is a solution of fractional integral equation*

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} h(t)\,dt - \frac{(x-a)}{(b-a)\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} h(t)\,dt - \frac{(x-b)}{(b-a)} y_a + \frac{(x-a)}{(b-a)} y_b \qquad (6)$$

if and only if y *is a solution of the fractional boundary value problem*

$$ {}^{c}D_{x}^{\alpha} y(x) = f(x,y(x)) \quad , \ 1<\alpha\le 2 \tag{7}$$

$$ y(a)=y_a , \qquad y(b)=y_b \tag{8}$$

Proof: By using definition (2.3), equation (7) can be written as

$$ {}_{a}^{x}I^{-\alpha} y(x) = f(x,y(x)) \quad , \ 1<\alpha\le 2 \tag{9}$$

Operating on both sides of equation (9) by the operator ${}_{a}^{x}I^{-\alpha}$ *,we obtain :*

$$ {}_{a}^{x}I^{\alpha}\, {}_{a}^{t}I^{-\alpha}\, y(x) = {}_{a}^{x}I^{\alpha} f(x,y(x))$$

by using Lemma (2.3), we get:

$$ y(x) = y(a) + y'(a)(x-a) + {}_{a}^{x}I^{\alpha} f(x,y(x)) \tag{10}$$

Using the boundary condition (10), we obtain:

$$ y(a) = y_a \tag{11}$$

$$ y(b) = y_a + y'(a)(b-a) + {}_{a}^{b}I^{\alpha} f(x,y(x)) \tag{12}$$

$$ y'(a) = \frac{y_b}{(b-a)} - \frac{y_a}{(b-a)} - \frac{1}{(b-a)}\, {}_{a}^{b}I^{\alpha} f(x,y(x)) \tag{13}$$

Substituting equation (13) and (11) in equation (10) , we obtain the final form of $\boldsymbol{y(x)}$ is :

$$ y(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} h(t)\,dt - \frac{(x-a)}{(b-a)\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} h(t)\,dt - \frac{(x-b)}{(b-a)} y_a + \frac{(x-a)}{(b-a)} y_b \tag{14}$$

Theorem 3.2 *Let the right hand side* $f(x,y)$ *of the fractional differential equation (4) be such that:*

i. *it satisfies the Lipschitz condition in y with Lipschitz constant K, that is,*

$$ \left| f(x,y_2) - f(x,y_1) \right| \le K \left| y_2 - y_1 \right| \tag{15}$$

on the domain D *, where:*

$$D=\{\ (x,y):\ \ |x-x_0|\le a\ ,\ |y-y_0|\le b\ \} \qquad (16)$$

ii. *it is a square –intgerable* $(f\in L_2(a,b))$ *as a function of* $x\in(a,b)$*. Then If*

$$\frac{2k(b-a)^{\alpha}}{\Gamma(\alpha)}\sqrt{\frac{1+2\alpha}{2\alpha(2\alpha-1)}}<1 \qquad (17)$$

There exists a square –integrable solution for the fractional differential equation (7) with the boundary condition (8) on (a,b) .

Proof. Let the mapping T on $L_2(a,b)$ be defined as**:**

$$T(y(x))=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-a)}{(b-a)\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-b)}{(b-a)}y_a+\frac{(x-a)}{(b-a)}y_b \qquad (18)$$

we claim that T takes every function $g\in L_2(a,b)$ into a function which belongs to $L_2(a,b)$. Let:

$$h(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-a)}{(b-a)\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-b)}{(b-a)}y_a+\frac{(x-a)}{(b-a)}y_b \qquad (19)$$

Then

$$|h(x)|^2=\left|\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-a)}{(b-a)\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-b)}{(b-a)}y_a+\frac{(x-a)}{(b-a)}y_b\right|^2 \qquad (20)$$

By the lemma $(2.4(b))$, , the first term of equation (19) exists and it is absolutely continuous so it is continuous for all $x\in[a,b]$.also the other terms are continuous for all $x\in[a,b]$.

Thus $h(x)$ is continuous and measurable in D. Hence $|\boldsymbol{h(x)}|^2$ is measurable.

Now, we have to show that $|\boldsymbol{h(x)}|^2$ is Lebsegue integrable. Since $|f+g|^p\le 2^p\left(|f|^p+|g|^p\right)$ and from equation (20), we have

$$|h(x)|^2\le 4\left|\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t,y(t))dt\right|^2+4\left|\frac{(x-a)}{(b-a)\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t,y(t))dt-\frac{(x-b)}{(b-a)}y_a+\frac{(x-a)}{(b-a)}y_b\right|^2$$

the term $\left| \frac{(x-a)}{(b-a)\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t,y(t))\,dt - \frac{(x-a)}{(b-a)} y_a + \frac{(x-a)}{(b-a)} y_b \right|$

is Lebsegue integrable.

Let

$$(Z(x))^2 = \left[\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t,y(t))\,dt \right]^2 \qquad (21)$$

We have to show that $(Z(x))^2$ is Lebesgue integrable. Since $(x-t)^{\alpha-1} \in L_2(a,b)$ and by hypothesis (ii) $f(t,g(t)) \in L_2(a,b)$ as a function of t, then by HÖlder inequality $(p=q=2)$ and from equation (18) we obtain:

$$(Z(x))^2 \le \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{2(\alpha-1)}\,dt \int_a^x f^2(t,y(t))\,dt$$

$$= \frac{(x-a)^{2\alpha-1}}{(2\alpha-1)\,\Gamma(\alpha)} \int_a^x f^2(t,y(t))\,dt \qquad (22)$$

For $x \in [a,b]$ and by using definition (2.1) and Lemma (2.4(a)) we have:

$$\int_a^x \left(\int_a^t f^2(t,y(t))\,ds \right) dt = \int_a^x \left(\int_a^t (t-s)^{1-1} f^2(t,y(t))\,ds \right) dt$$

$$= \int_a^x \left({}_a^x I^1 f^2 \right) dt$$

$$= {}_a^x I^2 f^2$$

Thus by Lemma (2.2), it follows that $\int_a^x f^2\,dt$ is Lebesgue integrable for all $x \in [a,b]$.

If there exist a Lebesgue integrable function $g(x)$ on $[a,b]$ such that $|f(x)| \le g(x),$ a.e on $[a,b]$, where $f(x)$ is measurable then $f(x)$ is Lebesgue integrable function. Hence from inequality (22), $(Z(x))^2$ is Lebesgue integrable. Thus $|h(x)|^2$ Lebesgue integrable and therefore T maps $L_2(a,b)$ into itself.

To prove that T is a contraction mapping, Let g_1, g_1 be any two function that belong to $L_2(a,b)$, and consider:

$$\left\| T(y_2(t)) - T(y_1(t)) \right\|_2^2 = \left\| \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left[f(t,y_2(t)) - f(t,y_1(t)) \right] dt - \right.$$

$$\left. - \frac{(x-a)}{(b-a)\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left[f(t,y_2(t)) - f(t,y_1(t)) \right] dt \right\|_2^2$$

$$\int_a^x \left| \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left[f(t,y_2(t)) - f(t,y_1(t)) \right] dt - \frac{(x-a)}{(b-a)\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left[f(t,y_2(t)) - f(t,y_1(t)) \right] dt \right|^2 dx$$

$$\leq \frac{4}{(\Gamma(\alpha))^2} \int_a^b \left(\int_a^x (x-t)^{\alpha-1} \left| f(t,y_2(t)) - f(t,y_1(t)) \right| dt \right)^2 dx + \frac{4(x-a)^2}{(b-a)^2 (\Gamma(\alpha))^2} \int_a^b \left(\int_a^b (b-t)^{\alpha-1} \left| f(t,y_2(t)) - f(t,y_1(t)) \right| dt \right)^2 dx$$

Using lipschitz condition (15),we have :

$$\left\| T(y_2(t)) - T(y_1(t)) \right\|_2^2 \leq \frac{4K^2}{(\Gamma(\alpha))^2} \int_a^b \left(\int_a^x (x-t)^{\alpha-1} \left| y_2(t) - y_1(t) \right| dt \right)^2 dx +$$

$$+ \frac{4(x-a)^2 K^2}{(b-a)^2 (\Gamma(\alpha))^2} \int_a^b \left(\int_a^b (b-t)^{\alpha-1} \left| y_2(t) - y_1(t) \right| dt \right)^2 dx \qquad (23)$$

Since $y_2, y_1 \in L_2(a,b)$ and $L_2(a,b)$ is a linear space, then $\left| y_2 - y_1 \right| \in L_2(a,b)$, $(x-t)^{\alpha-1} \in L_2(a,b)$ and $(b-t)^{\alpha-1} \in L_2(a,b)$ by using HÖlder inequality, where $(p = q = 1)$, inequality (23) becomes

$$\left\| T(y_2(t)) - T(y_1(t)) \right\|_2^2 \leq \frac{4K^2}{(\Gamma(\alpha))^2} \int_a^b \left[\left(\int_a^x (x-t)^{2(\alpha-1)} dt \right) \left(\int_a^x \left| y_2(t) - y_1(t) \right|^2 dt \right) \right] dx +$$

$$+ \frac{4(x-a)^2 K^2}{(b-a)^2 (\Gamma(\alpha))^2} \int_a^b \left[\left(\int_a^b (b-t)^{2(\alpha-1)} dt \right) \left(\int_a^x \left| y_2(t) - y_1(t) \right|^2 dt \right) \right] dx \qquad (24)$$

To integrate the right hand side of inequality (21):

Let $\quad r(x) = \int_a^x \left| y_2(t) - y_1(t) \right|^2 dt \quad$ and $\quad r'(x) = \left| y_2(t) - y_1(t) \right|^2 \quad a.e$

$$\left\| T(g_2(t))-T(g_1(t))\right\|_2^2 \le \frac{4K^2}{(2\alpha-1)(\Gamma(\alpha))^2}\int_a^b (x-a)^{2\alpha-1} r(x)dx + \frac{4(x-a)^2 K^2}{(b-a)^2(2\alpha-1)(\Gamma(\alpha))^2}\int_a^b (b-a)^{2\alpha-1} r(x)dx$$

$$\left\| T(y_2(t))-T(y_1(t))\right\|_2^2 \le \frac{4K^2}{2\alpha(2\alpha-1)(\Gamma(\alpha))^2}\left[r(b)(b-a)^{2\alpha} - \int_a^b (x-a)^{2\alpha} r'(x)dx + \frac{4(x-a)^2 K^2 (b-a)^{2\alpha-1}}{(b-a)^2(2\alpha-1)(\Gamma(\alpha))^2}\int_a^b r(x)dx\right.$$

$$\le \frac{4K^2}{2\alpha(2\alpha-1)(\Gamma(\alpha))^2}\left[(b-a)^{2\alpha}\int_a^b |y_2-y_1|^2 dx - \int_a^b (x-a)^{2\alpha}|y_2-y_1|^2 dx + \right] \frac{4(x-a)^2 K^2 (b-a)^{2\alpha-1}}{(b-a)^2(2\alpha-1)(\Gamma(\alpha))^2}\int_a^b \left(\int_a^b |y_2-y_1|^2 dt\right)dx \quad (25)$$

Since $\int_a^b (x-a)^{2\alpha} |y_2-y_1|^2 dx$, $a \le x \le b$, then inequality (25) gives

$$\left\| T(y_2(t))-T(y_1(t))\right\|_2^2 \le \frac{4K^2(b-a)^{2\alpha}}{2\alpha(2\alpha-1)(\Gamma(\alpha))^2}\int_a^b |y_2-y_1|^2 dx +$$

$$+\frac{4K^2(b-a)^{2\alpha}}{(2\alpha-1)(\Gamma(\alpha))^2}\int_a^b |y_2-y_1|^2 dx$$

$$\left\| T(y_2(t))-T(y_1(t))\right\|_2^2 \le \frac{2K(b-a)^{\alpha}}{\Gamma(\alpha)}\sqrt{\frac{1+2\alpha}{2\alpha(2\alpha-1)}}\ \left\|y_2-y_1\right\|_2^2$$

Consequently by (17) T is a contraction mapping, as a consequence of Banach fixed point theorem, we deduce that T has a unique fixed point $y(x) \in L^2(a,b)$, that is $\boldsymbol{T\, y(x) = y(x)}$ .Therefore from equation (18) we have

$$y(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t,g(t))dt - \frac{(x-a)}{(b-a)\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} f(t,g(t))dt - \frac{(x-b)}{(b-a)}y_a + \frac{(x-a)}{(b-a)}y_b$$

This completes the proof.

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