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D-Compact Sets in Random n-normed Linear Space

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Abstract

The aim of this paper is to introduce the notion of D-bounded sets and D-compact sets in Random n-normed linear space. Also we prove some results in relation between D-bounded and D-compact sets in random n-normed linear spaces.

Keywords: *D-bounded set, D-compact set, partially D-closed set.*

1 Introduction

The theory of probabilistic metric spaces was introduced in 1942 by K. Menger [18], in connection with some measurements problems in physics. The positive number expressing the distance between two points p, q of a metric space is replaced by a distribution function (in the sense of probability theory) $F_{p,q} : R \longrightarrow [0,1]$, whose value $F_{p,q}(t)$ at the point $t \in R$ can be interpreted as the probability that the distance between p and q be less than t. Since then the subject developed in various directions, see [2], [12] and [22]. A clear and thorough presentation of the results in probabilistic metric spaces in the book by Schweizer and Sklar [22].

In [21], A.N.Serstnev endowed a set having an algebraic structure of linear space with a random norm. He used K.Menger's idea, this idea led to a large development of the theory of random normed space in various directions. Applications to systems having hysteresis, mixture processes, the measuring error were also given. For an extensive review of this subject we refer to [1], [2], [13] and [17].

In [5] and [6] S.Gähler introduced an attractive theory of 2-norm and n-norm on a linear space, for more information see [3], [7] and [11].

In [16] I. Jebril introduced the notion of bounded sets in random n-normed linear space.

The aim of this paper is to introduce the notion of D-compact sets in Random n-normed linear space. Also we prove some results in relation between D-bounded and D-compact sets in random n-normed linear spaces.

2 Preliminaries

Definition 2.1. *[7] Let $n \in N$ and let X be a real linear space of dimension $\geq n$. A real valued function $\|\bullet, \bullet, \dots, \bullet\|$ on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ satisfying the following conditions*

nN_1: $\|x_1, x_2, \dots, x_n\| = 0$ *if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,*

nN_2: $\|x_1, x_2, \dots, x_n\|$ *is invariant under any permutation of $x_1, x_2, \dots, x_n$,*

nN_3: $\|x_1, x_2, \dots, x_{n-1}, \alpha x_n\| = |\alpha|\ \|x_1, x_2, \dots, x_n\|$ *for all $\alpha \in R$*

nN_4: $\|x_1, x_2, \dots, x_{n-1}, y+z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$ *for all $y, z, x_1, x_2, \dots, x_{n-1} \in X$,*
then the function $\|\bullet, \bullet, \dots, \bullet\|$ is called an n-norm on X and the pair $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is called an n-normed linear space.

Definition 2.2. *[22] A distance distribution function (briefly, a d.d.f.), is a function F defined from the extended interval $[0, +\infty]$ into the unit interval $I = [0, 1]$, that is non-decreasing and left continuous on $(0, +\infty)$ with $F(0) = 0$ and $F(+\infty) = 1$.*

The family of all distance distribution functions will be denoted by Δ^+ . We denote $D^+ = \{F \in \Delta^+ \backslash \lim_{x \to \infty} F(x) = 1\}$.

One introduces a natural ordering in D^+, by setting $F \leq G$ whenever $F(x) \leq G(x)$, for all $x \in R^+$. If $a \in R^+$, then ε_a will be an element of D^+, defined by

$$\varepsilon_a = 0 \quad if\ x \leq a,$$

$$= 1 \quad if\ x > a,$$

It is obvious that $\varepsilon_a \geq F$ if $x > a$ for all $F \in D^+$. The maximal element in this order is the distribution function given by

$$\varepsilon_0 = 0 \quad if\ x \leq 0,$$

$$= 1 \quad if\ x > 0,$$

The set D^+ will be endowed with the natural topology defined by the modified Levy metric d_L [22].

A t-norm T is a two-place function $T : I \times I \longrightarrow I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$.

Definition 2.3. *[22] A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, non-decreasing and which has ε_0 as unit, viz. for all $F, G, H \in \Delta^+$, we have*

$$\tau(\tau(F, G), H) = \tau(F, \tau(G, H)),$$

$$\tau(F, G) = \tau(G, F),$$

$$\tau(F, H) \leq \tau(G, H) \quad if\ F \leq G,$$

$$\tau(F, \varepsilon_0) = F.$$

Continuity of triangle function means continuity with respect to the topology of weak convergence in Δ^+. Triangular functions are recursively defined by

$$\tau^1(F, G) = \tau(F, G),$$

$$\tau^2(F, G, H) = \tau(\tau(F, G), H),$$

$$\tau^n(F_1, F_2, \ldots, F_{n+1}) = \tau(\tau^{n-1}(F_1, F_2, \ldots, F_n), F_{n+1}), \quad for \ \ n \geq 3.$$

Particular triangle functions are the functions τ_T which for any continuous t-norm T and any $x \geq 0$, are given by

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t))$$

for all F, G in Δ^+ and all x in R. Here T is a continuous t-norm, i.e., a continuous binary operation on that is associative, commutative, non-decreasing and has 1 as identity [22].

In some papers the probabilistic 2-metric space and random 2-normed space were also considered and some results are obtained [8], [9], [10], and [22].

Definition 2.4. *[10] A probabilistic 2-metric space is a triple (S, F, τ), where S is a nonempty set whose elements are the points in the space, F is a mapping from $S \times S \times S$ into D^+. $F(x, y, z)$ will be denoted by $F_{x,y,z}$, τ is a triangular function and the following conditions are satisfied, for all $x, y, z, u \in S$.*

$P - 2M_1$: *To each pair of distinct points x, y in S there exists a point z in S such that $F_{x,y,z} \neq \varepsilon_0$*

$P - 2M_2$: *$F_{x,y,z} = \varepsilon_0$ if at least two of x, y, z are equal,*

$P - 2M_3$: *$F_{x,y,z} = F_{x,z,y} = F_{y,z,x}$*

$P - 2M_4$: *$F_{x,y,z} \geq \tau(F_{x,y,u}, F_{x,u,z}, F_{u,y,z})$.*

Definition 2.5. *[9] Let L be a linear space of a dimension greater than one over a real field. Let τ be a triangle function and let v be a mapping from $L \times L$ into D^+. If the following conditions are satisfied.*
$R-2N_1$: $v_{x,y} = \varepsilon_0$ *if and only if x and y are linearly dependent,*
$R-2N_2$: $v_{x,y} \neq \varepsilon_0$ *if and only if x and y are linearly independent,*
$R-2N_3$: $v_{x,y} = v_{y,x}$ *for every x and y in L,*
$R-2N_4$: $v_{\alpha x,y} = v_{y,x}\left(\frac{t}{|\alpha|}\right)$, *for every $t > 0$, $\alpha \neq 0$, $\alpha \in R$, $x, y \in L$,*
$R-2N_5$: $v_{x+y,z} \geq \tau(v_{x,z}, v_{y,z})$, *whenever $x, y, z \in L$*
then v is called a random 2-norm on L and (L, v, τ) is called a random 2-normed linear space (briefly R-2-NLS).

3 D-Bounded Sets in Random n-Normed Linear Space

The diameter of a nonempty set in a probabilistic metric space was defined in Egbert [4], Lafuerza-Guillén et al. [17] have introduced definition of probabilistic radius and boundedness sets in probabilistic normed space. A. Pourmoslemi and M. Salimi are introduced the notion of D-bounded sets in generalized probabilistic 2-normed space [19]. By generalizing definition 2.1, we obtain a satisfactory notion of random n-normed space as follows, see [14], [15] and [20].

Definition 3.1. *[14] Let L be a linear space of a dimension greater than one over a real field. Let τ be a triangle function and let v be a mapping from $\underbrace{L \times L \times \cdots \times L}_{n} \times R = L^n \times R$ (R set of real numbers) into D^+. If the following conditions are satisfied.*
$R-nN_1$: $v_{(x_1,x_2,\ldots,x_n)} = \varepsilon_0$ *if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,*
$R-nN_2$: $v_{(x_1,x_2,\ldots,x_n)}$ *is invariant under any permutation of $x_1, x_2, \ldots, x_n$,*
$R-nN_3$: $v_{(x_1,x_2,\ldots,\alpha x_n)}(t) = v_{(x_1,x_2,\ldots,x_n)}(\frac{t}{|\alpha|})$, *for every $t > 0$, $\alpha \neq 0$, $\alpha \in R$,*
$R-nN_4$: $v_{(x_1,x_2,\ldots,x_n+x'_n)} \geq \tau(\ v_{(x_1,x_2,\ldots,x_n)}\ ,\ v_{(x_1,x_2,\ldots,x'_n)}\ )$,
then (L, v, τ) is called a random n-normed linear space (briefly R-n-NLS).

Remark 3.2. *[15] From $(R-nN_3)$, it follows that in an R-n-NLS and $(R-nN_4)$ for all $t, s \in R$ with $t > 0$*

$$v_{(x_1,x_2,\ldots,\alpha x_i,\ldots,x_n)}(t) = v_{(x_1,x_2,\ldots,x_i,\ldots,x_n)}\left(\frac{t}{|\alpha|}\right), \ \text{if } \alpha \neq 0,$$

$$v_{(x_1,x_2,\ldots,x_i+x'_i,\ldots,x_n)}(s+t) \geq \tau\left(v_{(x_1,x_2,\ldots,x_i,\ldots,x_n)}(s), v_{(x_1,x_2,\ldots,x'_i,\ldots,x_n)}(t)\right).$$

Remark 3.3. *From $(R-nN_3)$, Let (L, v, τ) be a R-n-NLS. If $|\alpha| \leq |\beta|$, then*

$$v_{(x_1,x_2,\ldots,\beta x_i,\ldots,x_n)} \leq v_{(x_1,x_2,\ldots,\alpha x_i,\ldots,x_n)},$$

with $t > 0$ *and* $\alpha, \beta \in R - \{0\}$

$$v_{(x_1,x_2,\ldots,\alpha x_i,\ldots,x_n)}(t) = v_{(x_1,x_2,\ldots,x_i,\ldots,x_n)}\left(\frac{t}{|\alpha|}\right)$$

and

$$v_{(x_1,x_2,\ldots,\beta x_i,\ldots,x_n)}(t) = v_{(x_1,x_2,\ldots,x_i,\ldots,x_n)}\left(\frac{t}{|\beta|}\right)$$

Hence $|\alpha| \le |\beta|$, *which implies that*

$$\frac{t}{|\beta|} \le \frac{t}{|\alpha|}.$$

Therefore

$$v_{(x_1,x_2,\ldots,x_i,\ldots,x_n)}\left(\frac{t}{|\beta|}\right) \le v_{(x_1,x_2,\ldots,x_i,\ldots,x_n)}\left(\frac{t}{|\alpha|}\right)$$

Then

$$v_{(x_1,x_2,\ldots,\beta x_i,\ldots,x_n)} \le v_{(x_1,x_2,\ldots,\alpha x_i,\ldots,x_n)}.$$

Definition 3.4. *Let* (L, v, τ) *be a R-n-NLS and* $\underbrace{A \times A \times \cdots \times A}_{n}$ *be a nonempty subset of* $\underbrace{L \times L \times \cdots \times L}_{n}$. *The probabilistic radius of* $\underbrace{A \times A \times \cdots \times A}_{n}$ *is the function* $R_{A\times A\times\cdots\times A}$ *defined by*

$$R_{A\times A\times\cdots\times A}(t) = l^{-}\varphi_{A\times A\times\cdots\times A}(t) \quad \text{if } t \in [0, +\infty),$$
$$= 1 \quad \text{if } t = +\infty,$$

where $\varphi_{A\times A\times\cdots\times A}(t) = \inf\{v_{(x_1,x_2,\ldots,x_n)}(t) : x_1, x_2, \ldots, x_n \in A\}$ *and* $l^{-}f(t)$ *denote the left limit of the function* f *at the point* t.

Definition 3.5. *A nonempty set* $\underbrace{A \times A \times \cdots \times A}_{n}$ *in a R-n-NLS* (L, v, τ) *is said to be*

(a) Certainly bounded if $R_{A\times A\times\cdots\times A}(t_0) = 1$, *for some* $t_0 \in (0, +\infty)$.

(b) Perhaps bounded if one has $R_{A\times A\times\cdots\times A}(t) < 1$, *for every* $t \in (0, +\infty)$ *and* $l^{-}R_{A\times A\times\cdots\times A}(+\infty) = 1$.

(c) Perhaps unbounded if $R_{A\times A\times\cdots\times A}(t_0) > 0$, *for some* $t_0 \in (0, +\infty)$ *and* $l^{-}R_{A\times A\times\cdots\times A}(+\infty) \in (0, 1)$.

(d) Certainly unbounded if $l^{-}R_{A\times A\times\cdots\times A}(+\infty) = 0$, *i.e.,* $R_{A\times A\times\cdots\times A}(+\infty) = \varepsilon_\infty$.

Moreover $\underbrace{A \times A \times \cdots \times A}_{n}$ will be said to be distributionally bounded or simply D-bounded if either (a) or (b) holds. i.e., $R_{A\times A\times\cdots\times A} \in D^{+}$. Otherwise (i.e., if $R_{A\times A\times\cdots\times A} \in \Delta^{+}\backslash D^{+}$) A is said to D-unbounded.

Note that in the definition 3.5, we can have used $\inf\{v_{(x_1,x_2,\ldots,x_n)}(t) : x_1, x_2, \ldots, x_n \in A\}$ instead of $R_{A\times A\times\cdots\times A}$.

Definition 3.6. *A nonempty set* $\underbrace{A \times A \times \cdots \times A}_{n}$ *in a R-n-NLS* (L, v, τ) *is said to be*
(a) Certainly bounded if $\inf\{v_{(x_1,x_2,\ldots,x_n)}(t_0) : x_1, x_2, \ldots, x_n \in A\} = 1$*, for some* $t_0 \in (0, +\infty)$.
(b) Perhaps bounded if one has $\inf\{v_{(x_1,x_2,\ldots,x_n)}(t) : x_1, x_2, \ldots, x_n \in A\} < 1$*, for every* $t \in (0, +\infty)$ *and* $l^- \inf\{v_{(x_1,x_2,\ldots,x_n)}(+\infty) : x_1, x_2, \ldots, x_n \in A\} = 1$.
(c) Perhaps unbounded if $\inf\{v_{(x_1,x_2,\ldots,x_n)}(t_0) : x_1, x_2, \ldots, x_n \in A\} > 0$*, for some* $t_0 \in (0, +\infty)$ *and* $l^- \inf\{v_{(x_1,x_2,\ldots,x_n)}(+\infty) : x_1, x_2, \ldots, x_n \in A\} \in (0, 1)$.
(d) Certainly unbounded if $l^- \inf\{v_{(x_1,x_2,\ldots,x_n)}(+\infty) : x_1, x_2, \ldots, x_n \in A\} = 0$*, i.e.,* $\inf\{v_{(x_1,x_2,\ldots,x_n)}(+\infty) : x_1, x_2, \ldots, x_n \in A\} = 0$.

Example 3.7. *Let* $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ *be an n-normed linear space defined*

$$v_{(x_1,x_2,\ldots,x_n)}(t) = 0 \quad \text{when} \;\; t \leq \|x_1, x_2, \ldots, x_n\|$$

$$= 1 \quad \text{when} \;\; \|x_1, x_2, \ldots, x_n\| < t$$

and τ_M *is the minimum t-norm. Then* (L, v, τ_M) *is an R-n-NLS.*

Proof. $(R{-}nN_1):$ For all $t \in R$, $v_{(x_1,x_2,\ldots,x_n)}(t) = \varepsilon_0 \Leftrightarrow \|x_1, x_2, \ldots, x_n\| < t \Leftrightarrow \|x_1, x_2, \ldots, x_n\| = 0 \Leftrightarrow x_1, x_2, \ldots, x_n$ are linearly dependent.
$(R{-}nN_2):$ As $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$. It follows that $v_{(x_1,x_2,\ldots,x_n)}$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.
$(R - nN_3):$ For all $t \in R$ with $t > 0$ and $\alpha \in R - \{0\}$,
$v_{(x_1,x_2,\ldots,\alpha x_n)}(t) = 0 \Leftrightarrow t \leq \|x_1, x_2, \ldots, \alpha x_n\| \Leftrightarrow t \leq |\alpha| \; \|x_1, x_2, \ldots, x_n\| \Leftrightarrow \left(\frac{t}{|\alpha|}\right) \leq \|x_1, x_2, \ldots, x_n\| \Leftrightarrow v_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{|\alpha|}\right) = 0$, and $v_{(x_1,x_2,\ldots,\alpha x_n)}(t) = 1 \Leftrightarrow \|x_1, x_2, \ldots, \alpha x_n\| < t \Leftrightarrow |\alpha| \; \|x_1, x_2, \ldots, x_n\| < t \Leftrightarrow \|x_1, x_2, \ldots, x_n\| < \left(\frac{t}{|\alpha|}\right) \Leftrightarrow v_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{|\alpha|}\right) = 1$. Thus $v_{(x_1,x_2,\ldots,\alpha x_n)}(t) = v_{(x_1,x_2,\ldots,x_n)}\left(\frac{t}{|\alpha|}\right)$.
$(R - nN_4):$ For all $s, t \in R$, $v_{(x_1,x_2,\ldots,x_n+x'_n)}(s + t) = 0 \Leftrightarrow s + t \leq \|x_1, x_2, \ldots, x_n + x'_n\| \leq \|x_1, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x'_n\|$.
If $\|x_1, x_2, \ldots, x_n\| < s$ then $\|x_1, x_2, \ldots, x_n\| \not< t$. That is $v_{(x_1,x_2,\ldots,x_n)}(s) = 1$ then $v_{(x_1,x_2,\ldots,x'_n)}(t) = 0$. Thus

$$v_{(x_1,x_2,\ldots,x_n+x'_n)}(s+t) = 0 \Rightarrow \min\left(v_{(x_1,x_2,\ldots,x_n)}(s), v_{(x_1,x_2,\ldots,x'_n)}(t)\right) = 0.$$

Similarly, $v_{(x_1,x_2,\ldots,x_n+x'_n)}(s+t) \geq \min\left(v_{(x_1,x_2,\ldots,x_n)}(s), v_{(x_1,x_2,\ldots,x'_n)}(t)\right)$.

A nonempty set $\underbrace{A \times A \times \cdots \times A}_{n}$ in a R-n-NLS (L, v, τ) is
(a) Certainly bounded, since for some $t_0 \in (0, +\infty)$ where $t_0 > \|x_1, x_2, \ldots, x_n\|$ then

$$\inf\{v_{(x_1,x_2,\ldots,x_n)}(t_0) : x_1, x_2, \ldots, x_n \in A\} = 1$$

(b) Not perhaps bounded, since for all $t \in (0, +\infty)$

$$\inf\{v_{(x_1,x_2,\ldots,x_n)}(t) : x_1, x_2, \ldots, x_n \in A\} = 1 \not< 1.$$

(c) Not perhaps unbounded, since

$$\lim_{t\to\infty} \inf\{v_{(x_1,x_2,\ldots,x_n)}(t) : x_1, x_2, \ldots, x_n \in A\} = 1 \notin (0, 1)$$

(d) Not certainly unbounded, since

$$\inf\{v_{(x_1,x_2,\ldots,x_n)}(+\infty) : x_1, x_2, \ldots, x_n \in A\} = 1 \neq 0.$$

Theorem 3.8. *Let (L, v, τ) be a R-n-NLS and $\underbrace{A \times A \times \cdots \times A}_{n}$ be a nonempty subset of $\underbrace{L \times L \times \cdots \times L}_{n}$. Then $\underbrace{A \times A \times \cdots \times A}_{n}$ is a D-bounded if and only if $\lim_{t\to+\infty} \varphi_{A\times A\times\cdots\times A}(t) = 1$.*

Proof. If $\underbrace{A \times A \times \cdots \times A}_{n}$ is a D-bounded set then it is clear that $\lim_{t\to+\infty} \varphi_{A\times A\times\cdots\times A}(t) = 1$. Conversely, if $\lim_{t\to+\infty} \varphi_{A\times A\times\cdots\times A}(t) = 1$. Then we have $\forall\ \delta > 0,\ \exists\ M > 0,\ \forall\ t_0 > M \Rightarrow 1 - \delta < \varphi_{A\times A\times\cdots\times A}(t_0) \leq 1 \Rightarrow\ \exists\ s$ such that $t_0 > s > M$, $1 - \delta < \varphi_{A\times A\times\cdots\times A}(s) \leq 1 \ \Rightarrow 1 - \delta < \lim_{s\to t_0} \varphi_{A\times A\times\cdots\times A}(s) \leq 1 \Rightarrow 1 - \delta < R_{A\times A\times\cdots\times A}(t_0) \leq 1 \Rightarrow \lim_{t_0\to+\infty} R_{A\times A\times\cdots\times A}(t_0) = 1$. Therefore $\underbrace{A \times A \times \cdots \times A}_{n}$ is a D-bounded set.

Theorem 3.9. *A subset $\underbrace{A \times A \times \cdots \times A}_{n}$ in a R-n-NLS (L, v, τ) is a D-bounded if and only if there exists a distance distribution function $G \in D^+$ such that $v_{(p_1,p_2,\ldots,p_n)} \geq G$ for every $p_1, p_2, \ldots, p_n \in A$.*

Proof. Let $G = \varphi_{A\times A\times\cdots\times A}$ therefore $G(0) = 0$ and by theorem 3.8. If $\underbrace{A \times A \times \cdots \times A}_{n}$ is a D-bounded set, then we have $\lim_{t\to+\infty} G(t) = 1$. Then $G(t) = \varphi_{A\times A\times\cdots\times A}(t) \leq v_{(p_1,p_2,\ldots,p_n)}(t)$, for every $p_1, p_2, \ldots, p_n \in A$. Conversely, let $G \in D^+$ such that, if $G(t) = v_{(p_1,p_2,\ldots,p_n)}(t)$ for every $p_1, p_2, \ldots, p_n \in A$, then $\underbrace{A \times A \times \cdots \times A}_{n}$ is a D-bounded set. But if $G(t) < v_{(p_1,p_2,\ldots,p_n)}(t)$ then $G(t) < \varphi_{A\times A\times\cdots\times A}(t) \leq v_{(p_1,p_2,\ldots,p_n)}(t) \Rightarrow \lim_{t\to+\infty} G(t) = 1$, since $G \in D^+ \Rightarrow \lim_{t\to+\infty} \varphi_{A\times A\times\cdots\times A}(t) = 1$.

If $\underbrace{A \times A \times \cdots \times A}_{n}$ be a D-bounded set then $\underbrace{A \times A \times \cdots \times \alpha A \times \cdots \times A}_{n}$ need not be D-bounded set, but this will hold under suitable conditions, as shown in the next theorem.

Theorem 3.10. *Let* (L, v, τ) *be a R-n-NLS and* $\underbrace{A \times A \times \cdots \times A}_{n}$ *be a D-bounded set in* $\underbrace{L \times L \times \cdots \times L}_{n}$*. Then*

$$\underbrace{A \times A \times \cdots \times \alpha A \times \cdots \times A}_{n} = \{(p_1, p_2, \ldots, \alpha p_i, \ldots, p_n) : p_1, p_2, \ldots, p_i, \ldots, p_n \in A\}$$

is also D-bounded for every fixed $\alpha \in R - \{0\}$*, where* D^+ *is closed set under* τ*. i.e.,* $\tau(D^+ \times D^+) \subseteq D^+$*.*

Proof. Since (L, v, τ) be a R-n-NLS and by remark 3.3, we have it sufficient to consider the case $\alpha > 0$, because $R - nN_3$. In case $\alpha \in (0, 1)$, then

$$v_{(p_1,p_2,\ldots,\alpha p_i,\ldots,p_n)}(t) = v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}\left(\frac{t}{|\alpha|}\right) \geq v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}(t) \geq R_{A\times A\times\cdots\times A}(t).$$

This shows that $\underbrace{A \times A \times \cdots \times \alpha A \times \cdots \times A}_{n}$ is D-bounded set. In case $\alpha = 1$ then $\underbrace{A \times A \times \cdots \times \alpha A \times \cdots \times A}_{n}$ is D-bounded set. In case $\alpha > 1$ then let $k = [\alpha] + 1$, then remark 3.3, we have $v_{(p_1,p_2,\ldots,\alpha p_i,\ldots,p_n)} \geq v_{(p_1,p_2,\ldots,kp_i,\ldots,p_n)}$. Now let $G_\alpha = \tau^{k-1}\left(R_{A\times A\times\cdots\times A}, R_{A\times A\times\cdots\times A}, \ldots, R_{A\times A\times\cdots\times A}\right)$, one has by induction

$$\begin{aligned}
v_{(p_1,p_2,\ldots,\alpha p_i,\ldots,p_n)}(t) &\geq \tau\left(v_{(p_1,p_2,\ldots,(k-1)p_i,\ldots,p_n)}, v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}\right)(t)\\
&\geq \tau\left(\tau\left(v_{(p_1,p_2,\ldots,(k-2)p_i,\ldots,p_n)}, v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}\right), v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}\right)(t)\\
&\geq \tau^{k-1}\left(v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}, v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}, \ldots, v_{(p_1,p_2,\ldots,p_i,\ldots,p_n)}\right)(t)\\
&\geq \tau^{k-1}\left(R_{A\times A\times\cdots\times A}, R_{A\times A\times\cdots\times A}, \ldots, R_{A\times\cdots\times A}\right)(t)
\end{aligned}$$

and hence $v_{(p_1,p_2,\ldots,\alpha p_i,\ldots,p_n)}(t) \geq G_\alpha$. Finally, one can say that $R_{A\times A\times\cdots\times\alpha A\times\cdots\times A} \geq G_\alpha$ and since $G_\alpha \in D^+$ then $\underbrace{A \times A \times \cdots \times \alpha A \times \cdots \times A}_{n}$ is D-bounded.

4 D-Compact Sets in Random n-Normed Linear Space

In this section we introduce the concept of distributional compactness (briefly D-compactness) in a random n-normed space.

Definition 4.1. *A sequence* $\{x_k\}$ *in a R-n-NLS* (L, v, τ) *is called D-convergent to* x *and denoted by* $x_k \longrightarrow x$ *as* $k \longrightarrow \infty$ *if* $\lim_{k\to\infty} v_{(x_k - x, w_2, \ldots, w_n)}(t) = \varepsilon_0$*, for all* $w_2, w_3, \ldots, w_n \in L$*,* $t > 0$*.*

Definition 4.2. *A subset* $\underbrace{A \times A \times \cdots \times A}_{n}$ *in a R-n-NLS* (L, v, τ) *is called D-compact subset if for every sequence* $\{y_k\}$ *in* A*, there exists a subsequence of* $\{y_k\}$ *which D-convergent to an element* $y \in A$*.*

Lemma 4.3. *Let* $x_1, x_2, \ldots, x_n$ *be an element of a R-n-NLS* (L, v, τ) *and* $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in R$*, then*

$$v_{(x_1,x_2,\ldots,x_{n-1},x_n)} = v_{(x_1,x_2,\ldots,x_{n-1},x_n+\gamma_1 x_1+\gamma_2 x_2+\cdots+\gamma_{n-1}x_{n-1})}.$$

Proof. It is sufficient to prove that

$$v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)} = v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)}$$

First we show that $v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)} \geq v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)}$, because

$$\begin{aligned} v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)} &= v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i-\gamma x_i,\ldots,x_n)} \\ &\geq \tau\left(v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)}, v_{(x_1,x_2,\ldots,x_i,\ldots,-\gamma x_i,\ldots,x_n)}\right) \\ &= \tau\left(v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)}, \varepsilon_0\right) = v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)}. \end{aligned}$$

On the other hand $v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)} \geq v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)}$, because

$$\begin{aligned} v_{(x_1,x_2,\ldots,x_i,\ldots,x_j+\gamma x_i,\ldots,x_n)} &\geq \tau\left(v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)}, v_{(x_1,x_2,\ldots,x_i,\ldots,\gamma x_i,\ldots,x_n)}\right) \\ &= \tau\left(v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)}, \varepsilon_0\right) = v_{(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)}. \end{aligned}$$

Hence the lemma.

Lemma 4.4. *Let* (L, v, τ) *be a R-n-NLS and* $\underbrace{A \times A \times \cdots \times A}_{n}$ *be a D-compact subspace of* (L, v, τ)*. For* $w_1, w_2, \ldots, w_n \in (L, v, \tau)$*. If*

$$\sup_{y \in A} v_{(w_1-y,w_2-y,\ldots,w_n-y)} = \varepsilon_0$$

then there exists an element $y_0 \in A$ *such that* $v_{(w_1-y_0,w_2-y_0,\ldots,w_n-y_0)} = \varepsilon_0$*.*

Proof. For each positive integer k, there exists an element $y_k \in A$ such that $v_{(w_1-y_k,w_2-y_k,\ldots,w_n-y_k)} > \varepsilon_{(\frac{1}{k})}$. Since $\{y_k\}$ is a sequence in a D-compact space A, then for every $\delta > 0$, there exists a positive integer K with $\left(\frac{1}{K}\right) < \delta$ such that $k > K$ implies that $v_{(y_k-y_0,w_2,\ldots,w_n)} > \varepsilon_\delta$, for every $w_i \in L$ $(i = 1, 2, \ldots, n)$.

$$\text{Let} \quad w^1 := (w_1 - y_k, w_2 - y_0, \ldots, w_n - y_0)$$

$$w^i := (w_1 - y_k, w_2 - y_k, \ldots, w_{i-1} - y_k, w_i - y_k, w_j - y_0, \ldots, w_n - y_0),$$

for $i = 2, 3, \ldots, n-1$, $j = i+1, 2, \ldots, n$,

$$w^n := (w_1 - y_k, w_2 - y_k, \ldots, w_n - y_k),$$

and

$$u^1 := (y_k - y_0, w_2 - y_0, \ldots, w_n - y_0)$$
$$u^i := (w_1 - y_k, w_2 - y_k, \ldots, w_{i-1} - y_k, y_k - y_0, w_j - y_0, \ldots, w_n - y_0)$$
$$u^n := (w_1 - y_k, w_2 - y_k, \ldots, w_{n-1} - y_k, y_k - y_0).$$

By previous lemma 4.3, if $k > K$, then we have

$$\begin{aligned}
v_{(w_1-y_0,w_2-y_0,\ldots,w_n-y_0)} &\geq \tau\left(v_{u^1}, v_{w^1}\right) \geq \tau\left(v_{u^1}, \tau\left(v_{u^2}, v_{w^2}\right)\right) \\
&\geq \tau\left(v_{u^1}, \tau\left(v_{u^2}, \tau\left(v_{u^3}, v_{w^3}\right)\right)\right) \\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
&\geq \tau^{n-1}\left(v_{u^1}, v_{u^2}, v_{u^3}, \ldots, v_{u^{n-1}}, v_{w^{n-1}}\right) \\
&\geq \tau^{n-1}\left(v_{u^1}, v_{u^2}, v_{u^3}, \ldots, v_{u^{n-1}}, \tau\left(v_{u^n}, v_{w^n}\right)\right) \\
&\geq \tau^{n}\left(v_{u^1}, v_{u^2}, v_{u^3}, \ldots, v_{u^{n-1}}, v_{u^n}, v_{w^n}\right) \\
&> \tau^{n}\left(\varepsilon_\delta, \varepsilon_\delta, \ldots, \varepsilon_\delta, \varepsilon_{\frac{1}{k}}\right) > \tau^{n}\left(\varepsilon_\delta, \varepsilon_\delta, \ldots, \varepsilon_\delta, \varepsilon_{\frac{1}{K}}\right) \\
&> \tau^{n}\left(\varepsilon_\delta, \varepsilon_\delta, \ldots, \varepsilon_\delta, \varepsilon_\delta\right).
\end{aligned}$$

Then $v_{(w_1-y_0,w_2-y_0,\ldots,w_n-y_0)} = \varepsilon_0$.

Lemma 4.5. *Let A and Z be subspaces of R-n-NLS (L, v, τ) and $\underbrace{A \times A \times \cdots \times A}_{n}$ be a D-compact proper subset of Z with $dim A \geq n$. Then*

$$\sup_{y \in A} v_{(w_1-y,w_2-y,\ldots,w_n-y)}(t) < \varepsilon_0(t),$$

for every $w_1, w_2, \ldots, w_n \in Z - A$ and $t > 0$.

Proof. Suppose $w_1, w_2, \ldots, w_n \in Z - A$ be linearly independent and

$$B = \sup_{y \in A} v_{(w_1-y,w_2-y,\ldots,w_n-y)}(t), \quad (t > 0).$$

If $B = \varepsilon_0(t)$, then by lemma 4.4, there exists $y_0 \in A$ such that $v_{(w_1-y_0,w_2-y_0,\ldots,w_n-y_0)} = \varepsilon_0$. If $y_0 = 0$ then $w_1, w_2, \ldots, w_n$ are linearly dependent, which is a contradiction. So $y_0 \neq 0$. Hence $w_1, w_2, \ldots, w_n, y_0$ are linearly independent. On the other hand, it follows from the definition and above equation that $w_1 - y_0, w_2 - y_0, \ldots, w_n - y_0$ are linearly dependent. Thus there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all of which are zero such that

$$\alpha_1(w_1 - y_0) + \alpha_2(w_2 - y_0) + \cdots + \alpha_n(w_n - y_0) = 0$$
$$\Rightarrow \quad \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n + (-1)(\alpha_1 + \alpha_2 + \cdots + \alpha_n) y_0 = 0$$

Then $w_1, w_2, \ldots, w_n, y_0$ are linearly dependent, which is a contradiction. Hence $B < \varepsilon_0(t)$.

Definition 4.6. *A subset* $\underbrace{A \times A \times \cdots \times A}_{n}$ *of R-n-NLS* (L, v, τ) *is called a partially D-closed subset if for linear independent elements* $x_1, x_2, \ldots, x_n$ *in* (L, v, τ) *there exists a sequence* $\{y_k\}$ *in* A *such that* $v_{(x_1-y_k, x_2-y_k, \ldots, x_n-y_k)}(t) \longrightarrow \varepsilon_0(t)$ *as* $k \longrightarrow \infty$ *then* $x_j \in A$ *for some* j.

Theorem 4.7. *Let* A *and* Z *be subspaces of R-n-NLS* (L, v, τ) *and* $\underbrace{A \times A \times \cdots \times A}_{n}$ *be a partially D-closed proper subset of* Z. *Assume that* $dimZ \geq n$. *Then*

$$\sup_{y \in A} v_{(w_1-y, w_2-y, \ldots, w_n-y)}(t) < \varepsilon_0(t),$$

for every $w_1, w_2, \ldots, w_n \in Z - A$ *and* $t > 0$.

Proof. Let $w_1, w_2, \ldots, w_n \in Z - A$ be linearly independent and

$$B = \sup_{y \in A} v_{(w_1-y, w_2-y, \ldots, w_n-y)}(t), \qquad (t > 0).$$

If $B = \varepsilon_0(t)$, then there is a sequence $\{y_k\}$ in A such that $v_{(w_1-y_k, w_2-y_k, \ldots, w_n-y_k)} \longrightarrow \varepsilon_0$ as $k \longrightarrow \infty$. Since A is partially D-closed, then $w_j \in A$ for some j, which is a contradiction. Hence $B < \varepsilon_0$.

Lemma 4.8. *Let* (L, v, τ) *be a R-n-NLS such that* $v(L^n) \subseteq D^+$ *and* $\tau(D^+ \times D^+) \subseteq D^+$ *then the set of all convergent sequences in* L *is a D-bounded subset of* (L, v, τ).

Proof. Let $A \subseteq L$ and $A = \{(p_m, w_2, \ldots, w_n) : p_m \in A, m \in \mathtt{N}, w_i \in L, 2 \leq i \leq n\}$ and $p_m \longrightarrow p$, then there exist a positive integer N such that for every $m > N$ we have $v_{(p_m-p, w_2, \ldots, w_n)} = \varepsilon_0$, it means $v_{(p_m-p, w_2, \ldots, w_n)} \geq G$, for every $G \in D^+$. Now we have

$$v_{(p_m, w_2, \ldots, w_n)} \geq \tau\left(v_{(p_m-p, w_2, \ldots, w_n)}, v_{(p, w_2, \ldots, w_n)}\right) \geq \tau\left(G, v_{(p, w_2, \ldots, w_n)}\right).$$

Now let $K = \min\left\{v_{(p_1, w_2, \ldots, w_n)}, v_{(p_2, w_2, \ldots, w_n)}, \ldots, v_{(p_{N-1}, w_2, \ldots, w_n)}, \tau\left(G, v_{(p, w_2, \ldots, w_n)}\right)\right\}$ then $K \in D^+$. Also $v_{(p_m, w_2, \ldots, w_n)} \geq K$, for every $m \in \mathtt{N}$. Therefore by theorem 3.9, A is D-bounded.

Theorem 4.9. *A D-compact subset of a R-n-NLS* (L, v, τ) *in which* $v(L^n) \subseteq D^+$ *and* $\tau(D^+ \times D^+) \subseteq D^+$ *is D-bounded and partially D-closed.*

Proof. Suppose that $A \subseteq L$ be D-compact. If A is D-unbounded, it contains a D-unbounded sequence $\{p_m\}$, by lemma 4.8, this sequence could not have a convergent subsequence, therefore A is not a D-compact set and it is a contradiction. The partially D-closeness of A comes from Definition 4.2 and 4.6.

The converse of the above theorem need not be true. The D-bounded and partially D-closed subset of a R-n-NLS (L, v, τ) is not D-compact in general.

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