Gen. Math. Notes, Vol. 5, No. 1, July 2011, pp.1-6
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Dependance of Solution of Difference Equation on Initial Conditions and Parameters

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(Received:26-5-11/Accepted:20-6-11)

Abstract

In this paper we consider the problem of continuity of solutions $x(t, t_0, x_0)$ *of system*

$$\Delta x(t) = f(t,\, x(t)),\ x(t_0) = x_0,\ t_0 \geq 0,$$

with respect to the initial values (t_0, x_0).

Keywords: *Difference Equation, Existence of solution, Fixed Point Theorem.*

1 Introduction

Let $J = \{t_0, t_0 + 1, ..., t_0 + a\},\ t_0 \in R$ and E be an open subset of R. Consider the difference equations with an initial condition,

$$\Delta u(t) = g(t,\, u(t)),\ u(t_0) = u_0. \tag{1}$$

where $u_0 \in E$, $u : J \to E$, $g : J \times E \to R$.
The function $\phi : J \to R$ is said to be a solution of initial value problem (1), if it satisfies

$$\Delta\phi(t) = g(t, \phi(t));\ \ \phi(t_0) = u_0.$$

The initial value problem (1) is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

By summation convetion $\sum_{s=t_0}^{t_0-1} g(s,u(s)) = 0$ and so $u(t)$ given above is the solution of (1).
Now we define the maximal and minimal solution of of (1).

Definition 1.1 *Let $r(t)$ be any solution of (1) on J. Then $r(t)$ is said to be maximal solution of (1), if every solution $u(t)$ of (1) existing on J, the inequality $u(t) \leq r(t)$ holds for $t \in J$.*
A solution $\rho(t)$ of (1) is said to be minimal solution of (1), if $\rho(t) \leq u(t)$ for $t \in J$.

Theorem 1.2 [4] *Suppose $g : R_0 \to R$, where $R_0 = \{(t,u) \in J \times E$ with $|u-u_0| \leq b\}$; $|g(t,u)| \leq M$ on R_0 and $g(t,u)$ is nondecreasing in u for all $t \in J$. Let $m : J \to R$ such that*
(i) $(t, m(t)) \in R$,
(ii) $m(t_0) \leq u_0$,
(iii) $\Delta m(t) \leq g(t, m(t))$
for $t \in [t_0,\ t_0+\alpha],\ \alpha = min\{a,\ b/2M + b\}$. If $r(t)$ is maximal solution of (1) on $[t_0,\ t_0+\alpha]$, then $m(t) \leq r(t)$ on $[t_0,\ t_0+\alpha]$.

Theorem 1.3 [2] *Assume that*
(i) the function $g(t,u)$ is continuous and nonnegative for $t_0 \leq t \leq t_0 + a,\ 0 \leq u \leq 2b$, and, for every t^*, $t_0 < t^* < t_0 + a$, $u(t) \equiv 0$ is the only function on $t_0 \leq t < t^*$, which satisfies

$$\Delta u(t) = g(t,\, u(t)),\ u(t_0) = 0$$

for $t_0 \leq t < t^*$;
(ii) $f : R_0 \to R$, where $R_0 = \{t \in [t_0, t_0 + a] :\ |x - x_0| \leq b\}$, and for $(t,\, x), (t,\, y) \in R_0$,

$$|f(t,\, x) - f(t,\, y)| \leq g(t,\, |x-y|).$$

Then the difference equation

$$\Delta x(t) = f(t,\, x),\ x(t_0) = x_0$$

has atmost one solution on $t_0 \leq t \leq t_0 + a$.

Theorem 1.4 [3] *Let $g : J \times E \to R$ and let J be the largest interval of the existence of the maximal solution $r(t)$ of (1). Suppose $[t_0,\ t_1]$ is a compact subinterval of J. Then there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the maximal solution $r(t,\epsilon)$ of*

$$\Delta u(t) = g(t,u) + \epsilon,\ u(t_0) = u_0 + \epsilon$$

exists over $[t_0,\ t_1]$, and $\lim_{\epsilon \to 0} r(t,\epsilon) = r(t)$ uniformly on $[t_0,\ t_1]$.

2 Main Results

Lemma 2.1 *Let $f : J \times R \to R$ be continuous and let*

$$G(t,r) = \max_{|x-x_0|\leq r} |f(t,x)|.$$

Assume that $r^*(t,t_0,0)$ is the maximal solution of

$$\Delta u(t) = G(t,u(t)),$$

through $(t_0,0)$. Let $x(t,t_0,x_0)$ be any solution of

$$\Delta x(t) = f(t,x),\ x(t_0) = x_0,\ t_0 \geq 0. \tag{2}$$

Then

$$|x(t,t_0,x_0) - x_0| \leq r^*(t,t_0,0),\ t \geq t_0.$$

Proof: Define $m(t) = |x(t,t_0,x_0) - x_0|$. Then

$$\begin{aligned} \Delta m(t) &\leq |\Delta x(t,t_0,x_0)| \\ &= |f(t,x(t,t_0,x_0))| \\ &\leq \max_{|x-x_0|\leq m(t)} |f(t,x)| \\ &= G(t,m(t)). \end{aligned}$$

This implies by Theorem 1.2 , that

$$m(t) = |x(t,t_0,x_0) - x_0| \leq r^*(t,t_0,0),\ t \geq t_0,$$

and this proves lemma.

Theorem 2.2 *Let $f : J\times R \to R$ be continuous and for $(t,x),(t,y) \in J\times R$,*

$$|f(t,x) - f(t,y)| \leq g(t,|x-y|), \tag{3}$$

where $g : J \times R \to$ is continuous mapping. Assume that $u(t) \equiv 0$ is the unique solution of difference equation

$$\Delta u(t) = g(t,u(t)) \tag{4}$$

such that $u(t) = 0$. Then if the solutions $u(t,t_0,u_0)$ of (4) through every point (t_0,u_0) are continuous with respect to initial conditions (t_0,u_0), the solutions $x(t,t_0,x_0)$ of (2)are unique and continuous with respect to the initial values (t_0,u_0).

Proof: Since the uniqueness of the solutions follows froem the Theorm 1.3, we have to prove the continuity part only. To that end, let $x(t,t_0,u_0)$ and $(y(t,t_0,u_0)$ be the solutions of (2) through (t_0,x_0) and (t_0,y_0) respectively. Defining $m(t) = |x(t,t_0,x_0) - y(t,t_0,y_0)|$, the condition (3) implies the inequality

$$\Delta m(t) \leq g(t,m(t)),$$

and by Theorem 1.2, we obtain

$$m(t) \leq r(t,t_0,|x_0 - y_0|),\ t \geq t_0,$$

where $r(t,t_0,|x_0-y_0|)$ is the maximal solution of (4) such that $u(t_0) = |x_0-y_0|$. Since the solutions $u(t,t_0,u_0)$ of (4) are assumed to be continuous with respect to the initial values, it follows that

$$\lim_{x_0 \to y_0} r(t,t_0,|x_0 - y_0|) = r(t,t_0,0),$$

and, by hypothesis, $r(t,t_0,0) \equiv 0$. This is in view of the definition of $m(t)$, yields that

$$\lim_{x_0 \to y_0} x(t,t_0,x_0) = y(t,t_0,y_0),$$

which shows the continuity of $x(t,t_0,x_0)$ with respect to x_0.
We shall next prove the continuity with respect to initial time t_0.
If $x(t,t_0,x_0),\ y(t,t^*,x_0),\ t^* > t_0$, are the solutions of (2) through $(t_0,x_0),\ (t^*,x_0)$, respectively, then, as before we obtain the inequality

$$\Delta m(t) \leq g(t,m(t)),$$

where $m(t) = |x(t,t_0,x_0) - y(t,t^*,x_0)|$. Also, $m(t^*) = |x(t^*,t_0,x_0) - x_0|$. Hence by Lema (2.1), $m(t^*) \leq r^*(t^*,t_0,0)$, and consequently, $m(t) \leq \bar{r}(t)$, $t > t^*$, where $\bar{r}(t) = \bar{r}(t,t^*,r^*(t^*,t_0,0))$ is the maximal solution of (4) through $(t^*,r^*(t^*,t_0,0))$. Since $r^*(t_0,t_0,0) = 0$, we have

$$\lim t^* \to t_0 \bar{r}(t,t^*,r^*(t^*,t_0,0)) = \bar{r}(t,t_0,0),$$

and, by hypothesis, $\bar{r}(t,t_0,0)$ is identically zero, thus proving the continuity of $x(t,t_0,x_0)$ with respect to t_0.

Theorem 2.3 *Let $f : E \to R$, where E is an open (t,x,μ)-set in $R \times R \times R$, and for $\mu = \mu_0$, let $x_0(t) = x(t,t_0,x_0,\mu_0)$ be a solution of*

$$\Delta x(t) = f(t,x,\mu_0),\ x(t_0) = x_0, \tag{5}$$

existing for $t \geq t_0$. *Assume further that*

$$\lim_{\mu \to \mu_0} f(t, x, \mu) = f(t, x, \mu_0), \tag{6}$$

uniformly in (t, x), *and for* (t, x_1, μ), $(t, x_2, \mu) \in E$,

$$|f(t, x_1, \mu) - f(t, x_2, \mu)| \leq g(t, |x_1 - x_2|) \tag{7}$$

where $g : J \times R_+ \to R_+$. *Suppose that* $u(t) \equiv 0$ *is the unique solution of (4) such that* $u(t_0) = 0$. *Then given* $\epsilon > 0$, *there exists a* $\delta(\epsilon) > 0$ *such that, for every* μ, $|\mu - \mu_0| < \delta(\epsilon)$, *the system*

$$\Delta x(t) = f(t, x, \mu), \;\; x(t_0) = x_0 \tag{8}$$

admits a unique solution $x(t) = x(t, t_0, x_0, \mu)$ *satisfying*

$$|x(t) - x_0(t)| < \epsilon, \;\; t \geq t_0.$$

Proof: The uniqueness of solutions is obvious from Theorem 1.3. From the assumption that $u(t) = 0$ is the unique solution of (4), it follows, by Theorem 1.4, that, given any compact interval $[t_0, t_0 + a]$ contained in J and any $\epsilon > 0$, there exist a positive number $\eta = \eta(\epsilon)$ such that the maximal solution $r(t, t_0, 0, \eta)$ of

$$\Delta u(t) = g(t, u) + \eta$$

exists on $t_0 \leq t \leq t_0 + a$ and satisfies

$$r(t, t_0, 0, \eta) < \epsilon, \;\; t \in [t_0, t_0 + a].$$

Furthermore, because of the condition (6), given $\eta > 0$, there exists a $\delta = \delta(\eta) > 0$ such that $|f(t, x, \mu) - f(t, x, \mu_0)| < \eta$ provided $|\mu - \mu_0| < \delta$.
Now, let $\epsilon > 0$ be given and define $m(t) = x(t) - x_0(t)$, where $x(t), x_0(t)$ are the solutions of (8) and (5) respectively. Then using the assumption (7), we get

$$\Delta m(t) \leq g(t, m(t)) + |f(t, x_0(t), \mu) - f(t, x_0(t), \mu_0)|.$$

From this it turns out that whenever $|\mu - \mu_0| < \delta$,

$$\Delta m(t) \leq g(t, m(t)) + \eta.$$

By Theorem 1.2, we have

$$\text{m(t)} \leq r(t, t_0, 0, \eta), \;\; t \geq t_0$$

and hence

$$|x(t) - x_0(t)| < \epsilon, \;\; t \geq t_0$$

provided that $|\mu - \mu_0| < \delta$.
Clearly δ depends on ϵ since η does. The proof is complete.

References

[1] R. Agarwal, *Difference Equations and Inequalities*, Markel Dekkar, New York, (1991).

[2] K.L. Bondar, Existence and uniqueness results for first order difference equation, *Journal of Modern Methods in Numerical Mathematics*, 2(1,2) (2011),16-20.

[3] K.L. Bondar, On uniform convergence of maximal solution for difference equation, *Bulletin of Pure and Applied Sciences*, 30(1) (2011).

[4] K.L. Bondar, V.C. Borkar and S.T. Patil, Comparison theorems for nonlinear difference equations, *Mathematics Education*, XLIV(4) (2010).

[5] K.L. Bondar, V.C. Borkar and S.T. Patil, Existence and uniqueness results for difference phi Laplacian, boundary value problems, *ITB Journal of Science*, 43(A)(1) (2011), 51-58.

[6] K.L. Bondar, V.C. Borkar and S.T. Patil, Some existence and uniqueness results for difference boundary value problems, *Bulletin of Pure and Applied Sciences*, 29(F)(2) (2010), 295-301.

[7] H. Dang and S.F. Oppenheimer, Existence and uniqueness results for some nonlinear boundary value problems, *J. Math. Anal. Appl.* , 198(1996), 35-48.

[8] Kelley and Peterson, *Difference Equations*, Academic Press, (2001).

[9] V. Laxmikantham and S. Leela, *Differential and Integral Inequalities-Theory and Applications*, Academic Press, (1969).