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Inverse Theorems of Approximation Theory in $L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$

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Abstract

In this paper, we prove analogues of some inverse theorems of Stechkin , for the Jacobi harmonic analysis, using the function with bounded spectrum.

Keywords: *Generalized continuity modulus, Function with bounded spectrum, Best approximation.*

1 Introduction

Yet by the year 1912, S. Bernstein obtained the estimate inverse to Jakson's inequality in the space of continuous functions for some special cases [2], later S.B.Stechkin [4], M.Timan [7], etc, proved such inverse estimation , including the case of the space L^p, $1 < p < \infty$.
In this paper, we obtain the estimate inverse to Jakson's inequality in the space $L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ (see [1], theorem 4.2) ,where the modulus of smoothness is constructed on the basis of the Jacobi generalized translation.

2 The Jacobi Transform and its Basic Properties

Let $\alpha \geq \frac{-1}{2}$, $\alpha > \beta \geq \frac{-1}{2}$, $\rho = \alpha + \beta + 1$ and let

$$L := \frac{d^2}{dx^2} + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\frac{d}{dx}.$$

be the Jacobi differential operator and denote by $\varphi_\lambda^{(\alpha,\beta)}(x)$ $(\lambda, x \in \mathbb{R}^+)$ the Jacobi function of order (α, β), the function $\varphi_\lambda^{(\alpha,\beta)}(x)$ satisfier the differential equation

$$L\varphi + (\lambda^2 + \rho^2)\varphi = 0,$$

with the initial conditions $\varphi(0) = 1$ and $\varphi'(0) = 0$

Lemma 2.1 *The following inequalities are valid for Jacobi functions* $\varphi_\lambda^{(\alpha,\beta)}(x)$
a) $\mid \varphi_\lambda^{(\alpha,\beta)}(x) \mid \leq 1.$
b) $1 - \varphi_\lambda^{(\alpha,\beta)}(x) \leq x^2(\lambda^2 + \rho^2).$
c) There exists a positive constant c_1 *such that if* $\lambda x > 1$ *, then*

$$1 - \varphi_\lambda^{(\alpha,\beta)}(x) \geq c_1.$$

Proof: Analogue (see[1], lemmas 3.1-3.2-3.3)
Consider the Hilbert space $L^2_{(\alpha,\beta)}(\mathbb{R}^+) = L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ with the norm

$$\|f\|_2 = \left(\int_0^{+\infty} \mid f(x) \mid^2 J^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

where

$$J^{\alpha,\beta}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}.$$

The Jacobi transform of a function $f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ is defined by

$$\mathfrak{J}^{\alpha,\beta}(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda^{(\alpha,\beta)}(x)J^{\alpha,\beta}(x)dx.$$

The inversion formula is

$$f(x) = \int_0^{+\infty} \mathfrak{J}^{\alpha,\beta}(f)(\lambda)\varphi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda).$$

where $d\mu(\lambda) := \frac{1}{2\pi} \mid C(\lambda) \mid^{-2} d\lambda$ and the C-function $C(\lambda)$ is defined by

$$C(\lambda) = \frac{2^\rho\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

The Plancherel formula for Jacobi transform (see [8]) is written as

$$\| f \|_{L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} = \|\mathfrak{J}^{\alpha,\beta}(f)\|_{L^2(\mathbb{R}^+, d\mu(\lambda))}. \quad (1)$$

We note the important property of the Bessel transform

$$\mathfrak{J}^{\alpha,\beta}(Lf)(\lambda) = -(\lambda^2 + \rho^2)\mathfrak{J}^{\alpha,\beta}(f)(\lambda). \quad (2)$$

The generalized translation operator was defined by Flented-Jensen and Koornwinder [6] given by

$$T^h f(x) = \int_0^{+\infty} f(z)K(x,h,z)J^{\alpha,\beta}(z)dz.$$

where the kernel K is explicitly known (see [8]).

In[3], we have

$$\mathfrak{J}^{\alpha,\beta}(T^h f)(\lambda) = \varphi_\lambda^{(\alpha,\beta)}(h)\mathfrak{J}^{\alpha,\beta}(f)(\lambda), h > 0. \quad (3)$$

In [6], we have

$$\|T^h f\|_2 \leq \|f\|_2. \quad (4)$$

A function $f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ is called a function with bounded spectrum of order $\nu > 0$ if $\mathfrak{J}^{\alpha,\beta}(f)(\lambda) = 0$ for $\lambda > \nu$. The set of all such functions is denoted by $\mathcal{I}_\nu$.

Lemma 2.2 *For every function $f \in \mathcal{I}_\nu$, $\nu \geq 1$ and any number $m \in \mathbb{N}$ we have*

$$\|L^m f\|_2 \leq c_2 \nu^{2m} \|f\|_2.$$

where $c_2 = (1+\rho^2)^m$ is a constant.

Proof: (see corollary 1.6 in [1])
The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = T^h f(x) - f(x) = (T^h - I)f(x).$$

where I is the identity operator in $L^2_{(\alpha,\beta)}(\mathbb{R}^+)$, and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (T^h - I)^k f(x).$$

The generalized continuity modulus of order k in $L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ is defined as follows

$$\omega_k(f,\delta)_2 := \sup_{0<h\leq\delta} \|\Delta_h^k f\|_2, \delta > 0, f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+).$$

The best approximation of a function $L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ by functions in $\mathcal{I}_\nu$ is the quantity

$$E_\nu(f)_2 := \inf_{g\in\mathcal{I}_\nu} \|f-g\|_2.$$

Let $W^m_{2,(\alpha,\beta)}$ be the Sobolev space of order $m \in \mathbb{N}^*$ constructed from the differential operator L, that is,

$$W^m_{2,(\alpha,\beta)} = \{f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+) : L^j f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+), j = 1, 2, ..., m\}.$$

where $L^j f = L(L^{j-1}f)$, and $L^0 f = f$.

3 Main Results

In this section we give the main results of this paper

Lemma 3.1 *The modulus of smoothness $\omega_k(f,t)_2$ has the following properties.*
i) $\omega_k(f+g,t)_2 \le \omega_k(f,t)_2 + \omega_k(g,t)_2$.
ii) $\omega_k(f,t)_2 \le 2^k\|f\|_2$.
iii) If $f \in W^k_{2,(\alpha,\beta)}$, then

$$\omega_k(f,t)_2 \le t^{2k}\|L^k f\|_2.$$

Proof:
Prperty 1 follow from the definition of $\omega_k(f,t)_2$.
Prperty 2 follow from the fact that $\|T^h f\|_2 \le \|f\|_2$.
Assume that $h \in (0,t]$. From formulas (1), (2) and (3), we have

$$\| \Delta^k_h f \|_{L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} = \|(1-\varphi^{\alpha,\beta}_\lambda(h))^k \mathfrak{J}^{\alpha,\beta}(f)(\lambda)\|_{L^2(\mathbb{R}^+,d\mu(\lambda))} \quad (5)$$

$$\| L^k f \|_{L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} = \|(\lambda^2+\rho^2)^k \mathfrak{J}^{\alpha,\beta}(f)(\lambda)\|_{L^2(\mathbb{R}^+,d\mu(\lambda))}. \quad (6)$$

Formula (5) implies the equality

$$\| \Delta^k_h f \|_{L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} = h^{2k}\left\|\frac{(1-\varphi^{\alpha,\beta}_\lambda(h))^k}{h^{2k}(\lambda^2+\rho^2)^k}(\lambda^2+\rho^2)^k\mathfrak{J}^{\alpha,\beta}(f)\right\|_{L^2(\mathbb{R}^+,d\mu(\lambda))}.$$

From lemma (2.1), we obtain

$$\begin{aligned}\| \Delta^k_h f \|_{L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} &\le h^{2k}\|(\lambda^2+\rho^2)^k\mathfrak{J}^{\alpha,\beta}(f)\|_{L^2(\mathbb{R}^+,d\mu(\lambda))} \\ &= h^{2k} \| L^k f \|_{L^2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} .\end{aligned}$$

Calculating the supremum with respect to all $h \in (0,t]$, we obtain

$$\omega_k(f,t)_2 \le t^{2k}\|L^k f\|_2.$$

Lemma 3.2 *For $j \geq 1$ we have*

$$2^{2k(j-1)} E_{2^j}(f)_2 \leq \sum_{l=2^{j-1}+1}^{2^j} l^{2k-1} E_l(f)_2.$$

Proof: Note that

$$\sum_{l=2^{j-1}+1}^{2^j} l^{2k-1} \geq (2^{j-1})^{2k-1} 2^{j-1} = 2^{2k(j-1)}.$$

Since $E_l(f)_2$ is monotonically decreasing, we conclude that

$$2^{2k(j-1)} E_{2^j}(f)_2 \leq \sum_{l=2^{j-1}+1}^{2^j} l^{2k-1} E_l(f)_2.$$

Lemma 3.3 *For $n \in \mathbb{N}^*$ we have*

$$2^k E_n(f)_2 \leq \frac{c_3}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_2.$$

Proof: Note that

$$\sum_{j=0}^{n} (j+1)^{2k-1} \geq \sum_{j \geq \frac{n}{2}-1}^{n} (j+1)^{2k-1} \geq \left(\frac{n}{2}\right)^{2k-1} \frac{n}{2} = 2^{-2k} n^{2k}.$$

Since $E_j(f)_2$ is monotonically decreasing, we conclude that

$$2^k E_n(f)_2 \leq \frac{c_3}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_2.$$

Lemma 3.4 *If $\Phi_\nu \in \mathcal{I}_\nu$ such that $\|f - \Phi_\nu\|_2 = E_\nu(f)_2$ For every $\nu \in \mathbb{N}$, then*

$$\|L^k \Phi_{2^{j+1}} - L^k \Phi_{2^j}\|_2 \leq c_2 2^{2k(j+1)+1} E_{2^j}(f)_2.$$

In particular

$$\|L^k \Phi_1\|_2 = \|L^k \Phi_1 - L^k \Phi_0\|_2 \leq c_2 2^{4k+1} E_0(f)_2.$$

Proof: By lemma (2.2) and the fact that $E_\nu(f)_2$ is monotonically decreasing with respect to ν, we obtain

$$\begin{aligned}
\|L^k \Phi_{2^{j+1}} - L^k \Phi_{2^j}\|_2 &\leq c_2 2^{2k(j+1)} \|\Phi_{2^{j+1}} - \Phi_{2^j}\|_2 \\
&= c_2 2^{2k(j+1)} \|(f - \Phi_{2^j}) - (f - \Phi_{2^{j+1}})\|_2 \\
&\leq c_2 2^{2k(j+1)} (E_{2^j}(f)_2 + E_{2^{j+1}}(f)_2) \\
&\leq c_2 2^{2k(j+1)+1} E_{2^j}(f)_2,
\end{aligned}$$

and

$$\begin{aligned}\|L^k\Phi_1 - L^k\Phi_0\|_2 &\leq c_2\|\Phi_1 - \Phi_0\|_2 = c_2\|(f-\Phi_1)-(f-\Phi_0)\|_2\\ &\leq c_2(E_1(f)_2 + E_0(f)_2)\\ &\leq 2c_2E_0(f)_2 \leq c_2 2^{4k+1}E_0(f)_2.\end{aligned}$$

The following theorems are analogues of the classical inverse theorems of approximation theory due to Stechkin in the case $p=\infty$ and A.F.Timan in the case $1 \leq p < \infty$ (see [4] and [5]).

Theorem 3.5 *For every function $f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ and $n \in \mathbb{N}^*$ we have*

$$\omega_k(f, \frac{1}{n})_2 \leq \frac{c}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2k-1}E_j(f)_2,$$

where $c = c(k,\alpha,\beta)$ is a positive constant.

Proof: Let $2^m \leq n \leq 2^{m+1}$ for any integer $m \geq 0$.
For every $\nu \in \mathbb{N}$, let $\Phi_\nu \in \mathcal{I}_\nu$ such that $\|f-\Phi_\nu\|_2 = E_\nu(f)_2$. By formulas (i) and (ii) of lemma (3.1), we obtain

$$\omega_k(f,\frac{1}{n})_2 \leq \omega_k(f-\Phi_{2^{m+1}},\frac{1}{n})_2 + \omega_k(\Phi_{2^{m+1}},\frac{1}{n})_2 \leq 2^k\|f-\Phi_{2^{m+1}}\|_2 + \omega_k(\Phi_{2^{m+1}},\frac{1}{n})_2.$$

Therefore

$$\omega_k(f,\frac{1}{n})_2 \leq 2^kE_{2^{m+1}}(f)_2 + \omega_k(\Phi_{2^{m+1}},\frac{1}{n})_2 \leq 2^kE_n(f)_2 + \omega_k(\Phi_{2^{m+1}},\frac{1}{n})_2. \quad (7)$$

Now with the aid of lemmas (3.2), (3.4) and formula (iii) of lemma (3.1), we conclude that

$$\begin{aligned}\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_2 &\leq \frac{1}{n^{2k}}\|L^k\Phi_{2^{m+1}}\|_2\\ &\leq \frac{1}{n^{2k}}\left(\|L^k\Phi_1 - L^k\Phi_0\|_2 + \sum_{j=0}^{m}\|L^k\Phi_{2^{j+1}} - L^k\Phi_{2^j}\|_2\right)\\ &\leq \frac{c_2}{n^{2k}}\left(2^{4k+1}E_0(f)_2 + \sum_{j=0}^{m}2^{2k(j+1)+1}E_{2^j}(f)_2\right)\\ &\leq \frac{c_2}{n^{2k}}2^{4k+1}\left(E_0(f)_2 + \sum_{j=0}^{m}2^{2k(j-1)}E_{2^j}(f)_2\right)\\ &\leq \frac{c_2}{n^{2k}}2^{4k+1}\left(E_0(f)_2 + E_1(f)_2 + \sum_{j=1}^{m}\sum_{l=2^{j-1}+1}^{2^j}l^{2k-1}E_l(f)_2\right)\\ &\leq \frac{c_2}{n^{2k}}2^{4k+1}\left(E_0(f)_2 + E_1(f)_2 + \sum_{j=2}^{2^m}(j+1)^{2k-1}E_j(f)_2\right).\end{aligned}$$

Whence

$$\omega_k(\Phi_{2^{m+1}}, \frac{1}{n})_2 \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{2^m} (j+1)^{2k-1} E_j(f)_2. \tag{8}$$

Thus from (7) and (8) we derive the estimate

$$\omega_k(f, \frac{1}{n})_2 \leq 2^k E_n(f)_2 + \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_2. \tag{9}$$

By lemma (3.3) and formula (9), we have

$$\omega_k(f, \frac{1}{n})_2 \leq \frac{c}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_2.$$

Theorem 3.6 *Suppose that $f \in L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ and*

$$\sum_{j=1}^{\infty} j^{2m-1} E_j(f)_2 < \infty.$$

Then $f \in W^m_{2,(\alpha,\beta)}$ and for $n \in \mathbb{N}^*$, we have

$$\omega_k(L^m f, \frac{1}{n})_2 \leq C \left(\frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2(k+m)-1} E_j(f)_2 + \sum_{j=n+1}^{\infty} j^{2m-1} E_j(f)_2 \right).$$

where $C = c(k, m, \alpha, \beta)$ is a positive constant.

Proof: Let $2^m \leq n \leq 2^{m+1}$ for any integer $m \geq 0$.
By lemmas 3.4 and 3.2, we have for $r \leq m$

$$\begin{aligned}
\sum_{j=0}^{\infty} \|L^r \Phi_{2^{j+1}} - L^r \Phi_{2^j}\|_2 &\leq c_2 \sum_{j=0}^{\infty} 2^{2r(j+1)+1} E_{2^j}(f)_2 \\
&= c_2 2^{2r+1} E_1(f)_2 + c_2 2^{4r+1} \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2^j}(f)_2 \\
&\leq c_2 2^{4r+1} \left(E_1(f)_2 + \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2^j}(f)_2 \right) \\
&\leq c_2 2^{4r+1} \left(E_1(f)_2 + \sum_{j=1}^{\infty} \sum_{l=2^{j-1}+1}^{2^j} l^{2r-1} E_l(f)_2 \right) \\
&\leq c_2 2^{4r+1} \sum_{j=1}^{\infty} j^{2r-1} E_j(f)_2 < \infty.
\end{aligned}$$

Note that

$$f=\Phi_1+\sum_{j=0}^{\infty}(\Phi_{2^{j+1}}-\Phi_{2^j}).$$

Since the series $\sum_{j=0}^{\infty}L^r\Phi_{2^{j+1}}-L^r\Phi_{2^j}$ converges in $L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ and L is a closed operator, we have

$$L^rf=L^r\Phi_1+\sum_{j=0}^{\infty}(L^r\Phi_{2^{j+1}}-L^r\Phi_{2^j}).$$

Whence $L^rf\in L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ for $r\leq m$ and $f\in W^m_{2,(\alpha,\beta)}$.
By formula (i) of lemma (3.1), we obtain

$$\omega_k(L^mf,\frac{1}{n})_2\leq\omega_k(L^mf-L^m\Phi_{2^{s+1}},\frac{1}{n})_2+\omega_k(L^m\Phi_{2^{s+1}},\frac{1}{n})_2.$$

Using lemmas (3.1) , (3.2) and (3.4), we get

$$\begin{aligned}\omega_k(L^mf-L^m\Phi_{2^{s+1}},\frac{1}{n})_2 &\leq 2^k\|L^mf-L^m\Phi_{2^{s+1}}\|_2\\ &\leq 2^k\sum_{j=s+1}^{\infty}\|L^m\Phi_{2^{j+1}}-L^m\Phi_{2^j}\|_2\\ &\leq 2^kc_2\sum_{j=s+1}^{\infty}2^{2m(j+1)+1}E_{2^j}(f)_2\\ &\leq c_22^{k+4m+1}\sum_{j=s+1}^{\infty}2^{2m(j-1)}E_{2^j}(f)_2\\ &\leq c_22^{k+4m+1}\sum_{j=s+1}^{\infty}\sum_{l=2^{j-1}+1}^{2^j}l^{2m-1}E_l(f)_2\end{aligned}$$

$$\omega_k(L^mf-L^m\Phi_{2^{s+1}},\frac{1}{n})_2 \leq c_22^{k+4m+1}\sum_{j=2^s+1}^{\infty}j^{2m-1}E_j(f)_2.$$

Whence

$$\omega_k(L^mf-L^m\Phi_{2^{s+1}},\frac{1}{n})_2 \leq c_5\sum_{j=2^s+1}^{\infty}j^{2m-1}E_j(f)_2. \tag{10}$$

Now with the aid of lemmas (3.2) , (3.4) and by formula (*iii*) of lemma (3.1), we conclude that

$$
\begin{aligned}
\omega_k(L^m\Phi_{2^{s+1}}, \frac{1}{n})_2 &\leq \frac{1}{n^{2k}}\|L^{m+k}\Phi_{2^{s+1}}\|_2 \\
&\leq \frac{1}{n^{2k}}\left(\|L^{m+k}\Phi_1 - L^{m+k}\Phi_0\|_2 + \sum_{j=0}^{s}\|L^{m+k}\Phi_{2^{j+1}} - L^{m+k}\Phi_{2^j}\|_2\right) \\
&\leq \frac{c_2}{n^{2k}}\left(2^{4(k+m)+1}E_0(f)_2 + \sum_{j=0}^{s}2^{2(k+m)(j+1)+1}E_{2^j}(f)_2\right) \\
&\leq \frac{c_2}{n^{2k}}2^{4(k+m)+1}\left(E_0(f)_2 + \sum_{j=0}^{s}2^{2(k+m)(j-1)}E_{2^j}(f)_2\right) \\
&\leq \frac{c_2}{n^{2k}}2^{4(k+m)+1}\left(E_0(f)_2 + E_1(f)_2 + \sum_{j=1}^{s}\sum_{l=2^{j-1}+1}^{2^j}l^{2(k+m)-1}E_l(f)_2\right) \\
&\leq \frac{c_2}{n^{2k}}2^{4(k+m)+1}\left(E_0(f)_2 + E_1(f)_2 + \sum_{j=2}^{2^s}(j+1)^{2(k+m)-1}E_j(f)_2\right).
\end{aligned}
$$

Whence

$$
\omega_k(B^m\Phi_{2^{s+1}}, \frac{1}{n})_2 \leq \frac{c_6}{n^{2k}}\sum_{j=0}^{2^s}(j+1)^{2(k+m)-1}E_j(f)_2. \tag{11}
$$

Thus from (10) and (11) we derive the estimate

$$
\omega_k(B^m f, \frac{1}{n})_2 \leq C\left(\sum_{j=n+1}^{\infty}j^{2m-1}E_j(f)_2 + \frac{1}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2(k+m)-1}E_j(f)_2\right).
$$

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