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An Analysis of the Modified Lotka-Volterra Predator-Prey Model

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Abstract

One of the most ecological applications of differential equations systems is predator-prey problem. In fact, differential equations are very useful in many areas of applied sciences. However, most of the nature problems involve with some unknown function. In this paper, an environmental case containing two related populations of prey species and predator species is studied. It is expected that two population make influence on the size of each other.

Keywords: *Equilibrium Point, Stability, Prey-Predator Model, Rate Growth.*

1 Introduction

In the 1920s, the Italian mathematician Vito Volterra proposed a differential equations model to describe the population dynamics of two interacting species of a predator and its prey. He hoped to explain the increasing in predator fish (and so,decreasing in prey fish) in the Adriatic Sea during World War I. Independently, this equations is studied by Volterra were derived by Alfred Lotka to describe a hypothetical chemical reaction in which the chemical concentrations oscillate [1], in the United States. There are many species of animals in nature where one species feeds on another species. The first species and the second one are called predator and prey respective[2, 3]. Some of the text books in this area may be found in [4, 5, 6].

Theoretically, the predator can destroy all of the prey and so, the latter it will extinct. However, if this happens the predator will also become extinct. It is maybe that the above cyclic leads into the fact that "Prey species will be increased by predator in during of time". An important problem of ecology is to investigate the coexistence questions of the two species [1]. To this end, it is natural to seek a mathematical formulation of this predator-prey problem and to use it to be predict populations behavior of various species at different times. For this purpose, the model may be based on the following two assumptions:

1- Prey dies iff it is eaten by predator;
2- Predator only die because of natural causes.
Consider the following initial value problem:

$$\begin{cases} \frac{dx}{dt} = ax - bx^2 - cxy & x(0) = x_0 \\ \frac{dy}{dt} = -ky + dxy & y(0) = y_0 \end{cases} \tag{1}$$

The functions $x(t)$ and $y(t)$ represent the populations of prey and predator at time t, respectively. And also, the numbers x_0, y_0 represent the initial sizes of prey and predator, respectively. Reader can find useful and related articles in this area in [7, 8, 9].

2 Main Results

First, we consider

$$\frac{dx}{dt} = ax - bx^2 - cxy;$$

The derivative $\frac{dx}{dt}$, growth rate of prey population, is influenced by three different terms. In fact, the prey species is increased by its current population shown as ax, where a is a non-negative constant. Indeed, the term of ax is the birth rate of prey. It is negatively influenced by the natural death rate of prey which is shown by the term of $-bx^2$, where b is non-negative constant and bx^2 is the natural death rate of prey. It is also decreased by the death rate of prey shown by the term $-cxy$, where c is non-negative constant.

Now consider $\frac{dy}{dt} = -ky + dxy$. The derivative $\frac{dy}{dt}$, the growth rate of predator population, is influenced by two different terms. In deed, the predator species is decreased by current predator population shown by the term $-ky$, where k is a non-negative constant real number. It is increased by prey-predator interactions which is shown by the term dxy, where d is non-negative constant. let:

$$\frac{dx}{dt} = x(a - bx - cy) \quad , \quad \frac{dy}{dt} = y(-k + dx)$$

Now, we are going to solve the following equations:

$$\begin{cases} \frac{dx}{dt} = 0: & x = 0 \quad or \quad x = \frac{a-cy}{b}, \\ \frac{dy}{dt} = 0: & y = 0 \quad or \quad x = \frac{k}{d}, \end{cases} \tag{2}$$

The first equilibrium point is origin. The second of equilibrium point is $(\frac{a}{b}, 0)$. To obtain the third point of equilibrium, we should make equal the variable x in system (1.2).

And so, we see that

$$y = \frac{-kb + ad}{cd}.$$

Thus tertiary of equilibrium point is $(\frac{k}{d}, \frac{-kb+ad}{cd})$. Now, to study the stability of the equilibrium points, we first need to find the Jacobian matrix which may be found as follows:

$$J(x, y) = \begin{bmatrix} a - 2bx - cy & -cx \\ dy & -k + dx \end{bmatrix}$$

2.1 The Stability of Equilibrium Points

i)origin:

Since for origin one eigenvalue is negative $(-k)$ and the other positive $(+a)$, origin is unstable.

ii)$(\frac{a}{b}, 0)$:

$$\lambda_1 = -a, \lambda_2 = \frac{-kb + ad}{cd}$$

If $\lambda_2 = \frac{-kb+ad}{cd} < 0$ (i.e. $ad < kb$) then the above equilibrium point is stable but if $\lambda_2 = \frac{-kb+ad}{cd} > 0$ (i.e. $ad > kb$) then above equilibrium point is unstable.

iii) $(\frac{k}{d}, \frac{-kb+ad}{cd})$:

$$\lambda_1 = \frac{1}{2d}(-kb + \sqrt{k^2b^2 + 4dk^2b - 4akd^2}) = \frac{-1}{2}(\frac{kb - i\sqrt{-k^2b^2 - 4dk^2b + 4akd^2}}{d})$$

$$\lambda_2 = \frac{1}{2d}(-kb - \sqrt{k^2b^2 + 4dk^2b - 4akd^2}) = \frac{-1}{2}(\frac{kb + i\sqrt{-k^2b^2 - 4dk^2b + 4akd^2}}{d})$$

This point is stable, because both of the real parts are negative. The imaginary numbers implies that it will be periodic.

Case 1: Stability of prey $(ad > kb)$:

In this case, $\frac{a}{b}$ is the stable point for the prey population in a predator-free world, and $\frac{k}{d}$ the term of is the critical point for the prey population living with predator. This case holds true with out initial conditions until $x_0 > 0$, and $y_0 > 0$.

Case 2: Stability of prey $(ad < kb)$

For the second case, when $\frac{a}{b}$ is less than $\frac{k}{d}$, we arrive at an interesting conclusion. When $\frac{a}{b} < \frac{k}{d}$, then for the predator equation, the critical point becomes a negative number. so $y = \frac{-kb+ad}{cd}$ results in a negative number. So, in the second case, the predator population will die out regardless of the initial conditions. Therefore, the solution would be converged to the predator-free stable point.

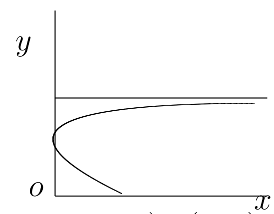

Fig 1: $u(x, y) = x(2 - 6x - 4y)$ $v(x, y) = y(-3 + 5x)$ $x_0 = 1$ $y_0 = 0.5$

Case 3: For the third case, we pay attention the question: What will be happen if all of the constants are same? A simple glance at the equations tells us that this would be similar to the second case: We get $\frac{a}{b} = \frac{k}{d} = 1$, yet the critical point would be $(1, 0)$ which is on the y-axis and identical to the stable point for the prey population in predator-free world. So here the predator die out again.

Case 4: stability of both species$(b = 0)$

let $b = 0$, Which implies it is the model into the simplest form of the prey-predator model. The new equations look like this:

$$\frac{dx}{dt} = x(a - cy)\ , \frac{dy}{dt} = y(-k + dx)$$

So the critical point becomes $(\frac{k}{d}, \frac{a}{c})$ and we get an ellipse around the above critical point. The shape and size depend on the constants and initial conditions. So, both the prey and predator populations increase and decrease in the above cyclic pattern.

Case 5: Now for the fifth case, we suppose the question: What does happens if $x(t) = y(t)$? This is pretty much similar to the first case, since $\frac{a}{b} > \frac{k}{d}$. The only significant impact is the location of the critical point.

3 Conclusion

This Lotka-Volterra Predator-Prey Model is a rudimentary model of the complex ecology of this world. It assumes just one prey for the predator, and vice versa and It also assumes no outside influences like disease, changing conditions, pollution, and so on. However, the model can be expanded to include other variables,We can polish the equations by adding more variables and get a better picture of the ecology. But with more variables, the model becomes more complex and would require more brains or computer resources. It also shows a special relationship between biology and mathematics.

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