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Kirk's Fixed Point Theorem in Generating Spaces of Semi-Norm Family

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Abstract

In this paper, we deal with a modified slightly form of generating space of quasi-norm family and extend the Kirk's type fixed point theorem in this setting which serves as a unified form of such fixed point theorem both of classical functional analysis as well as fuzzy functional analysis.

Keywords: *Generating space of quasi-norm family, normal structure, non-expansive mapping, Kirk's fixed point theorem.*

1 Introduction

J.X.Fan [3] dealt with generating spaces of quasi-metric family while he was studying on generalizations of Ekeland's variational principle and Caristi's fixed point theorem. Later Chang et al. [2], Jung et al.[4], Lee et al. [8] studied generating spaces of quasi-metric family. In 2006, Xiao & Zhu [15] introduced the concept of Generating spaces of Quasi-Norm family (GSQ-NF) and extended Schauder fixed point theorem for continuous mapping in such spaces. Observing the significance of the generating spaces of quasi-metric (quasi-norm) family in unifying the results of classical, probabilistic and fuzzy settings, we have studied some properties of this spaces and extended Hahn-Banach theorem in GSQ-NF ([9], [11], [12]).

In this paper, we extend the celebrated Kirk's fixed point theorem for non-expansive mapping in G.S.Q-N.F setting. For this, we introduce the concept of weakly convergent sequences, weakly Cauchy sequences and weakly compact sets in GSQ-NF. The definitions of diameter, radius, Chebyshev radius, Chebyshev centre are given in this space. The concept of diametral point, non-diametral point, normal structure and non-expansive mapping are also introduced in GSQ-NF.

The organization of the paper is as follows:
Section 2 comprises some preliminary results.
In Section 3, we give definitions of weakly convergent sequences, weakly Cauchy sequences, and weakly compact sets.
Some geometric properties are studied in section 4.
We establish Kirk-type fixed point theorem in Section 5.

2 Some Preliminary Results

In this section some preliminary results are given which are related to this paper.

Definition 2.1 *[15] Let X be a linear space over E and θ be the origin of X. Let*

$$Q = \{\, |.|_\alpha : \alpha \in (0,1] \,\}$$

be a family of mappings from X into $[0,\ \infty)$. $(X,\ Q)$ is called a generating space of quasi-norm family and Q a quasi-norm family if the following conditions are satisfied:
(QN-1) $|x|_\alpha = 0\ \ \forall \alpha \in (0,\ 1]$ *iff* $x = \theta$;
(QN-2) $|ex|_\alpha = |e||x|_\alpha$ *for* $x \in X$ *and* $e \in E$;
(QN-3) for any $\alpha \in (0,\ 1]$ *there exists a* $\beta \in (0,\ \alpha]$ *such that*

$$|x+y|_\alpha \leq |x|_\beta + |y|_\beta \quad \text{for } x,\ y \in X;$$

(QN-4) for any $x \in X$, $|x|_\alpha$ *is non-increasing and left-continuous for* $\alpha \in (0,\ 1]$.
$(X,\ Q)$ is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3t) and (QN-3e), where
(QN-3u) for any $\alpha \in (0,\ 1]$ *there exists* $\beta \in (0,\ \alpha]$ *such that*

$$|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta \text{ for any } n \in Z^+,\ x_i \in X (i = 1,\ 2, ...,\ n);$$

(QN-3t) for any $\alpha \in (0,\ 1]$ *there exists a* $\beta \in (0,\ \alpha]$ *such that*

$$|x+y|_\alpha \leq |x|_\alpha + |y|_\beta \quad \text{for } x,\ y \in X;$$

(QN-3e) for any $\alpha \in (0,\ 1]$, *it holds that* $|x+y|_\alpha \leq |x|_\alpha + |y|_\alpha \quad$ *for* $x,\ y \in X$.

Waiving the left-continuity property of $|x|_\alpha$ in (QN-4) and taking the family $Q = \{|.|_\alpha : \alpha \in (0,1)\}$, in [10], we have dealt with a slightly generalized form of definition of GSQ-NF [8]. Our definition is as follows:

Definition 2.2 *[10] Let X be a linear space over E(Real or Complex) and θ be the origin of X. Let*

$$Q = \{\, |.|_\alpha : \alpha \in (0,\ 1)\,\}$$

be a family of mappings from X into $[0,\ \infty)$. $(X,\ Q)$ is called a generating space of quasi-norm family and Q, a quasi-norm family, if the following conditions are satisfied:
(QN1) $|x|_\alpha = 0 \ \ \forall \alpha \in (0,\ 1)$ *iff* $x = \theta$;
(QN2) $|ex|_\alpha = |e||x|_\alpha \quad \forall\ x \in X,\ \forall \alpha \in (0,\ 1)$ *and* $\forall\ e \in E$;
(QN3) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that
$$|x+y|_\alpha \leq |x|_\beta + |y|_\beta \quad \text{for } x,\ y \in X;$$
(QN4) for any $x \in X$, $|x|_\alpha$ is non-increasing for $\alpha \in (0,\ 1)$.
$(X,\ Q)$ is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3t) and (QN-3e), where
(QN-3u) for any $\alpha \in (0,\ 1)$ there exists $\beta \in (0,\ \alpha]$ such that
$$|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta \text{ for any } n \in Z^+,\ x_i \in X (i = 1,\ 2, ...,\ n);$$
(QN-3t) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that
$$|x+y|_\alpha \leq |x|_\alpha + |y|_\beta \quad \text{for } x,\ y \in X;$$
(QN-3e) for any $\alpha \in (0,\ 1)$, it holds that $|x+y|_\alpha \leq |x|_\alpha + |y|_\alpha \quad$ for $x, y \in X$.

Approaching as in [15], it can be shown that in a GSQ-NF (X, Q), the collection $\{N(\epsilon, \alpha) : \epsilon > 0,\ \alpha \in (0,1)\}$, where $N(\epsilon,\ \alpha) = \{x\ :\ |x|_\alpha < \epsilon\}$ from a neighborhood base of $\theta \in X$. Further the associated topology τ_Q is a first countable Hausdroff linear topological space. If in addition, $(X\ ,\ Q)$ is a generating space of semi-norm family(GSS-N), then $(X\ ,\ \tau_Q)$ is a locally convex space.

Definition 2.3 *[10] Let (X, Q) be a GSQ-NF.*
(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said
(a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_\alpha = 0$ for each $\alpha \in (0,\ 1)$;
(b) to be a Cauchy sequence if $\lim_{m,\ n\to\infty} |x_n - x_m|_\alpha = 0$ for each $\alpha \in (0,\ 1)$.
(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.

Definition 2.4 *[10] Let $(X,\ Q)$ be a GSS-NF where Q satisfies*
(QN-6)*: if $x (\neq \theta) \in\ X$ then $|x|_\alpha\ > 0\ \forall \alpha \in (0,1)$. Then $(X,\ Q)$ is said to be a generating space of norm family (GSNF).*

Definition 2.5 *[10] Let (X, Q) be a GSS-NF.*
(a) A subset A of X is said to be bounded if for each $\alpha \in (0,\ 1)$ there exists a real number $M(\alpha)$ such that $|x|_\alpha \leq\ M(\alpha)\ \ \forall\ x \in\ A$.

(b) A subset A of X is said to be α-level bounded for some $\alpha \in (0,\ 1)$ if there exists a real number $M(\alpha)$ such that $|x|_\alpha \leq\ M(\alpha)\ \ \forall\ x \in\ A$.

(c) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim\limits_{n\to\infty} x_n = x$ implies $x \in\ A$.

(d) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Remark 2.6 *In a generating space of quasi-norm family (X, Q), intersection of two closed sets is a closed set.*

Definition 2.7 *[9]Let (X, Q) be a generating space of quasi-norm family.*
(a) The closure of a subset A of X is denoted by $\overline{A}$ and is defined by
$\overline{A} = \{x : if\ \exists$ *a sequence* $\{x_n\}$ *in* A *such that* $\lim\limits_{n\to\infty} x_n = x\}$.

(b) The set of all convex combinations of points of a subset A of X is denoted by $convA$ and is defined by
$convA = \{\lambda\ x + (1-\lambda)\ y\ \ \forall\ x,\ y \in\ A\ \forall\lambda \in [0,\ 1]\}$.

Definition 2.8 *[10] Let $Q_1 = \{\ |.|^1_\alpha : \alpha \in (0,1)\}$ and $Q_2 = \{\ |.|^2_\alpha : \alpha \in (0,1)\}$ be two quasi-norm families on X_1 and X_2 respectively and $T : (X_1, Q_1)\ \to\ (X_2, Q_2)$ be an operator. Then T is said to be continuous at $x \in\ X_1$ if for any sequence $\{x_n\}$ of X_1 with $x_n \to\ \ x$ i.e. with $\lim\limits_{n\to\infty} |x_n - x|^1_\alpha = 0\ \ \forall\alpha \in (0,1)$ implies $T(x_n) \to\ T(x)$.*
i.e. $\lim\limits_{n\to\infty} |T(x_n) - T(x)|^2_\alpha = 0\ \ \forall\alpha \in (0,1)$. If T is continuous at each point of X_1, then T is said to be continuous on X_1.

Definition 2.9 *[10] Let $T : (X_1, Q_1)\ \to\ (X_2, Q_2)$ be an operator. Then*
(a) T is said to be bounded if corresponding to each $\alpha \in (0,1)$, $\exists\ M_\alpha > 0$ such that

$$|T(x)|^2_\alpha \leq\ M_\alpha |x|^1_{1-\alpha}\ \ \forall x \in\ X_1.$$

(b) T is said to be $\alpha-$ level bounded for some $\alpha \in (0,1)$ if $\exists\ \ M_\alpha > 0$ such that
$|T(x)|^2_\alpha \leq\ M_\alpha |x|^1_{1-\alpha}\ \ \forall x \in\ X_1$.

Theorem 2.10 *[10] Let $T : (X_1, Q_1)\ \to\ (X_2, Q_2)$ be a linear operator. If T is bounded then it is continuous but not conversely.*

Definition 2.11 *[13] Let X be a topological vector space(TVS). The vector space of all continuous linear functionals on X is said to the dual space of X and is denoted by X^*. The addition and scalar multiplication in X^* are defined*

by:
$(T_1 + T_2)x = T_1x + T_2x$
and
$(\lambda\ T)x = \alpha\ (Tx) \qquad \forall\ T_1,\ T_2,\ T \in\ X^*$ *and* $\forall\ x \in\ X$ *and* λ *is a scalar.*

Remark 2.12 *[13] If X is locally convex space then X^* separates points on X.*

Remark 2.13 *[13] Let X be a TVS with topology τ whose dual space is X^* separates points on X. Then the X^*-topology on X makes X into a locally convex space whose dual space is X^*. Let it be denoted by τ_w. Since every $T \in\ X^*$ is τ-continuous and since τ_w is the weakest topology on X with respect to which every $T \in\ X^*$ is continuous, we have $\tau_w \subset\ \tau$. In this context τ is called original topology.*

Theorem 2.14 *[13] Let E be a convex subset of a locally convex vector space X. Then the weak closure $\overline{E_w}$ of E is equal to its original closure $\overline{E}$.*

Definition 2.15 *Let $(X_1,\ Q_1)$ and $(X_2,\ Q_2)$ be two generating spaces of quasi-norm families. We denote by $L(X_1,\ X_2)$ the set of all linear functions from X_1 to X_2. Then $L(X_1,\ X_2)$ is a linear space with respect to usual addition and scalar multiplication of operators. The vector space of all continuous linear functional on X_1 is said to be the dual space of X_1 and is denoted by X_1^*.*

Note 2.16 *As the underlying topological vector space is first countable and the induced topology is Hausdorff topology the continuity as defined in Definition 2.8 is same as the continuity with respect to the topological vector space. So all the results in the topological vector space $(X,\ \tau_Q)$ are valid in a generating space of semi-norm family $(X,\ Q)$. In this case $X_2 = R$ and $Q_2 = \{\ |.|_\alpha = |.| : \alpha \in (0,\ 1]\}$.*

Remark 2.17 *Let $(X,\ Q)$ be a generating space of quasi-norm family and $\alpha \in (0, 1)$. Then any α-level bounded linear functional on X is a continuous linear functional on X.*

Theorem 2.18 *[11] Let $(X,\ Q)$ be a generating space of semi-norm family and $x_0 \in\ X$ such that $|x_0|_{1-\alpha} \neq\ 0$ for some $\alpha \in\ (0, 1)$. Then there exists an α-level bounded linear functional $\hat{f}_\alpha$ i.e. a continuous linear functional on X such that*
$|\hat{f}_\alpha|_\alpha^s = 1$ *and* $\hat{f}_\alpha(x_0) = |x_0|_{1-\alpha}$.

3 Weakly Convergent Sequence and Weakly Compact Set

In this section, we introduce the concept of weakly convergent sequences, weakly Cauchy sequences and weakly compact set in GSS-NF.

Definition 3.1 *Let (X, Q) be a GSS-NF:*
(a) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said
(i) to weakly convergent to $x \in X$ denoted by $x_n \to^w x$ if $\lim\limits_{n\to\infty} |f(x_n) - f(x)| = 0$ for each $f \in X^*$. In this case x is called the weak limit of the sequence $\{x_n\}$;
(ii) to be a weakly Cauchy sequence if $\lim\limits_{m,\ n\to\infty} |f(x_n) - f(x_m)| = 0$ for each $f \in X^*$;
(b) A subset $B \subset X$ is said to be weakly complete if every weakly Cauchy sequence in B weakly converges in B;
(c) A subset A of X is said to be weakly closed if for any sequence $\{x_n\}$ of points of A with $x_n \to^w x$ implies $x \in A$;
(d) A subset A of X is said to be weakly compact if for any sequence $\{x_n\}$ of points of A has a weakly convergent subsequence which weakly converges to a point in A.

Proposition 3.2 *Let $(X,\ Q)$ be a GSS-NF:*
(a) The weak limit of a sequence $\{x_n\}$ in X if exists is unique;
(b) Every subsequence of a weakly convergent sequence converges weakly to the same weak limit;
(c) Every convergent sequence in X is weakly convergent;
(d) Every weakly convergent sequence in X is a weakly Cauchy sequence;
(e) Every weakly compact set in X is weakly closed and closed.

Proof(a). If possible let a sequence $\{x_n\}$ in $(X,\ Q)$ converges weakly to two different limits x and y. Then
$\lim\limits_{n\to\infty} |f(x_n) - f(x)| = 0$ and
$\lim\limits_{n\to\infty} |f(x_n) - f(y)| = 0$ for each $f \in X^*$.
Now $|f(x) - f(y)| \leq |f(x) - f(x_n)| + |f(x_n) - f(y)|$
$\Rightarrow \lim\limits_{n\to\infty} |f(x) - f(y)| \leq \lim\limits_{n\to\infty} |f(x) - f(x_n)| + \lim\limits_{n\to\infty} |f(x_n) - f(y)| = 0$
$\Rightarrow |f(x - y)| = 0$
$\Rightarrow f(x - y) = 0$ for each $f \in X^*$.
We claim that $x = y$, if not, there exists an $\alpha \in (0,\ 1)$ such that $|x-y|_{1-\alpha} \neq 0$ and by Theorem 2.18, there exists a continuous linear functional $f_\alpha \in X^*$ such that
$|f_\alpha|_\alpha^s = 1$ and $f_\alpha(x - y) = |x - y|_{1-\alpha} \neq 0$, which is a contradiction.

Proof(b). Let the sequence $\{x_n\}$ converges weakly to x. Then
$\lim\limits_{n\to\infty} |f(x_n) - f(x)| = 0$ for each $f \in X^*$
$\Rightarrow \lim\limits_{n_k\to\infty} |f(x_{n_k}) - f(x)| = 0$ for all $f \in X^*$ and for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Proof(c). Let the sequence $\{x_n\}$ converges to x. Then
$\lim\limits_{n\to\infty} |x_n - x|_\alpha = 0 \quad \forall\alpha \in (0,\ 1)$.
If $f \in X^*$ then, since f is continuous
$\lim\limits_{n\to\infty} |f(x_n) - f(x)| = 0$.
Hence the the sequence $\{x_n\}$ converges weakly to x.

Proof(d). Proof is straightforward.

Proof(e). Let $A \subset X$ be weakly compact and $\{x_n\}$ be a weakly convergent sequence in A which converges weakly to x. Since A is weakly compact, $\{x_n\}$ has a weakly convergent subsequence which converges weakly to some point in A. But all the subsequences of $\{x_n\}$ converges weakly to x, since $\{x_n\}$ is weakly convergent sequence and converges weakly to x. Hence $x \in A$. So A is weakly closed.
Let $\{x_n\}$ be a convergent sequence in A which converges to x. Then $\{x_n\}$ is a weakly convergent sequence which converges weakly to x. Since A is weakly closed, $x \in A$. So A is closed.

4 Some Geometric Properties in GSQ-NF

In this section, we give the definitions of radius, Chebyshev radius, Chebyshev center, diametral point and normal structure in GSQ-NF.

Definition 4.1 *Let (X, Q) be a generating space of quasi-norm family. A subset A of X is said to be strongly bounded if there exists a real number $M > 0$ such that $|x|_\alpha \leq M \ \ \forall\, x \in A \ \ \forall\alpha \in (0,\ 1)$.*

Proposition 4.2 *In a GSQ-NF $(X,\ Q)$ every strongly bounded subset of X is bounded.*

The converse of the above Proposition is not always true, which is justified by the following example.

Example 4.3 *Let $X = R^2$ be a linear space. For $x = (x_1,\ x_2) \in X$ define*

$$|x|_\alpha = \frac{1}{\alpha}\sqrt{x_1^2 + x_2^2} \qquad \forall\alpha \in (0,\ 1).$$

Then clearly

$$Q = \{\, |.|_\alpha : \alpha \in (0,\ 1)\,\}$$

is a quasi-norm family and $(X,\ Q)$ *is a GSQ-NF.*
Let us consider the set $A = \{x = (x_1,\ x_2) \in\ X : x_1^2 + x_2^2 \leq\ 1\}$. *Then it is easy to verify that* A *is bounded but not strongly bounded.*

Proposition 4.4 *In a generating space of quasi-norm family* $(X,\ Q)$, *every compact set is closed and bounded.*

Proof. Let A be any compact subset of X. Let $\{x_n\}$ be a sequence in A such that $lim\ x_n = x$. Since A is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point in A. But $lim\ x_n = x$ implies $lim\ x_{n_k} = x$ and hence $x \in\ A$. So A is closed.
If possible let A be unbounded. Then there exists $\alpha_0 \in (0,\ 1)$ and for each $n \in\ N$, $\exists\ x_n \in\ A$ such that $|x_n|_{\alpha_0} \geq\ n$. So $\{x_n\}$ is a sequence in A. Since A is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $lim\ x_{n_k} = x \in\ A$.
Now $|x_{n_k}|_{\alpha_0} = |x_{n_k} - x + x|_{\alpha_0} \leq\ |x_{n_k} - x|_{\beta_0} + |x|_{\beta_0}$, where $0 < \beta_0 \leq \alpha_0$
$\Rightarrow\ |x|_{\beta_0} \geq\ |x_{n_k}|_{\alpha_0} - |x_{n_k} - x|_{\beta_0}$.
Taking limit $n \to \infty$ we get
$|x|_{\beta_0} \geq\ \lim\limits_{n\to\infty} |x_{n_k}|_{\alpha_0} = \infty$, which is a contradiction. Hence A is bounded.

Definition 4.5 *Let* $(X,\ Q)$ *be a GSQ-NF and* D , H *are two strongly bounded subsets of* X. *Set:*

(i) $\delta(D) = \bigvee\limits_{\alpha\in(0,1)} [\bigvee\{|x-y|_\alpha,\ \forall x,\ y \in D\}]$.

(ii) $r_u(D) = \bigvee\limits_{\alpha\in(0,1)} [\bigvee\{|u-x|_\alpha,\ \forall x \in D\}],\ (u\ \in H)$.

(iii) $r_H(D) = \bigwedge\limits_{u\in H} \{r_u(D)\}$.

(iv) $C_H(D) = \{u\ \in H :\ r_u(D) = r_H(D)\}$.

The number $\delta(D)$ *is called the diameter of* D , $r_u(D)$ *is called the radius of* D *relative to* u, $r_H(D)$ *and* $C_H(D)$ *are called respectively the Chebyshev radius and the Chebyshev center of* D *relative to* H. *When* $H = D$ *the notations* $r(D)$ *and* $C(D)$ *are used for* $r_H(D)$ *and* $C_H(D)$ *respectively.*

Proposition 4.6 *Let* $(X,\ Q)$ *be a GSQ-NF and* $D \subset\ X$. *Then for any* $u \in\ D$,
$\delta(D)\ \geq\ r_u(D)\ \geq\ r(D)$.

Proof. The proof is straightforward.

Definition 4.7 *Let $(X,\ Q)$ be a GSQ-NF and D is a strongly bounded subset of X. A point $u \in D$ is said to be a diametral point if $r_u(D) = \delta(D)$. If u is not a diametral point of D, then it is called a fuzzy non-diametral point of D.*

Definition 4.8 *Let $(X,\ Q)$ be a GSQ-NF. A nonempty strongly bounded, convex subset K of X is said to have normal structure if each strongly bounded, convex subset S of K with $\delta(S) > 0$ contains a non-diametral point.*

Definition 4.9 *Let $(X,\ Q)$ be a GSQ-NF and D, H are two bounded subsets of X.*
For $\alpha \in (0,\ 1)$ set:

(i) $\delta^{\alpha}(D) = \bigvee\{|x-y|_{\alpha},\ \forall x,\ y \in D\}$.

(ii) $r_u^{\alpha}(D) = \bigvee\{|u-x|_{\alpha},\ \forall x \in D\},\ (u \in H)$.

(iii) $r_H^{\alpha}(D) = \bigwedge_{u \in H}\{r_u^{\alpha}(D)\}$.

(iv) $C_H^{\alpha}(D) = \{u \in H:\ r_u^{\alpha}(D) = r_H^{\alpha}(D)\}$.

The numbers $\delta^{\alpha}(D)$, $r_u^{\alpha}(D)$, $r_H^{\alpha}(D)$ and $C_H^{\alpha}(D)$ are called respectively the α-diameter of D, α-radius of D relative to u, the Chebyshev α-radius and the Chebyshev α-center of D relative to H.
When $H = D$, then the notations $r^{\alpha}(D)$ and $C^{\alpha}(D)$ are used for $r_H^{\alpha}(D)$ and $C_H^{\alpha}(D)$ respectively.

Proposition 4.10 *Let $(X,\ Q)$ be a GSQ-NF. Then for any subset strongly bounded subset D of X with $u \in X$, $r_u(D) = \bigvee\{r_u^{\alpha}(D):\ \alpha \in (0\ ,\ 1)\}$.*

Proof. The proof is straightforward.

Remark 4.11 *From the above relation it is obvious that*
$r_u(D) \geq r_u^{\alpha}(D)\ \forall\ \alpha \in (0,\ 1)$.

Proposition 4.12 *Let $(X,\ Q)$ be a GSQ-NF and D be a strongly bounded subset of X. If $\delta\ (D) > r(D)$ then $\exists \alpha_0 \in (0,\ 1)$ such that*
$\delta^{\alpha_0}(D) > r^{\alpha}(D)\ \forall \alpha \in (0,\ 1)$.

Proof. Let $\delta\ (D) > r(D)$.

Then $\delta(D) = \bigvee_{\alpha \in (0,1)} [\bigvee\{|x-y|_{\alpha},\ \forall x,\ y \in D\}] > \bigwedge_{u \in D}\{r_u(D)\} = r(D)$.

Thus $\exists \alpha_0 \in (0,\ 1)$ and $u_0 \in D$ such that

$\bigvee\{|x-y|_{\alpha_0},\ \forall x,\ y \in D\} > r_{u_0}(D) \geq r_{u_0}^{\alpha}(D)\ \forall\ \alpha \in (0,\ 1)$
$\geq r^{\alpha}(D)\ \forall \alpha \in (0,\ 1)$
$\Rightarrow \delta^{\alpha_0}(D) > r^{\alpha}(D)\ \forall \alpha \in (0,\ 1)$.

Remark 4.13 *Let $(X,\ Q)$ be a GSQ-NF and $\alpha \in (0,\ 1)$. Let K be a nonempty, strongly bounded, convex subset of X having normal structure. Then for each convex subset S of K with $\delta(S) > 0$, $\exists \alpha_S \in (0,\ 1)$ such that $\delta^{\alpha_s}(S) > r^{\alpha}(S)\ \forall \alpha \in (0,\ 1)$.*

5 Kirk's Fixed Point Theorem

In this section, we define non-expansive mapping and establish Kirk's fixed point theorem.

Definition 5.1 *Let $(X,\ Q)$ be a GSQ-NF and $T : X \to X$. The operator T is said to be non-expansive if*
$|Tx - Ty|_{\alpha} \leq |x - y|_{\alpha}\ \ \forall \alpha \in (0,\ 1),\ \forall\ x,\ y \in X$.

Definition 5.2 *Let $(X,\ Q)$ be a GSQ-NF and $T : X \to X$. A nonempty, closed, convex subset D of X is said to be T-invariant if $T(D) \subset D$.*

Definition 5.3 *Let $(X,\ Q)$ be a GSQ-NF and $T : X \to X$. A nonempty, closed, convex subset D of X is said to be minimal T-invariant if $T(D) \subset D$ and D has no nonempty, closed, convex subset which is T-invariant.*

Theorem 5.4 *Let $(X,\ Q)$ be a generating space of semi-norm family. Let K be a nonempty, weakly compact, convex subset of X. Then for any mapping $T : K \to K$ there exists a nonempty, closed, convex subset of K which is minimal T-invariant.*

Proof. Let us consider the family M of all nonempty, closed and convex subsets of K which are T-invariant and order this family by set inclusion. For $K_1,\ K_2 \in \mu$, $K_1 \leq K_2$ provided $K_2 \subset K_1$.
Let us consider a chain C of nonempty, closed and convex subsets of K. Since C is a chain it has finite intersection property. Further every member of M , being closed and convex, is weakly closed subset of the weakly compact set K and hence weakly compact. So $\cap_{i=1}^{\infty} K_i$ is nonempty closed and convex and which is the upper bound of C. By Zorn's Lemma, M has a maximal element which is the minimal T-invariant.

Lemma 5.5 *Let $(X,\ Q)$ be a generating space of semi-norm family and $T : X \to X$. Then if $K \subset X$ is a nonempty, closed, convex and minimal T-invariant set, then*

$$K = \overline{convT(K)}.$$

Proof. Let $K \subset X$ is a T-invariant set, then $T(K) \subset K$. Since K is convex, $convT(K) \subset K$ and $\overline{convT(K)} \subset K$, since K is closed. Now $T(\overline{convT(K)}) \subset T(K) \subset \overline{convT(K)}$. So clearly $\overline{convT(K)}$ is T-invariant and it is also closed and convex. Since K is the minimal T-invariant set, $K = \overline{convT(K)}$.

Theorem 5.6 *(Kirk's) Let (X, Q) be a generating space of semi-norm family and K be a nonempty, weakly compact, convex subset of X. If K has a normal structure then for any non-expansive mapping $T : K \to K$ has a fixed point.*

Proof. By Theorem 5.4, there exists a nonempty, closed, convex subset K_0 of K which is minimal T-invariant and by Lemma 5.5

$$K_0 = \overline{convT(K_0)}.$$

We claim that $\delta(K_0) = 0$. If not let $\delta(K_0) > 0$. Since K has a normal structure, K_0 has a non-diametral point. Since $\delta(K_0) > 0$, K_0 has more than one element.
Let $u \in C(K_0)$ i.e. $r_u(K_0) = r(K_0) > 0$.
Since T is non-expansive,
$|Tu - Tv|_\alpha \leq |u - v|_\alpha \ \forall \alpha \in (0, 1), \ \forall u \in K_0$
$\Rightarrow |Tu - Tv|_\alpha \leq |u - v|_\alpha \leq r(K_0) \ \forall \alpha \in (0, 1)$
$\Rightarrow T(K_0) \subset B(Tu, r(K_0)) = \{x \in X : |Tu - x|_\alpha \leq r(K_0) \ \forall \alpha \in (0, 1)\}$.
Next we shall show that
$K_0 = \overline{convT(K_0)} \subset B(Tu, r(K_0))$.
Let $z \in convT(K_0)$ then $z = \lambda x + (1 - \lambda) y$ for some $x, y \in T(K_0)$ and $\lambda \in [0, 1]$.
Now $|Tu - z|_\alpha = |\lambda Tu + (1 - \lambda)Tu - \lambda x - (1 - \lambda) y|_\alpha$
$\leq |\lambda||Tu - x|_\beta + |(1 - \lambda)||Tu - y|_\beta$ for some $\beta \in (0, \alpha]$
$\Rightarrow |Tu - z|_\alpha \leq \lambda r(K_0) + (1 - \lambda) r(K_0) = r(K_0)$.
Hence $z \in B(Tu, r(K_0))$ and $convT(K_0) \subset B(Tu, r(K_0))$.
Let $x \in \overline{convT(K_0)}$ then there exists a sequence $\{x_n\}$ of points of $convT(K_0)$ i.e. of $B(Tu, r(K_0))$ such that $\lim_{n \to \infty} x_n = x$.
Now $|Tu - x|_\alpha \leq |Tu - x_n|_\beta + |x_n - x|_\beta$ for some $\beta \in (0, \alpha], \ \forall n \in N$
$\Rightarrow |Tu - x|_\alpha \leq r(K_0) + |x_n - x|_\beta \ \forall n \in N$.
Taking limit $n \to \infty$ we get $|Tu - x|_\alpha \leq r(K_0) \ \forall \alpha \in (0, 1)$ which implies $x \in B(Tu, r(K_0))$.
Hence $K_0 = \overline{convT(K_0)} \subset B(Tu, r(K_0))$,
which shows that $r_{T_u}(K_0) = r(K_0)$ and hence $Tu \in C(K_0)$. This proves that $C(K_0)$ is T-invariant, which contradicts the fact that K_0 is the minimal T-invariant.
Hence $\delta(K_0) = 0$ and since K_0 is nonempty it is a singleton set and is fixed under T.

6 Conclusion

Generating spaces of quasi-norm family (GSQ-NF) has the strength of unifying results on classical, fuzzy and probabilistic functional analysis. Not much study has yet been made in fixed point in this setting. In this paper, we have attempted to extend the fixed point theory of nonexpansive mappings in GSQ-NF. For this we have introduced Chebyshev radius, Chebyshev centre, diametral point, non-diametral point, normal structure in GSQ-NF. Finally we have extended the Kirk's fixed point theorem in GSS-NF.
We have a lot of scope for studying fixed point theorems in this settings.

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