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Differential Equations Generating Densifying Curves

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Abstract

In this paper we are interessted in generating the α-dense curves in a hyper-rectangle of $\mathbb{R}^n$ using the periodic solutions of some ordinary differential equations of high order. The solutions are defined by periodic functions. Some applications and numerical results are introduced to illustrate the idea of this technique.

Keywords: *α-dense curve; Differential equation; Periodic solution; Global optimization.*

1 Introduction

Global optimization has received a wide attraction from many scientific fields in the past few years, due to the success of new algorithms for addressing intractable problems from diverse areas as computational chemistry, biology, biomedicine, structural optimization, computer sciences, operations research, economics, engineering design and control. Very interesting results have been obtained in [2], [6, 7], [11, 12]. However, global optimization multidimensional problems are difficult to solve numerically and that there is an obstacle to use directly multidimensional methods, because of their complexity and the lengthy calculation times [13, 14], [18]. It exist performance methods in the case unidimensional; it is more reasonable to exploit those methods after some transformations.

In this optics the Alienor transformation method has been applied and it has proved their effectiveness in various situations [1], [10], [15]. It is based on the idea of reducing several variables of minimization problem to a single variable of minimization problem allowing the use of well known powerful methods and techniques available in the case of a single variable [3, 4], [9]. There is a useful notation for this approach which was developed in the eighties by Cherruault and al.. In this paper, we will show that some α-dense curves can be obtained from the solution of large class of ordinary differential equations, this is a new approach to find α-dense curves. This constructive method to determine the α-dense curves in a space of high dimension consists of using periodic solutions of ordinary differential equations.

In a general metric space (E, d) some subsets X can be densified with a degree of density $\alpha > 0$, by means of curves densified in E satisfying the called α-density property, which defined in the following theorem.

For $E = \mathbb{R}^n$ with $n \geq 2$, the case $\alpha = 0$ leads us to the Peano curves provided that the subset X to have positive Jordan content. The functions h of the following theorem are also known as space-filling curves [16], [19]; so they can be considered as a special subclass of α-dense curves for the limit case $\alpha = 0$.

Briefly speaking, the Alienor reducing transformation method can be summarized as follows.

It is asked for solving the global minimization problem:

$$\min_{(x_1,x_2,\ldots,x_n)\in {}_{i=1}^{n}[a_i,b_i]} f(x_1, x_2, ..., x_n) \tag{1}$$

where f is Lipschitz on ${}_{i=1}^{n}[a_i, b_i]$.

We construct a parameterized curve $h(\theta) = (h_1(\theta), h_2(\theta), ..., h_n(\theta))$, α-dense in ${}_{i=1}^{n}[a_i, b_i]$ for $\theta \in [0, \theta_{\max}]$, where $\theta_{\max}$ is the supremum the domain of definition of the function h when it "α-densifies" the hyper-rectangle.

The minimization problem (1) is then approximated by the problem

$$\min_{\theta\in[0,\theta_{\max}]} f^*(\theta)$$

where $f^*(\theta) = f(h_1(\theta), h_2(\theta), ..., h_n(\theta))$.

In the basic method, the unidimentional minimization problem (1) is solved by descretizing the interval $[0, \theta_{\max}]$ via a chosen $\Delta\theta$. Then we look for the minimum of the finite set $\{f^*(\theta_k),\ k = 1, ..., N\}$, where $\theta_0, \theta_1, ..., \theta_N$ are the discretized points. Obviously, the densification parameter α and the step $\Delta\theta$ are chosen such that the global minimum is obtained with the desired accuracy ε.[5], [17] and [20, 21].

The Alienor method has also proved to be a great efficiency when it mixed with some one dimentional methods as the covering algorithms (the method of Piyavskii-Shubert, the Brent algorithm and Evtushenko method).

For this reason, the generating methods for them are specially interesting for our purpose.

Consider the function $h(\theta) = (h_1(\theta), h_2(\theta), ..., h_n(\theta))$, defined on the interval closed and bounded A of $\mathbb{R}$ and the value in the indicated hyper-rectangle.

Definition 1.1 *A curve of* $\mathbb{R}^n$ *defined by*

$$h : A \to_{i=1}^{n} [a_i, b_i]$$

is called α*-dense in* ${}_{i=1}^{n} [a_i, b_i]$, *if for any* $x \in_{i=1}^{n} [a_i, b_i]$, *there exists* $\theta \in A$ *such that* $d(x, h(\theta) \leq \alpha$, *where* d *is the Euclidean distance in* $\mathbb{R}^n$.

Theorem 1.2 *Consider the function* $(h_1, h_2, ..., h_n) : A \to_{i=1}^{n} [a_i, b_i]$, *with* $\theta_1, \theta_2, ..., \theta_n, \alpha$ *are strictly positive numbers such that:*

a) for any $i = 1, 2, ..., n$, h_i *are continuous and surjective*

b) for any $i = 1, 2, ..., n-1$, h_i *are periodic, respectively of period* $\theta_1, \theta_2, ..., \theta_{n-1}$

c) for any interval I *of* A *and for any* $i \in \{1, 2, ..., n-1\}$, *we have:*

$$\mu(I) < \theta_i \Rightarrow \mu(h_{i+1}(I)) < \frac{\alpha}{\sqrt{n-1}}.$$

Then the curve defined by $h(\theta) = (h_1(\theta), h_2(\theta), ..., h_n(\theta))$, *for* $\theta \in A$, *is* α*-dense in* ${}_{i=1}^{n} [a_i, b_i]$.

Proof. The proof will be obtained in a recurrent way:

i) First consider the case $n = 2$

Let $x = (x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$ and consider the interval $[x_2 - \alpha, x_2 + \alpha]$. On the one hand the function h_2 is surjective, and consecently there exists a closed interval $I \subset A$ such that $h_2(I) = [x_2 - \alpha, x_2 + \alpha] \cap [a_2, b_2]$, but $\mu(h_2(I)) \geq \alpha$, which implies $\mu(I) \geq \theta_1$, and thus $h_1(I) = [a_1, b_1]$. On the other hand $x_1 \in [a_1, b_1]$, then there exists $\theta' \in I$ such that $x_1 = h_1(\theta')$; and since $h_2(\theta') \in [x_2 - \alpha, x_2 + \alpha] \cap [a_2, b_2]$, we have $|x_2 - h_2(\theta')| \leq \alpha$.

We deduce

$$\|(x_1, x_2) - (h_1(\theta'), h_2(\theta'))\| \leq \alpha.$$

ii) We suppose that the theorem is true for $(n-1)$.

Let $x = (x_1, x_2, ..., x_n) \in_{i=1}^{n} [a_i, b_i]$. Let us consider the interval $[x_n - \alpha, x_n + \alpha]$.

Inasmuch as h_n is surjective, there exists a closed interval $I \subset A$ such that $h_n(I) = [x_n - \alpha, x_n + \alpha] \cap [a_n, b_n]$, which yields $\mu(h_n(I)) \geq \frac{\alpha}{\sqrt{n-2}}$, and therefore $\mu(I) \geq \theta_{n-1}$, leading to $h_{n-1}(I) = [a_{n-1}, b_{n-1}]$.

Moreover, the sequence of numbers $\theta_1, \theta_2, ..., \theta_{n-1}$ is increasing, hence the functions $h_1, h_2, ..., h_{n-1}$, restricted to interval I, are surjective and satisfied the hypotheses of the theorem. Consequently, the curve defined by $(h_1(\theta), h_2(\theta), ..., h_{n-1}(\theta))$ for $\theta \in I$, is α-dense in $_{i=1}^{n-1} [a_i, b_i]$, but $(x_1, x_2, ..., x_{n-1}) \in_{i=1}^{n-1} [a_i, b_i]$ involving the existence of $\theta'' \in I$ such that:

$$\|(x_1, x_2, ..., x_{n-1}) - (h_1(\theta''), h_2(\theta''), ..., h_{n-1}(\theta''))\| \leq \alpha.$$

Since $|x_n - h_n(\theta'')| \leq \alpha$, we deduce:

$$\|(x_1, x_2, ..., x_n) - (h_1(\theta''), h_2(\theta''), ..., h_n(\theta''))\| \leq \alpha.$$

Remark 1.3 *For any $i \in \{1, 2, ..., n-1\}$, h_i is continuous and since it is θ_i-periodic and surjective, it reaches the bounds a_i and b_i in every closed interval of length θ_i.*

Now, we give a method to determine α-dense curves in hyper-rectangle of $\mathbb{R}^n$, based on the resolution of ordinary differential equations of the form

$$\frac{d^n x(t)}{dt^n} + a_n(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + ... + a_2(t)\frac{dx(t)}{dt} + a_1(t)x(t) = g(t) \tag{2}$$

where $x(t)$ is the unknown function, $a_k(t), k = 1, 2, ..., n$ and $g(t)$ are known continuous functions defined on $-\infty < t < \infty$ ..

2 Construction of α-Dense Curves

In this section we will discuss the periodic solution of nth order differential equation. In particular, for the third order differential equation

$$\frac{d^3x(t)}{dt^3} = p(t)\frac{d^2x(t)}{dt^2} + q(t)\frac{dx(t)}{dt} + r(t)x(t) \tag{3}$$

where $x = x(t)$ is an unknown function and $p(t)$, $q(t)$, $r(t)$ are continuous functions defined as follows:

$$q(t) = \frac{(\beta^2 - \gamma^2)\tan(t) + \beta^3(\gamma^2 - 1)\tan(\beta t) + \gamma^3(1 - \beta^2)\tan(\gamma t)}{(-\beta^2 + \gamma^2)\tan(t) + \beta(-\gamma^2 + 1)\tan(\beta t) + \gamma(-1 + \beta^2)\tan(\gamma t)},$$

$$p(t) = \frac{\gamma^3\tan(\gamma t) - \tan(t) + (\gamma\tan(\gamma t) - \tan(t))\,q(t)}{1 - \gamma^2},$$

$$r(t) = p(t) + (1 + q(t))\tan(t),$$

where $(\beta, \gamma) \in \mathbb{R}^2$ is a fixed couple, such that

$$\frac{2\pi.\sqrt{2}}{\alpha} < \beta < \frac{\alpha.\gamma}{2\pi.\sqrt{2}}.$$

The number α is strictly positive and belongs to a neighbourhood of 0. Then the fundamental system of solution of (3) is $(\sin(\gamma t), \sin(\beta t), \sin(t))$ generates an α-dense curve in $[-1,1]^3$.

Indeed the equation (3) can be written in the system form

$$\dot{X} = A(t)X \text{ , where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix}$$

There exists $x \in C^3(I)$ such that $\{x(\gamma t), x(\beta t), x(t)\}$ is a fundamental system of solutions of the equation (3). Using any method of solving high ordinary differential equations, we can find the fundamental set of solution of the homegeneous equation (3) which has the form $\{\sin(\gamma t), \sin(\beta t), \sin(t)\}$.

Consider the function

$$\begin{aligned} h(\theta) = (h_1(\theta), h_2(\theta), h_3(\theta)) &: [0, 2\pi] \to [-1,1]^3 \\ \theta \to (\sin(\gamma\theta), \sin(\beta\theta), \sin(\theta)) \end{aligned}$$

a) h_1, h_2, h_3 are continuous and surjective

b) h_1, h_2 are periodic, with period $\frac{T}{\alpha_1} = \frac{2\pi}{\gamma}, \frac{T}{\alpha_2} = \frac{2\pi}{\beta}$

c) h_1, h_2, h_3 are Lipshitzian, respectively of constants $c_1 = \gamma, c_2 = \beta, c_3 = 1$

d) according to the hypotheses, we have:

$$\frac{2\pi.\sqrt{2}}{\alpha} < \beta < \frac{\alpha.\gamma}{2\pi.\sqrt{2}}.$$

So, for any interval I of $A = [0, 2\pi]$ and for any $i = 1, 2$

$$\mu(I) < \frac{2\pi}{\alpha_i} \Rightarrow \mu(h_{i+1}(I)) < \frac{\alpha}{\sqrt{2}}.$$

Thus all the hypotheses of theorem (2) are satisfied. Then the curve defined by $h(\theta) = (\sin\gamma\theta, \sin\beta\theta, \sin\theta)$,for $\theta \in [0, 2\pi]$ is α-dense in $[-1,1]^3$.

As another example, let the differential equation

$$\frac{d^3x(t)}{dt^3} - p(t)\frac{d^2x(t)}{dt^2} - q(t)\frac{dx(t)}{dt} - r(t)x(t) = 0 \tag{4}$$

where $p(x)$, $q(x)$ and $r(x)$ are defined as follows:

$$q(x) = \frac{(\beta^2 - \gamma^2)\tan(x) + \beta^3(\gamma^2 - 1)\tan(\beta x) + \gamma^3(1 - \beta^2)\tan(\gamma x)}{(-\beta^2 + \gamma^2)\tan(x) + \beta(-\gamma^2 + 1)\tan(\beta x) + \gamma(-1 + \beta^2)\tan(\gamma x)},$$

$$p(x) = \frac{\gamma^3\tan(\gamma x) - \tan(x) + (\gamma\tan(\gamma x) - \tan(x))\,q(x)}{1 - \gamma^2},$$

$$r(x) = p(x) + (1 + q(x))\tan(x),$$

where $(\beta,\gamma)\in\mathbb{R}^2$ with $1<\beta<\gamma$. such that

$$\frac{2\pi.\sqrt{2}}{\alpha}<\beta<\frac{\alpha.\gamma}{2\pi.\sqrt{2}},$$

the number α is strictly positive and in the neighbourhood of 0. Then the fundamental system of solution of (4) is $(\cos(\gamma t),\cos(\beta t),\cos(t))$ generates an α-dense curve in $[-1,1]^3$.

Indeed the equation (4) can be written in the form $\dot{x}=A(t)x$ where:

$$A(t)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix}$$

There exists $x\in C^3(I)$ such that $\{x(\gamma t),x(\beta t),x(t)\}$ is a fundamental system of solutions of the equation (4). Using any method of solving high ordinary differential equations, we can find the set of solution of (4) that has the form $\{\cos(\gamma t),\cos(\beta t),\cos(t)\}$.

Consider the function

$$h(\theta)=(h_1(\theta),h_2(\theta),h_3(\theta)):[0,2\pi]\to[-1,1]^3$$
$$\theta\to(\cos(\gamma\theta),\cos(\beta\theta),\cos(\theta))$$

a) h_1,h_2,h_3 are continuous and surjective
b) h_1,h_2 are periodic,respectively of period $\frac{T}{\alpha_1}=\frac{2\pi}{\gamma},\frac{T}{\alpha_2}=\frac{2\pi}{\beta}$
c) h_1,h_2,h_3 are Lipshitzian, respectively of constants $c_1=\gamma,c_2=\beta,c_3=1$
d) for any interval I of $[0,2\pi]$ and for any $i=1,2$

$$\mu(I)<\frac{2\pi}{\alpha_i}\Rightarrow\mu(h_{i+1}(I))<\frac{\alpha}{\sqrt{2}}.$$

Then the curve defined by $h(\theta)=(\cos\gamma\theta,\cos\beta\theta,\cos\theta)$, for $\theta\in[0,2\pi]$ is α-dense in $[-1,1]^3$.

In the case of fourth order, we consider the linear ordinary differential equation

$$\frac{d^4x(t)}{dt^4}+\left(\lambda^2+\beta^2\right)\frac{d^2x(t)}{dt^2}+\left(\beta^2\lambda^2\right)x(t)=0.$$

The vector function of solution has the form $\{a_1\cos\lambda t,a_2\sin\lambda t,a_3\cos\beta t,a_4\sin\beta t\}$, a_1, a_2, a_3, a_4 are arbitrary constants

Set $h(\theta)=(h_1(\theta),h_2(\theta),h_3(\theta),h_4(\theta))=(a_1\cos\lambda\theta,a_2\sin\lambda\theta,a_3\cos\beta\theta,a_4\sin\beta\theta)$. where a_1, a_2, a_3, $a_4\in[0,1]$ and verified:

$$a_2<\frac{\alpha}{2\pi\sqrt{3}},\qquad a_3\beta<\frac{\alpha}{2\pi\sqrt{3}}\lambda\quad\text{and}\quad a_4<\frac{\alpha}{2\pi\sqrt{3}}.$$

The number α is strictly positive and in the neighbourhood of 0. It is easy to see that:

a) h_i are continuous and surjective, for $i = 1, ..., 4$

b) h_1, h_2, h_3 are periodic, respectively of period $\frac{T}{\alpha_1} = \frac{T}{\alpha_2} = \frac{2\pi}{\lambda}, \frac{T}{\alpha_3} = \frac{2\pi}{\beta}$

c) h_i are Lipshitzian of constants $c_1 = a_1\lambda$, $c_2 = a_2\lambda$, $c_3 = a_3\beta$, $c_4 = a_4\beta$, for $i = 1, ..., 4$

d) for any interval I of $A = [0, 2\pi]$ and for any $i = 1, 2, 3$

$$\mu(I) < \frac{2\pi}{\alpha_i} \Rightarrow \mu(h_{i+1}(I)) < \frac{\alpha}{\sqrt{3}}.$$

Then the curve difined by $h(\theta) = (a_1 \cos\lambda\theta, a_2 \sin\lambda\theta, a_3 \cos\beta\theta, a_4 \sin\beta\theta)$, for $\theta \in [0, 2\pi]$ is α-dense in $[-1, 1]^4$ of $\mathbb{R}^4$.

In the case of second order, consider the linear ordinary differential equation and homogeneous

$$\frac{d^2x(t)}{dt^2} + \beta^2 x(t) = 0.$$

The set of solution has the form $\{a_1 \cos\beta t, a_2 \sin\beta t\}$, a_1, a_2 are arbitrary constants

Set $h(\theta) = (h_1(\theta), h_2(\theta)) = (a_1 \cos\beta\theta, a_2 \sin\beta\theta)$, where a_1, a_2 are constants strictly positive numbers such that: $a_1 \in [0, 1]$ and a_2 verified

$$a_2 < \frac{\alpha}{2\pi}.$$

The number α is strictly positive and in the neighbourhood of 0. It is clear that

a) h_1, h_2 are continuous and surjective

b) $h_1$is periodic of period $\frac{T}{\alpha_1} = \frac{2\pi}{\beta}$

c) h_1, h_2 are Lipshitzian of constants $c_1 = a_1\beta., c_2 = a_2\beta$

d) for any interval I of $A = [0, 2\pi]$ and for $i = 1$

$$\mu(I) < \frac{2\pi}{\alpha_1} \Rightarrow \mu(h_2(I)) < \alpha.$$

Then this solution generates an α-dense curve in $[-1, 1]^2$.

3 Conclusion

α-dense curves can be generated by non-periodic solutions of ordinary differential equations. Indeed, the α-density of these curves is obtained from relationship existing between functions $h_1, h_2, ..., h_n$. However, periodic functions are often used because of their simplicity in calculations. It is why we often use sinusoidal functions which, in addition to the periodicity, generate curves of class C^∞. Other classes generating this type of curves, have been given in [8], [16] and [19]. This permits us to improve the Alienor method used in the multidimentional problems of optimization and for the approximation of several variables functions.

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