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On n-binormal Operators

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Abstract

In this paper we introduce n-binormal operators acting on a Hilbert space H. An *operator* $T \in L(H)$ *is n-binormal if* T^*T^n *commutes with* T^nT^* *or* $[T^*T^n, T^nT^*]=0$ *and it is denoted by* $[nBN]$. *We investigate some basic properties of such operators. In general a n-binormal operator need not be a normal operator. Further we study n-binormal composite integral operators.*

Keywords: *Normal, n-normal, binormal, n -isometry and Hilbert space.*

1 n-binormal Operators

Let H be a Hilbert space and $L(H)$ be the algebra of all bounded linear operators acting on H. An operator $T \in L(H)$ is called normal if $T^*T = TT^*$, n-normal if $T^*T^n = T^nT^*$, binormal if T^*T commutes with TT^*, isometry if $T^*T = I$, 2-isometry if $T^{*^2}T^2 - 2T^*T + I = 0$, 3-isometry if $T^{*^3}T^3 - 3T^{*^2}T^2 + 3T^*T - I = 0$, n-isometry if $\sum_{k=0}^{n}(-1)^k \binom{n}{k} T^{*^{n-k}}T^{n-k} = 0$ or $T^{*^n}T^n - \binom{n}{1}T^{*^{n-1}}T^{n-1} + \binom{n}{2}T^{*^{n-2}}T^{n-2} \ldots\ldots\ldots (-1)^{n-1}\binom{n}{n-1}T^*T + (-1)^n I = 0$ and n-binormal if $T^*T^nT^nT^* = T^nT^*T^*T^n$ (refer[1], [2], [3] and [8]). In this section we investigate some basic properties of n-binormal operators.

Theorem 1.1 *If $T \in [nBN]$ then so are*

(i) kT for any real number k.

(ii) any $S \in L(H)$ that is unitarily equivalent to T.

(iii) the restriction T/M of T to any closed subspace M of H that reduces T.

Proof . (i) The proof is straightforward.

(ii) Let $S \in L(H)$ be unitarily equivalent to T then there is a unitary operator $U \in L(H)$ such that $S = UTU^*$ which implies that $S^* = UT^*U^*$ and $S^n = UT^nU^*$

If T is n-binormal then $T^*T^nT^nT^* = T^nT^*T^*T^n$

now $S^*S^nS^nS^* = UT^*U^*UT^nU^*UT^nU^*UT^*U^* = UT^*T^nT^nT^*U^*$

and $S^nS^*S^*S^n = UT^nU^*UT^*U^*UT^*U^*UT^nU^* = UT^nT^*T^*T^nU^*$

Hence S is unitary equivalent to T (refer [5]).

(iii) If T is n-binormal then $T^*T^nT^nT^* = T^nT^*T^*T^n$

Consider
$$\left(T/M\right)^*\left(T/M\right)^n\left(T/M\right)^n\left(T/M\right)^* = \left(T^*/M\right)\left(T^n/M\right)\left(T^n/M\right)\left(T^*/M\right)$$
$$= \left(T^*T^nT^nT^*/M\right)$$
$$= \left(T^nT^*T^*T^n/M\right)$$
$$= \left(T^n/M\right)\left(T^*/M\right)\left(T^*/M\right)\left(T^n/M\right)$$
$$= \left(T/M\right)^n\left(T/M\right)^*\left(T/M\right)^*\left(T/M\right)^n$$

Hence $T/M \in [nBN]$.

Theorem 1.2 *If $T \in L(H)$ is n-normal then $T \in [nBN]$.*

Proof. If T is n-normal then $T^*T^n = T^nT^*$

Post multiply by T^nT^* on both sides

$$T^*T^nT^n T^* = T^nT^*T^nT^* = T^nT^*T^*T^n$$

Hence T is n-binormal.

The following example shows that the converse need not be true.

Example 1.3 Let $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ be an operator on R^2, which is $[3BN]$ but neither 3-normal nor normal.

Theorem 1.4 *Let $T \in [nBN]$ and $S \in [nBN]$. If T and S are doubly commuting then TS is n-binormal.*

Proof . $(TS)^n(TS)^*(TS)^*(TS)^n$

$$= S^nT^nS^*T^*S^*T^*S^nT^n$$

$$= S^nS^*T^nT^*T^*S^*T^nS^n$$

$$= S^nS^*T^nT^*T^*T^nS^*S^n$$

$$= S^nS^*T^*T^nT^nT^*S^*S^n \text{, since } T \text{ is } [nBN]$$

$$= S^nT^*S^*T^nT^nS^*T^*S^n$$

$$= T^*S^nT^nS^*S^*T^nS^nT^*$$

$$= T^*T^nS^nS^*S^*S^nT^nT^*$$

$$= T^*T^nS^*S^nS^nS^*T^nT^* \text{, since } S \text{ is } [nBN]$$

$$= T^*S^*T^nS^nS^nT^nS^*T^*$$

$$= S^*T^*S^nT^nS^nT^nS^*T^*$$

$$= (TS)^*(TS)^n(TS)^n(TS)^*$$

Hence TS is n-binormal.

Example 1.5 Let $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ be not commuting $[2BN]$ operators.Then ST is not $[2BN]$.

The following example shows that the sum and difference of two commuting n-binormal operators need not be n-binormal.

Example 1.6 Let $S=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $T=\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}$ on R^2. Then S and T are commuting $[2BN]$ operators but $S+T=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ is not $[2BN]$ and $S-T=\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$ is not $[2BN]$.

In the following theorem, we obtain sufficient condition for the sum of n-binormal operators be n-binormal(refer [7]).

Theorem 1.7 *Let* S *and* T *be commuting* $[nBN]$ *operators such that* $(S+T)^*$ *commutes with* $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$. *Then* $(S+T)$ *is n-binormal operator.*

Proof. Consider $(S+T)^*(S+T)^n(S+T)^n(S+T)^*$

$$=\left((S+T)^*\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k\right)\left(\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k(S+T)^*\right)$$

$$=\left((S+T)^*S^n+(S+T)^*\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+(S+T)^*T^n\right)\left(S^n(S+T)^*+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^*+T^n(S+T)^*\right)$$

$$=\left((S^*+T^*)S^n+(S+T)^*\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+(S^*+T^*)T^n\right)\left(S^n(S^*+T^*)+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^*+T^n(S^*+T^*)\right)$$

$$=\left(S^*S^n+T^*S^n+(S+T)^*\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+S^*T^n+T^*T^n\right)\left(S^nS^*+S^nT^*+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^*+T^nS^*+T^nT^*\right)$$

Since S and T are commuting $[nBN]$ operators such that $(S+T)^*$ commutes with $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$.

$$=\left(S^nS^*+S^nT^*+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^*+T^nS^*+T^nT^*\right)\left(S^*S^n+T^*S^n+(S+T)^*\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+S^*T^n+T^*T^n\right)$$

$$=\left(S^n(S^*+T^*)+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^*+T^n(S^*+T^*)\right)\left((S^*+T^*)S^n+(S+T)^*\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+(S^*+T^*)T^n\right)$$

$$=\left(\left(S^n+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+T^n\right)(S^*+T^*)\right)\left((S^*+T^*)\left(S^n+\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k+T^n\right)\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k(S+T)^*(S+T)^*\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k$$

$=(S+T)^n(S+T)^*(S+T)^*(S+T)^n$. Hence $(S+T)$ is n-binormal

Theorem 1.8 *Let* $T \in L(H)$ *with the Cartesian decomposition* $T = A + iB$. *Then* T *is binormal if and only if (i)* $AB^3 + B^3A = A^3B + BA^3$ *and*
(ii) $A^2BA + ABA^2 = B^2AB + BAB^2$.

Proof. Since T is binormal then $T^*TTT^* = T\ T^*T^*T$.

$$T^*TTT^* = (A - iB)(A + iB)(A + iB)(A - iB)$$
$$= (A^2 + iAB - iBA + B^2)(A^2 - iAB + iBA + B^2)$$
$$= A^4 - iA^3B + iA^2BA + A^2B^2 + iABA^2 + ABAB - ABBA + iAB^3$$
$$- iBA^3 - BAAB + BABA - iBAB^2 + B^2A^2 - iB^2AB + iB^3A + B^4$$

$$TT^*T^*T = (A + iB)(A - iB)(A - iB)(A + iB)$$
$$= (A^2 - iAB + iBA + B^2)(A^2 + iAB - iBA + B^2)$$
$$= A^4 + iA^3B - iA^2BA + A^2B^2 - iABA^2 + ABAB - ABBA - iAB^3$$
$$+ iBA^3 - BAAB + BABA + iBAB^2 + B^2A^2 + iB^2AB - iB^3A + B^4$$

It is easy to observe that T is binormal if and only if (i) and (ii) are true.

The following examples show that n-binormal and n-isometry operators are independent classes.

Example 1.9 Consider the operator $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ on R^2, which is 3-binormal but not 3-isometry.

Example 1.10 Consider the operator $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on R^2, which is 3-isometry but not 3-binormal.

2 n-binormal Compositie Integral Operators

Let (X, S, μ) be a σ-finite measure space and let $\phi: X \to X$ be a non-singular measurable transformation $(\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0)$. Then a composition transformation, for $1 \leq p \prec \infty, C_\phi : L^p(\mu) \to L^p(\mu)$ is defined by $C_\phi f = f \circ \phi$ for every $f \in L^p(\mu)$. In case C_ϕ is continuous, we call it a composition operator induced by ϕ. C_ϕ is bounded operator if and only if $\frac{d\mu\phi^{-1}}{d\mu} = f_0$. This is the Radon- Nikodym derivative of the measure $\mu\phi^{-1}$ w.r.to the measure μ and it is essentially bounded. For more details about composition operators refer [1] and[6].

A kernel $K \in L^p(\mu \times \mu)$ always induces a bounded integral operator $T_K : L^p(\mu) \rightarrow L^p(\mu)$

defined by $(T_K f)(x) = \int K(x, y) f(y) d\mu(y)$.

Given a kernel K and a non-singular measurable function $\phi : X \rightarrow X$, the composite integral operator T_{K_ϕ} induced by (K, ϕ) is a bounded linear operator $T_K : L^p(\mu) \rightarrow L^p(\mu)$ defined by

$$
\begin{aligned}
(T_{K\phi} f)(x) &= \int K(x, y) f(\phi(y)) d\mu(y) \\
&= \int K_\phi(x, y) f(y) d\mu(y)
\end{aligned}
$$

We note that

$$
\begin{aligned}
(T_{K_\phi}^n f)(x) &= \int K^n(x, y) f(\phi(y)) d\mu(y) \\
&= \int K_\phi^n(x, y) f(y) d\mu(y)
\end{aligned}
$$

where

$$
K_\phi^n(x, y) = \int\int\int .....\int K_\phi(x, z_1) K_\phi(z_{1,} z_2) ....K_\phi(z_{n-2}, z_{n-1}) K_\phi(z_{n-1,} y) dz_1 dz_2 .....dz_{n-1} \ .
$$

Theorem 2.1 *Let* $K_\phi \in L^2(\mu \times \mu)$ *. Then* T_{K_ϕ} *is n-binormal if and only if*

$$
\begin{aligned}
&\iiint K_\phi^*(x, y) K_\phi^n(y, z) K_\phi^n(z, t) K_\phi^*(t, p) d\mu(y) d\mu(z) d\mu(t) \\
&= \iiint K_\phi^n(x, y) K_\phi^*(y, z) K_\phi^*(z, t) K_\phi^n(t, p) d\mu(y) d\mu(z) d\mu(t) .
\end{aligned}
$$

Proof. Suppose the condition is true . For $f, g \in L^2(\mu)$, we have

$$
\begin{aligned}
\left\langle T_{K_\phi}^* T_{K_\phi}^n T_{K_\phi}^n T_{K_\phi}^* f, g \right\rangle &= \int \left(T_{K_\phi}^* T_{K_\phi}^n T_{K_\phi}^n T_{K_\phi}^* f\right)(x) \overline{g}(x) d\mu(x) \\
&= \iint \left[K_\phi^*(x, y) \left(T_{K_\phi}^n T_{K_\phi}^n T_{K_\phi}^* f\right)(y) d\mu(y)\right] \overline{g}(x) d\mu(x) \\
&= \iint K_\phi^*(x, y) \left(\int K_\phi^n(y, z) \left(T_{K_\phi}^n T_{K_\phi}^* f\right)(z) d\mu(z)\right) d\mu(y) \overline{g}(x) d\mu(x) \\
&= \iiint K_\phi^*(x, y) K_\phi^n(y, z) \left(\int K_\phi^n(z, t) T_{K\phi}^* f(t) d\mu(t)\right) d\mu(z) d\mu(y) \overline{g}(x) d\mu(x)
\end{aligned}
$$

$$
= \int\int\int\int K_\phi^*(x, y) K_\phi^n(y, z) K_\phi^n(z, t) \left(\int K_\phi^*(t, p) f(p) d\mu(p)\right) d\mu(t) d\mu(z) d\mu(y) \overline{g}(x) d\mu(x)
$$

$$
= \int\int\int\int\int K_\phi^*(x, y) K_\phi^n(y, z) K_\phi^n(z, t) K_\phi^*(t, p) f(p) d\mu(p) d\mu(t) d\mu(z) d\mu(y) \overline{g}(x) d\mu(x)
$$

$$=\int\int\int\int\int K_\phi^*(x,y)K_\phi^n(y,z)K_\phi^n(z,t)K_\phi^*(t,p)d\mu(y)d\mu(z)d\mu(t)f(p)d\mu(p)\overline{g}(x)d\mu(x)\ldots(1)$$

and $\left\langle T_{K_\phi}^n T_{K_\phi}^* T_{K_\phi}^* T_{K_\phi}^n f, g\right\rangle = \int\left(T_{K_\phi}^n T_{K_\phi}^* T_{K_\phi}^* T_{K_\phi}^n f\right)(x)\overline{g}(x)d\mu(x)$

$$=\int\int\int\int\int K_\phi^n(x,y)K_\phi^*(y,z)K_\phi^*(z,t)K_\phi^n(t,p)f(p)d\mu(p)d\mu(t)d\mu(z)d\mu(y)\overline{g}(x)d\mu(x)$$

$$=\int\int\int\int\int K_\phi^n(x,y)K_\phi^*(y,z)K_\phi^*(z,t)K_\phi^n(t,p)d\mu(y)d\mu(z)d\mu(t)f(p)d\mu(p)\overline{g}(x)d\mu(x)\ldots(2)$$

It follows from (1) and (2) that T_{K_ϕ} is n-binormal .

Conversely suppose T_{K_ϕ} is n-binormal .Take $f=\chi_E$ and $g=\chi_F$ we see that from (1) and (2)

$$\int_E\int_F\int K_\phi^n(x,y)K_\phi^n(y,z)K_\phi^n(z,t)K_\phi^*(t,p)d\mu(y)d\mu(z)d\mu(t)$$

$$=\int_E\int_F\int K_\phi^n(x,y)K_\phi^*(y,z)K_\phi^*(z,t)K_\phi^n(t,p)d\mu(y)d\mu(z)d\mu(t)$$

for all $E,F\in S\times S$. Hence the required condition holds.

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