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Topological Results About the Set of Generalized Rhaly Matrices

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Abstract

In this paper, we will investigate some topological properties of $\mathcal{R}_b$ set of all bounded generalized Rhally matrices given by b sequences in H^2. We will show that $\mathcal{R}_b$ is closed subspace of $B(H^2)$ and we will define an operator from $\mathbb{T}$ to $\mathcal{R}_b$ and show that this operator is injective and continuous in strong-operator topology.

Keywords: *Generalized Rhaly operators, weak-operator topology, strong-operator topology.*

1 Introduction

A Hilbert space has two useful topologies (weak and strong); the space of operators on a Hilbert space has several. The metric topology induced by the norm is one of them; to distinguish it from the others, it is usually called the norm topology or the uniform topology. The next two are natural outgrowths for operators of the strong and weak topologies for vectors. A subbase for the strong operator topology is the collection of all sets of the form

$$\{A : \|(A - A_0) f\| < \epsilon\};$$

correspondingly a base is the collection of all sets of the form

$$\{A : \|(A - A_0) f_i\| < \epsilon, i = 1, 2, \dots, k\}$$

Here k is a positive integer, $f_1, f_2, \ldots, f_k$ are vectors, and ϵ is a positive number. A subbase for the weak operator topology is the collection of all sets of the form

$$\{A : |\langle (A - A_0) f, g\rangle| < \epsilon\},$$

where f and g are vectors and $\epsilon > 0$; as above (as always) a base is the collection of all finite intersections of such sets. The corresponding concepts of convergence (for sequences and nets) are easy to describe: $A_n \to A$ strongly if and only if $A_n f \to Af$ strongly for each f (i.e., $\|(A_n - A) f\| \to 0$ for each f), and $A_n \to A$ weakly if and only if $A_n f \to Af$ weakly for each f (i.e., $\langle A_n f, g\rangle \to \langle Af, g\rangle$ for each f and g). For a slightly different and often very efficient definition of the strong and weak operator topologies (see [3, Problems 224 and 225]).

The easiest questions to settle are the ones about comparison. The weak topology is smaller (weaker) than the strong topology, and the strong topology is smaller than the norm topology. In other words, every weak open set is a strong open set, and every strong open set is norm open. In still other words: every weak neighborhood of each operator includes a strong neighborhood of that operator, and every strong neighborhood includes a metric neighborhood. Again: norm convergence implies strong convergence, and strong convergence implies weak convergence. These facts are immediate from the definitions. In the presence of uniformity on the unit sphere, the implications are reversible (see [3]).

Let $H(\mathbb{D})$ denotes the space of complex valued analytic function on the unit disk $\mathbb{D}$, for $1 \leq p < \infty$, H^p denotes the standard Hardy space on $\mathbb{D}$.

Integral operators have been studied of the greater part of the last century.

In [5], Scott W. Young gave definition of generalized Cesàro matrix as follows:

Definition 1.1 *Let g be analytic on the unit disk. The generalized Cesàro operator on H^2 with symbol g*

$$C_g(f)(z) = \frac{1}{z}\int_0^z f(t)\, g(t)\, dt \tag{1}$$

In [4], Pommerenke defined the following operator on H^2 :

$$I_G(f) = \int_0^z f(t)\, G'(t)\, dt.$$

He showed the following lemma:

Lemma 1.2 ([4, Lemma 1]) *I_G is bounded on H^2 iff $G \in BMOA$. Morever, there is a constant K_1 such that $\|I_G\|_{op} \leq K_1 \|G\|_*$.*

If $G(z) = \int\limits_0^z g(w)\,dw$, then Pommerenke [4] showed that C_g is bounded on the Hilbert space H^2 if and only if $G \in BMOA$. Aleman and Siskakis [1] extended Pommerenke's result to the Hardy spaces H^p for all p, $1 \leq p < \infty$, and show that C_g is compact on H^p if and only if $G \in VMOA$.

In [2], the authors defined generalized Rhaly matrix by using the generalized Cesàro matrix as following:

Definition 1.3 *Let $\{b_n\}$ be a scalar sequence and $g(z) = \sum\limits_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$. The matrix*

$$R_b^g = \begin{pmatrix} a_0b_0 & & & & \\ a_1b_1 & a_0b_1 & & & \\ a_2b_2 & a_1b_2 & a_0b_2 & & \\ a_3b_3 & a_2b_3 & a_1b_3 & a_0b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

is called generalized Rhaly matrix with symbol g on H^2.

From (2), we can write

$$(R_b^g)_{nj} = \begin{cases} a_{n-j}b_n & , \quad n \geq j \\ 0 & , \quad n < j \end{cases}. \tag{3}$$

The relation $R_b^g = D_b C_g$ is valid similar to on Rhaly matrix, where $D_b = diag\{(n+1)\,b_n\}_{n=0}^{\infty}$. We recall that $C_g = C_1$ for $g(z) = \frac{1}{1-z}$. Since $g(z) = \sum\limits_{k=0}^{\infty} z^k$, which fixes then $a_n = 1$ for all $n \in \mathbb{N}$. Thus, from (2) we get

$$R_b^g = \begin{pmatrix} b_0 & & & & \\ b_1 & b_1 & & & \\ b_2 & b_2 & b_2 & & \\ b_3 & b_3 & b_3 & b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = R_b.$$

On the other hand $R^g_{1/(n+1)} = C_g$. Therefore this definition could be regarded as a two-way generalizing both for Rhaly and Cesàro operator.

We define the following set $\mathcal{B}$:

$$\mathcal{B} := \left\{ g : \mathbb{D} \to \mathbb{C} : \int_0^z g(t)\, dt \in BMOA \right\}.$$

The authors gave for bounded of generalized Rhally matrices as follows:

Lemma 1.4 ([2, Lemma 3.1]) *If $g \in \mathcal{B}$ and $\{(n+1) b_n\}$ is bounded sequence, then R^g_b is bounded.*

For this paper, we make the convention that whenever β or β_j is notated, it is assumed that $|\beta| = |\beta_j| = 1$. Furthermore, $R^\beta_b = R_b^{1/(1-\beta z)}$.

The authors gave for unitarily equivalent of generalized Rhally matrices as follows:

Lemma 1.5 ([2, Theorem 5]) *If $g_\beta(z) = g(\beta z)$ with $|\beta| = 1$, then $R^{g_\beta}_b$ is unitarily equivalent to R^g_b. We denote this by $R^{g_\beta}_b \cong R^g_b$.*

2 Topological Results of Generalized Rhaly Matrices

In this part, we will investigate some topological properties of $\mathcal{R}_b$ set of all bounded generalized Rhally matrices given by b sequences in H^2. From Lemma 1.4, $\mathcal{R}_b$ is

$$\mathcal{R}_b = \{R^g_b : g \in \mathcal{B}\}$$

where $\{(n+1) b_n\}$ is bounded sequence. We will show that $\mathcal{R}_b$ is closed subspace of $B(H^2)$ and we will define an operator from $\mathbb{T} := \{z : \|z\| = 1\}$ to $\mathcal{R}_b$ and show that this operator is injective and continuous in strong-operator topology.

Definition 2.1 *The weak-operator topology, or WOT, on $B(H)$ is the locally convex topology generated by the seminorms, $T_{x,y} = |\langle Tx, y\rangle|$ for $x, y \in H$ and $T \in B(H)$.*

Equivalently, if $\{T_\alpha\}$ is a net of operators in $B(H)$; then, $T_\alpha \overset{\text{WOT}}{\to} T$ iff $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for each $x, y \in H$.

Definition 2.2 *The strong-operator topology, or SOT, on $B(H)$ is the locally convex topology generated by the seminorms, $T_x = \|Tx\|$ for $x \in H$ and $T \in B(H)$.*

As with WOT-convergence, SOT-convergence can be formulated as follows: if $\{T_\alpha\}$ is a net of operators in $B(H)$; then, $T_\alpha \overset{\text{SOT}}{\to} T$ iff $\|(T_\alpha - T)x\| \to 0$ for each $x \in H$.

Theorem 2.3 *$\mathcal{R}_b$ is a Weak-Operator closed subspace of $B(H^2)$.*

Proof Let $R_b^{g_1}, R_b^{g_2} \in \mathcal{R}_b$. Then $R_b^{g_1} + R_b^{g_2} = R_b^{g_1+g_2} \in \mathcal{R}_b$ since $g_1 + g_2 \in \mathcal{B}$ for all $g_1, g_2 \in \mathcal{B}$. We prove that $\mathcal{R}_b$ is closed in the weak operator topology.

Let $\{R_b^g\}_\alpha$ be a net of operators in $\mathcal{R}_b$ such that $\{R_b^g\}_\alpha \overset{\text{WOT}}{\to} S$. We must show that $S = R_b^h$ for some $h \in \mathcal{B}$. Let $\sum_{k=0}^{\infty} (a_k)_\alpha z^k$ be the Taylor series for g_α. $S \in B(H^2)$ since $B(H^2)$ is WOT-closed. Let $\left[\{R_b^g\}_\alpha\right]_{nj}$ be the matrix for $\{R_b^g\}_\alpha$ in the standart basis for H^2.

$\{R_b^g\}_\alpha \overset{\text{WOT}}{\to} S$. Thus, we get

$$\left[\{R_b^g\}_\alpha\right]_{nj} \to S_{nj} \text{ for each } n, j.$$

We know that for any bounded operator on a Hilbert space T, the $nj-$th entry in a matrix for T in an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ is $T_{nj} = \langle Te_n, e_j\rangle$. $\left[\{R_b^g\}_\alpha\right]_{nj} = (a_{n-j})_\alpha b_n$ if $n \geq j$. This implies that $S_{nj} = \lim_\alpha (a_{n-j})_\alpha b_n$ for $n \geq j$. For $n < j$, we know that $\left[\{R_b^g\}_\alpha\right]_{nj} = 0$. Therefore, $S_{nj} = 0$ for $n < j$. So S has the form of (3).

Define

$$h(z) := c + \sum_{n=1}^{\infty} \left[\{R_b^g\}_\alpha\right]_{n1} z^n,$$

where $c \in \mathbb{C}$. Then $S = R_b^h$. By Lemma 1.2, $h \in \mathcal{B}$. Thus $S \in \mathcal{R}_b$.

The WOT is weaker than the norm topology, thereby giving us that $\mathcal{R}_b$ is also norm closed. Therefore $\mathcal{R}_b$ is closed subspace of $B(H^2)$.

Now, we want to define a map from $\mathbb{T}$ to $\mathcal{R}_b$. If we let $\beta \in \mathbb{T}$, then define

$$\phi : \mathbb{T} \to \mathcal{R}_b \quad , \qquad \beta \to R_b^{\frac{1}{1-\beta z}}.$$

Theorem 2.4 *$\phi : \mathbb{T} \to \mathcal{R}_b$ is a strong-operator continuous bijective map.*

Proof Let $\beta_k \to \beta$. We must show that

$$\left\| \left(R_b^{\frac{1}{1-\beta_k z}} - R_b^{\frac{1}{1-\beta z}} \right)(f) \right\| \to 0$$

for each $f \in H^2$. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. From Lemma 1.5, we have

$$R_b^{\frac{1}{1-\beta_k z}} - R_b^{\frac{1}{1-\beta z}} \cong R_b^{\frac{1}{1-\overline{\beta}\beta_k z}} - R_b.$$

If $f \in H^2$, then

$$\begin{aligned}
\left\| \left(R_b^{\frac{1}{1-\overline{\beta}\beta_k z}} - R_b \right)(f) \right\| &= \left\| U_\beta \left(R_b^{\frac{1}{1-\beta_k z}} - R_b^{\frac{1}{1-\beta z}} \right) U_{\overline{B}}(f) \right\| \\
&= \left\| \left(R_b^{\frac{1}{1-\beta_k z}} - R_b^{\frac{1}{1-\beta z}} \right) U_{\overline{B}}(f) \right\| \\
&= \left\| \left(R_b^{\frac{1}{1-\beta_k z}} - R_b^{\frac{1}{1-\beta z}} \right)(g) \right\|
\end{aligned}$$

where $g = U_{\overline{B}}(f) \in H^2$. Thus,

$$R_b^{\frac{1}{1-\beta_k z}} \overset{\text{SOT}}{\to} R_b^{\frac{1}{1-\beta z}} \Leftrightarrow R_b^{\frac{1}{1-\overline{\beta}\beta_k z}} \overset{\text{SOT}}{\to} R_b.$$

We may assume $\beta = 1$.

$$\begin{aligned}
R_b^{\frac{1}{1-\beta_k z}}(f(z)) &= \sum_{n=0}^{\infty} \left(b_n \sum_{j=0}^{n} a_j \right) (\beta_k z)^n \\
R_b(f(z)) &= \sum_{n=0}^{\infty} \left(b_n \sum_{j=0}^{n} a_j \right) z^n.
\end{aligned}$$

Comparing these last equations, we get

$$\left(R_b^{\frac{1}{1-\beta_k z}} - R_b \right)(f) = \sum_{n=0}^{\infty} b_n \left(\sum_{j=0}^{n} a_j \left(\beta_k^j - 1 \right) \right).$$

Since $\beta_k^j - 1 = (\beta_k - 1)\left(1 + \beta_k + \beta_k^2 + \cdots + \beta_k^{j-1}\right) = (\beta_k - 1) \sum\limits_{r=0}^{j-1} \beta_k^r$, we obtain that

$$\begin{aligned}
\left\| \left(R_b^{\frac{1}{1-\beta_k z}} - R_b \right)(f) \right\|^2 &= \sum_{n=0}^{\infty} b_n^2 \left| \sum_{j=0}^{n} a_j \left(\beta_k^j - 1 \right) \right|^2 \\
&= \sum_{n=0}^{\infty} b_n^2 \left| \beta_k - 1 \right|^2 \left| \sum_{j=0}^{n} a_j \sum_{r=0}^{j-1} \beta_k^r \right|^2 \\
&= \left| \beta_k - 1 \right|^2 \sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} a_j b_n \sum_{r=0}^{j-1} \beta_k^r \right|^2.
\end{aligned}$$

$\left(R_b^{\frac{1}{1-\beta_k z}} - R_b \right)(f) \in H^2$ implies that for every $\epsilon > 0$, there is an A such that for every $n \geq A$, we have

$$\left| \beta_k - 1 \right|^2 \sum_{n=A}^{\infty} \left| \sum_{j=0}^{n} a_j b_n \sum_{r=0}^{j-1} \beta_k^r \right|^2 < \frac{\epsilon}{2}. \tag{4}$$

Let

$$B = \sum_{n=0}^{A-1} \left| \sum_{j=0}^{n} a_j b_n \sum_{r=0}^{j-1} \beta_k^r \right|^2 .$$

Since sum is finite, $B < \infty$. $\beta_k \to 1$ means that for every $\epsilon > 0$, there is a K such that if $k \geq K$, then $|\beta_k - 1| < \dfrac{\epsilon}{2B}$. Fix $\epsilon > 0$. If we take $n \geq A$ and $k \geq K$, then we obtain that

$$\begin{aligned}
|\beta_k - 1|^2 \sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} a_j b_n \sum_{r=0}^{j-1} \beta_k^r \right|^2 &= |\beta_k - 1|^2 \sum_{n=0}^{A-1} \left| \sum_{j=0}^{n} a_j b_n \sum_{r=0}^{j-1} \beta_k^r \right|^2 \\
&\quad + |\beta_k - 1|^2 \sum_{n=A}^{\infty} \left| \sum_{j=0}^{n} a_j b_n \sum_{r=0}^{j-1} \beta_k^r \right|^2 \\
&< \frac{\epsilon}{2B}.B + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So, $R_b^{\frac{1}{1-\overline{\beta}\beta_k z}} \overset{\text{SOT}}{\to} R_b$. Therefore, ϕ, SOT-continuous.

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