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Generalized (k, r)–Fibonacci Numbers

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Abstract

In this paper, and from the definition of a distance between numbers by a recurrence relation, new kinds of k–Fibonacci numbers are obtained. But these sequences differ among themselves not only by the value of the natural number k but also according to the value of a new parameter r involved in the definition of this distance. Finally, various properties of these numbers are studied.

Keywords: *Generalizations of Fibonacci numbers, k–Fibonacci numbers, r–distance Fibonacci numbers.*

1 Introduction

Classical Fibonacci numbers have been generalized in different ways [1, 2, 3, 4]. One of these generalizations that greater interest lately among mathematical researchers is that leads to the k–Fibonacci numbers [6, 7].

Then the k–Fibonacci numbers are defined.

Definition 1.1 *For every natural number k, the k–Fibonacci sequence $F_k = \{F_{k,n}\}$ is defined by the recurrence relation*

$$F_{k,n+1} = k\, F_{k,n} + F_{k,n-1} \text{ for } n \geq 1. \tag{1}$$

with initial conditions $F_{k,0} = 0;\ F_{k,1} = 1$

From this definition, the general expression of the first k–Fibonacci numbers is presented in the following table:

Table 1: *k–Fibonacci numbers*

$$\begin{array}{l} F_{k,0} = 0 \\ F_{k,1} = 1 \\ F_{k,2} = k \\ F_{k,3} = k^2 + 1 \\ F_{k,4} = k^3 + 2k \\ F_{k,5} = k^4 + 3k^2 + 1 \\ F_{k,6} = k^5 + 4k^3 + 3k \\ F_{k,7} = k^6 + 5k^4 + 6k^2 + 1 \\ F_{k,8} = k^7 + 6k^5 + 10k^3 + 4k \\ \cdots \end{array}$$

If $k = 1$ the clasical Fibonacci sequence is obtained $F_1 = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$ and if $k = 2$ that is the Pell sequence $F_2 = P = \{0, 1, 2, 5, 12, 29, 70, 169, \ldots\}$.

2 Generalized (k, r)–Fibonacci Numbers

In this section we apply the definition of r–distance to the k–Fibonacci numbers in such a way that generalize earlier results [5, 11]. Very interesting are the formulas used to calculate the general term of the sequences generated by the above definition, as well as which allows to find the sum of the first n terms.

Definition 2.1 *Let the natural numbers $k \geq 1,\ n \geq 0,\ r \geq 1$ be. We define the generalized (k, r)–Fibonacci numbers $F_{k,n}(r)$ by the recurrence relation*

$$F_{k,n}(r) = k\ F_{k,n-r}(r) + F_{k,n-2}(r) \quad for \quad n \geq r, \tag{2}$$

with the initial conditions $F_{k,n}(r) = 1,\ n = 0, 1, 2, \ldots r-1,$ except $F_{k,1}(1) = k$

So, if $F_k(r) = \{F_{k,n}(r)/n \in \mathcal{N}\}$, the expressions of the sequences obtained for $r = 1, 2, \ldots 7$ are:

$$\begin{array}{lcl} F_k(1) & = & \{1, k, 1+k^2, 2k+k^3, 1+3k^2+k^4+, 3k+4k^3+k^5, 1+6k^2+5k^4+k^6, \ldots\} \\ F_k(2) & = & \{1, 1, 1+k, 1+k, (1+k)^2, (1+k)^2, (1+k)^3, (1+k)^3, (1+k)^4, (1+k)^4, \ldots\} \\ F_k(3) & = & \{1, 1, 1, 1+k, 1+k, 1+2k, 1+2k+k^2, 1+3k+k^2, 1+3k+3k^2, \ldots\} \\ F_k(4) & = & \{1, 1, 1, 1, 1+k, 1+k, 1+2k, 1+2k, 1+3k+k^2, 1+3k+k^2, 1+4k+3k^2, \ldots\} \\ F_k(5) & = & \{1, 1, 1, 1, 1, 1+k, 1+k, 1+2k, 1+2k, 1+3k, 1+3k+k^2, 1+4k+k^2, \ldots\} \\ F_k(6) & = & \{1, 1, 1, 1, 1, 1, 1+k, 1+k, 1+2k, 1+2k, 1+3k, 1+3k, 1+4k+k^2, \ldots\} \\ F_k(7) & = & \{1, 1, 1, 1, 1, 1, 1, 1+k, 1+k, 1+2k, 1+2k, 1+3k, 1+3k, 1+4k, 1+4k+k^2, \ldots\} \end{array}$$

2.1 (k, r)–Fibonacci Numbers for $k = 1, 2, 3$

If we particularize the previous sequences for $k = 1, 2, 3, \ldots$, then obtain distinct integer sequences whose properties we study below.

For $k = 1$, the following sequences [11] are obtained:

Table 2: $(1, r)$–Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F_{1,n}(1)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	$\cdots$
$F_{1,n}(2)$	1	1	2	2	4	4	8	8	16	16	32	32	64	64	128	128	256
$F_{1,n}(3)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	65
$F_{1,n}(4)$	1	1	1	1	2	2	3	3	5	5	8	8	13	13	21	21	34
$F_{1,n}(5)$	1	1	1	1	1	2	2	3	3	4	5	6	8	9	12	14	17
$F_{1,n}(6)$	1	1	1	1	1	1	2	2	3	3	4	4	6	6	9	9	13
$F_{1,n}(7)$	1	1	1	1	1	1	1	2	2	3	3	4	4	5	6	7	9

$F_1(1)$ is the classical Fibonacci sequence $F = \{F_n\} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$. The first eight sequences of this relationship are listed in [10] (from now on OEIS).

$F_1(4)$ is the classical Fibonacci sequence, double.

For $k = 2$, the following sequences are obtained: $F_2(1)$ is the Pell sequence.

Table 3: $(2, r)$–Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_{2,n}(1)$	1	2	5	12	29	70	169	408	985	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_{2,n}(2)$	1	1	3	3	9	9	27	27	81	81	243	243	729	729	$\cdots$	$\cdots$
$F_{2,n}(3)$	1	1	1	3	3	5	9	11	19	29	41	67	99	149	233	347
$F_{2,n}(4)$	1	1	1	1	3	3	5	5	11	11	21	21	43	43	85	85
$F_{2,n}(5)$	1	1	1	1	1	3	3	5	5	7	11	13	21	23	35	45
$F_{2,n}(6)$	1	1	1	1	1	1	3	3	5	5	7	7	13	13	23	23

Finally, for $k = 3$ the following table is obtained:

3 Some Properties of the Generalized (k, r)–Fibonacci Sequences

In this section, we study the general properties of the (k, r)–Fibonacci sequences.

Table 4: $(3, r)$–Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_{3,n}(1)$	1	3	10	33	109	360	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_{3,n}(2)$	1	1	4	4	16	16	64	64	256	256	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_{3,n}(3)$	1	1	1	4	4	7	16	19	37	67	94	178	295	460	829	$\cdots$
$F_{3,n}(4)$	1	1	1	1	4	4	7	7	19	19	40	40	97	97	217	217
$F_{3,n}(5)$	1	1	1	1	1	4	4	7	7	10	19	22	40	43	70	100
$F_{3,n}(6)$	1	1	1	1	1	1	4	4	7	7	10	10	22	22	43	43

Proposition 3.1 *For $r \geq 2$ it is*

$$F_{k,r}(r) = 1 + k \tag{3}$$

Just apply Equation (2).

Proposition 3.2 *If r is even, the respective sequence is double:* $F_{k,2n}(2m) = F_{k,2n+1}(2m)$

Proof. By induction.
For $n = 0$ it is $F_{k,0}(2m) = 1,\ F_{k,1}(2m) = 1$ because definition of r–distance.
Let us suppose this formula is true until $2n + 1$. Then

$$\begin{aligned} F_{k,2n+2}(2m) &= k\ F_{k,2n+2-2m}(2m) + F_{k,2n}(2m) \\ &= k\ F_{k,2(n+1-m)}(2m) + F_{k,2n}(2m) \\ F_{k,2n+3}(2m) &= k\ F_{k,2n+3-2m}(2m) + F_{k,2n+1}(2m) \\ &= k\ F_{k,2(n+1-m)+1}(2m) + F_{k,2n+1}(2m) \end{aligned}$$

And both expressions are equal because
$F_{k,2n+1}(2m) = F_{k,2n}(2m) \to F_{k,2(n+1-m)+1}(2m) = F_{k,2(n+1-m)}(2m)$

Proposition 3.3 *For $n = r, r+1, \ldots, 2r-1$, it is*
$F_{k,n}(r) = \left(\left\lfloor \dfrac{n-r}{2} \right\rfloor + 1\right) k + 1.$

Proof. By induction.
Let us suppose r is even, $r = 2m$.
Then, for $n = r = 2m$, Left Hand Side (LHS) of this equation is
$LHS = F_{k,2m}(2m) = kF_{k,0}(2m) + F_{k,2m-2}(2m) = k + 1$ while Right Hand Side (RHS) is $RHS = \left(\left\lfloor \dfrac{n-r}{2} \right\rfloor + 1\right) k + 1 = \left(\left\lfloor \dfrac{0}{2} \right\rfloor + 1\right) k + 1 = k + 1.$
Let us suppose this formula is true until $n = 2r - 2 = 4m - 2$. That is,

$F_{k,4m-2}(2m) = mk+1 = \left(\left\lfloor \frac{n-r}{2} \right\rfloor + 1\right) k + 1 = \left(\left\lfloor \frac{4m-2-2m}{2} \right\rfloor + 1\right) k + 1 = mk+1$.
Then, if $n = 2r-1 = 4m-1$, it is

$$\begin{aligned} F_{k,4m-1}(2m) &= kF_{k,2m-1}(2m) + F_{k,4m-3}(2m) = k + \left(\left\lfloor \frac{2m-3}{2} \right\rfloor + 1\right) k + 1 \\ &= k + (m-1+1)k + 1 = (m+1)k+1 \end{aligned}$$

And of the other hand:
$\left(\left\lfloor \frac{n-r}{2} \right\rfloor + 1\right) k + 1 = \left(\left\lfloor \frac{2m-1}{2} \right\rfloor + 1\right) k + 1 = (m+1)k+1.$

If r is odd, the proof is similar.

For proofs that follow will take into account that $a < b \to \binom{a}{b} = 0$, $a < 0 \to \binom{a}{b} = 0$, and $\binom{a}{0} = 1, \forall a$. For these reasons will extend the following sums from $j = 0$ until the combinatorial number is null, without having to specify exactly where just the sum.

Following proposition shows the formulas used to calculate the general term of the sequence $F_k(r) = \{F_{k,n}(r)\}$, according to that $r \geq 2$ is odd or even (see [6, 7] for $r = 1$).

Theorem 3.4 *Main formula*

1. *If r is even, $r = 2p$:*

$$F_{k,2n}(2p) = F_{k,2n+1}(2p) = \sum_{j=0}^{n/p} \binom{n-(p-1)j}{j} k^j \tag{4}$$

2. *If r is odd, $r = 2p+1 \geq 3$:*

$$F_{k,2n}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-p-(2p-1)j}{2j+1} k^{2j+1} \right] \tag{5}$$

$$F_{k,2n+1}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-(p-1)-(2p-1)j}{2j+1} k^{2j+1} \right] \tag{6}$$

Proof. By induction.
Formula (4). Let $r = 2p$ be.
For $n = 0$, by definition it is $F_{k,0}(2p) = 1$ and SHR of (4) is $F_{k,0}(2p) = \sum_0^0 \binom{(1-p)j}{j} k^j = 1$.
For $n = 1$, by definition it is $F_{k,2}(2) = 1+k$ and $F_{k,2}(r) = 1$ for $r > 2$. In Formula (4) it is $F_{k,2}(2p) = \sum_0^2 \binom{1-(p-1)j}{j} k^j = 1 + \binom{1-(p-1)j}{1} k.$

Then, $F_{k,2}(2)=1+k$ y $F_{k,2}(2p)=1$ for $2p=4,6,8,\dots$
Let us suppose this formula is true until n. Then:

$$\begin{aligned}
F_{k,2n+2}(2p) &= k\,F_{k,2n+2-2p}(2p)+F_{k,2n}(2p)\\
&= \sum_{j=0}\binom{n-(p-1)j}{j}k^j + k\,F_{k,2(n+1-p)}(2p)\\
&= \sum_{j=0}\binom{n-(p-1)j}{j}k^j + k\sum_{j=0}\binom{n+1-p-(p-1)j}{j}k^j\\
&= 1+\sum_{j=1}\binom{n-(p-1)j}{j}k^j + \sum_{j=0}\binom{n-(p-1)(j+1)}{j}k^{j+1}\\
&= 1+\sum_{j=0}\left[\binom{n-(p-1)(j+1)}{j+1}k^{j+1}+\binom{n-(p-1)(j+1)}{j}k^{j+1}\right]\\
&= 1+\sum_{j=0}\binom{n-(p-1)(j+1)+1}{j+1}k^{j+1}\\
&= \sum_{j=0}\binom{n+1-(p-1)j}{j}k^j = F_{k,2n+2}(2p)
\end{aligned}$$

no more take into account the addition formula $\binom{a}{j+1}+\binom{a}{j}=\binom{a+1}{j+1}$ (see [9]).

We will prove Formulas (5) and (6) together.
For $n=0$ it is $F_{k,0}(2p+1)=1$ and SHR of (5) is
$$\sum_0^0\left[\binom{-(2p-1)j}{2j}k^{2j}+\binom{-p-(2p-1)j}{2j+1}k^{2j+1}\right]=1.$$

In the same way, it is $F_{k,1}(2p+1)=1$ and SHR of (6) is
$$\sum_0^0\left[\binom{-(2p-1)j}{2j}k^{2j}+\binom{1-p-(2p-1)j}{2j+1}k^{2j+1}\right]=1 \text{ because } 1-p<0.$$

For $n=1$ it is $F_{k,2}(2p+1)=1$ and SHR of (5) is
$$\sum_0^0\left[\binom{1-(2p-1)j}{2j}k^{2j}+\binom{1-p-(2p-1)j}{2j+1}k^{2j+1}\right]=1, \text{ because}$$
$p\geq 1\rightarrow 1-p-(2p-1)j<0$.
For Formula (6), if $p=1$, the LHS is $F_{k,3}(2p+1)=1+k$ while the RHS is
$$\sum_0^0\left[\binom{1-j}{2j}k^{2j}+\binom{1-j}{2j+1}k^{2j+1}\right]=1+k.$$
If $p>1$, the LHS of (6) is 1 as well the RHS.
Let us suppose the formula is true for $2n$ and $2n+1$. Then:

$$F_{k,2n+2}(2p+1) = k\,F_{k,2n+1-2p}(2p+1)+F_{k,2n}(2p+1)$$

$$
\begin{aligned}
&= k\,F_{k,2(n-p)+1}(2p+1) + F_{k,2n}(2p+1) \\
&= k\sum_{j=0}\left[\binom{n-p-(2p-1)j}{2j}k^{2j} + \binom{n-(2p-1)-(2p-1)j}{2j+1}k^{2j+1}\right] \\
&+ \sum_{j=0}\left[\binom{n-(2p-1)j}{2j}k^{2j} + \binom{n-p-(2p-1)j}{2j+1}k^{2j+1}\right] \\
&= \sum_{j=0}\left[\binom{n-p-(2p-1)j}{2j} + \binom{n-p-(2p-1)j}{2j+1}\right]k^{2j+1} \\
&+ \sum_{j=0}\binom{n-(2p-1)(j+1)}{2j+1}k^{2j+2} + 1 + \sum_{1}\binom{n-(2p-1)j}{2j}k^{2j} \\
&= \sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1} \\
&+ \sum_{j=1}\binom{n-(2p-1)j}{2j-1}k^{2j} + 1 + \sum_{1}\binom{n-(2p-1)j}{2j}k^{2j} \\
&= \sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1} + \sum_{j=1}\binom{n+1-(2p-1)j}{2j}k^{2j} + 1 \\
&= \sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1} + \sum_{j=0}\binom{n+1-(2p-1)j}{2j}k^{2j} \\
&= F_{k,2n+2}(2p+1)
\end{aligned}
$$

Similarly proved for $F_{k,2n+3}(2p+1)$.

If you want to implement Formulas (5) and (6) in Wolfram Mathematica, the limits of the sums would be as follows:

$$
\begin{aligned}
F_{k,2n}(2p+1) &= \sum_{j=0}^{n/r}\binom{n-(2p-1)j}{2j}k^{2j} + \sum_{j=0}^{(n-p-1)/r}\binom{n-p-(2p-1)j}{2j+1}k^{2j+1} \\
F_{k,2n+1}(2p+1) &= \sum_{j=0}^{n/r}\binom{n-(2p-1)j}{2j}k^{2j} + \sum_{j=0}^{(n-p)/r}\binom{n-(p-1)-(2p-1)j}{2j+1}k^{2j+1}
\end{aligned}
$$

We will then study the formula to find the sum of the terms of the sequence $F_k(r)$.

Proposition 3.5 Sum of the terms of the sequence $F_k(r)$
The sum of the first n terms of the sequence $F_k(r)$ is given by the formula

$$S_{k,n}(r) = \frac{1}{k}\Big(F_{k,n+r-1}(r) + F_{k,n+r}(r) - 2\Big) \tag{7}$$

Proof. Iteratly applying Formula (2:

$$F_{k,n+r}(r) + F_{k,n+r-1}(r) = k\,F_{k,n}(r) + F_{k,n+r-2}(r) + k\,F_{k,n-1}(r) + F_{k,n+r-3}(r)$$

$$\begin{aligned}
&= k\,F_{k,n}(r) + k\,F_{k,n-1}(r) + k\,F_{k,n-2}(r) + k\,F_{k,n-3}(r) + F_{k,n+r-4}(r) + F_{k,n+r-5}(r)\\
&= k[F_{k,n}(r) + F_{k,n-1}(r) + F_{k,n-2}(r) + \cdots + F_{k,1}(r) + F_{k,0}(r)] + F_{k,r-1}(r) + F_{k,r-2}(r)\\
&= k\sum_{j=0}^{n} F_{k,j}(r) + 2 \to \sum_{j=0}^{n} F_{k,j}(r) = \frac{1}{k}\Big(F_{k,n+r-1}(r) + F_{k,n+r}(r) - 2\Big)
\end{aligned}$$

In particular, for $r = 1$, the formula $S_{k,n}(1) = S_{k,n} = \frac{1}{k}\Big(F_{k,n} + F_{k,n+1} - 2\Big)$ is obtained, [6, 7, 8]. This formula shows the sum of the first k–Fibonacci numbers with $F_{k,0} = 0$

3.0.1 Special Sums

The formulas that indicate the sum of the even and odd terms of the above sequences are, respectively,

$$\sum_{j=0}^{n} F_{k,2j}(r) = \frac{1}{k}(F_{k,2n+r} - 1) \tag{8}$$

$$\sum_{j=0}^{n} F_{k,2j+1}(r) = \frac{1}{k}(F_{k,2n+1+r} - 1) \tag{9}$$

Proof. Proof of the first formula. If we apply iteratly Equation (2), it is:

$$\begin{aligned}
F_{k,2n+r}(r) &= k\,F_{k,2n}(r) + F_{k,2n+r-2}(r)\\
&= k\,F_{k,2n}(r) + k\,F_{k,2n-2}(r) + F_{k,2n+r-4}(r)\\
&= k\,F_{k,2n}(r) + k\,F_{k,2n-2}(r) + k\,F_{k,2n-4}(r) + \cdots + kF_{k,0}(r) + F_{k,r-2}(r)\\
&= k\sum_{j=0}^{n} F_{k,2j}(r) + 1 \to \sum_{j=0}^{n} F_{k,2j}(r) = \frac{1}{k}\big(F_{k,2n+r} - 1\big)
\end{aligned}$$

The second formula is proven in a similar way:

$$\begin{aligned}
F_{k,2n+1+r}(r) &= k\,F_{k,2n+1}(r) + F_{k,2n+r-1}(r)\\
&= k\,F_{k,2n+1}(r) + k\,F_{k,2n-1}(r) + k\,F_{k,2n-3}(r) + \cdots + kF_{k,1}(r) + F_{k,r-1}(r)\\
&= k\sum_{j=0}^{n} F_{k,2j+1}(r) + 1 \to \sum_{j=0}^{n} F_{k,2j+1}(r) = \frac{1}{k}\big(F_{k,2n+1+r} - 1\big)
\end{aligned}$$

4 Generating Functions of the (k, r)–Fibonacci Numbers

In this section we will study the generating functions of the different sequences $F_k(r)$, starting with the calculation of their formulas and continuating with their graphics.

Proposition 4.1 *Generating function of the sequence* $F_k(r) = \{F_{k,n}(r)\}$ *is* $f_k(r,x) = \dfrac{1+x}{1-x^2-k\ x^r}$

Proof.
If $f_k(r,x)$ is the generating function of the sequence $F_k(r)$, then

$$\begin{aligned} f_k(r,x) &= F_{k,0}(r) + F_{k,1}(r)x + F_{k,2}(r)x^2 + F_{k,3}(r)x^3 + \cdots \\ &\qquad + F_{k,r}(r)x^r + F_{k,r+1}(r)x^{r+1} + \cdots \\ x^2 f_k(r,x) &= \qquad F_{k,0}(r)x^2 + F_{k,1}(r)x^3 + \cdots \\ &\qquad + F_{k,r-2}(r)x^r + F_{k,r-1}(r)x^{r+1} + \cdots \\ k\ x^r f_k(r,x) &= \qquad k\ F_{k,0}(r)x^r + k\ F_{k,1}(r)x^{r+1} + \cdots \end{aligned}$$

So, $f_k(r,x)(1-x^2-k\ x^r) =$
$F_{k,0}(r) + F_{k,1}(r)x + (F_{k,2}(r) - F_{k,0}(r))x^2 + (F_{k,3}(r) - F_{k,1}(r))x^3 + \cdots$
$+(F_{k,r}(r) - F_{k,r-2}(r) - kF_{k,0}(r))x^r + (F_{k,r+1}(r) - F_{k,r-1}(r) - kF_{k,1}(r))x^{r+1} + \cdots$
$= 1 + x$ since for $n < r$ it is $F_{k,n}(r) = 1$ and for $n \geq r$ the coefficient of x^n of the SHC verifies the condition of distance, so it vanishes.
Finally, taking into account that for $r \geq 2$ it is $F_{k,0}(r) = F_{k,1}(r) = 1$, we obtain the indicated generating function.

4.1 Graphs of the Generating Functions

Below we show the graphs of the generating functions of the different (k,r)–Fibonacci numbers.

- If $r = 1$, the graph of the generating function $f_k(1,x) = f_k(x) = \dfrac{1+x}{1-k\ x-x^2}$ is shown below:

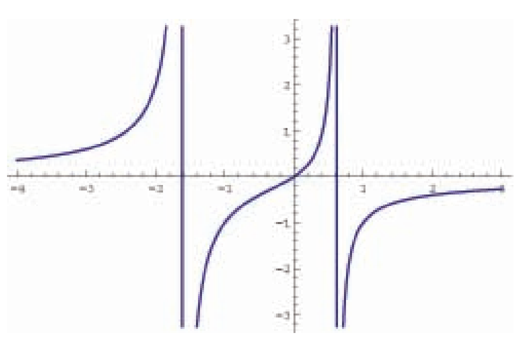

Figure 1: Generating function of the $(k,1)$–Fibonacci numbers $F_k(1)$

- If r is even, its graph is

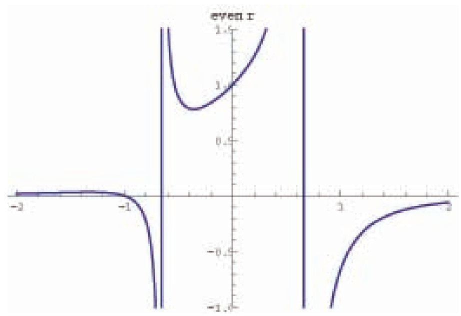

Figure 2: Generating function of the $(k,2m)$–Fibonacci numbers

As it increases r, this curve has a minimum and a maximum relative who tend to $(-1^+,0)$ and $(-1^-,0)$, respectively.

- Finally, if r is odd and greater than 1, the graph is of the form

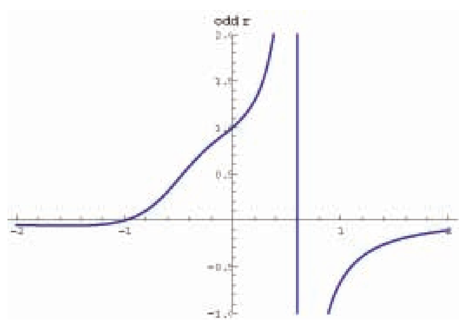

Figure 3: Generating function of the $(k,2m+1)$–Fibonacci numbers

This curve has a relative minimum that tends to $(-1,0)$ as r increases.

5 Conclusions

We have generalized the r–distance Fibonacci numbers to the case of k–Fibonacci numbers, getting more general formulas that previously found.
These formulas include which allows to find the general term of a sequence of this type according to r is even or odd.
It also shows the formula to find the sum of the terms of the generalized (k,r)–Fibonacci sequences as well as the sum of terms of even order and odd-order.
Finally we indicate another way of finding the generalized (k,r)–Fibonacci sequences from the generating function.

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