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Variations on the Projective Central Limit Theorem

Brian Knaeble

Department of Mathematics, Statistics and Computer Science
University of Wisconsin-Stout, WI Menomonie WI 54751, United States
E-mail: knaebleb@uwstout.edu

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Abstract

This expository article states and proves four, concrete, projective, central limit theorems. The results are known or suspected to be true by experts who are familiar with the more general central limit theorem for convex bodies, and related theory. Here we consider only four types of high dimensional geometric objects: spheres, balls, cubes, and boundaries of cubes. Each is capable of transforming uniform random variables into normal random variables through projection. This paper has been written to introduce new proof techniques, demonstrate how statistical simulation can be applied to geometry, and to build a foundation upon which recreational research projects can be built. The goal is to give the reader a better understanding of some of the mathematics at the juncture of probability theory, analysis, and geometry in high dimensions.

Keywords: *Concentration of measure, Dominated convergence, Law of large numbers, Convolution, Statistical simulation.*

1 Introduction

High-dimensional geometry is a fascinating subject. Techniques of modern, convex geometry are applicable in probability and statistics [2], and recently the projective central limit theorem for convex bodies has been proven [15] (see also [7]). Meanwhile, entertaining expository articles have been published that discuss some counter-intuitive phenomena in high dimensions (see [11] or [10]). This paper presents four concrete examples of how high-dimensional geometric objects can relate uniform random variables, through projection, to normal random variables.

The objects are the sphere, the ball, the cube, and the surface of the cube, and the associated results are presented here so as to highlight connections to established features of classical mathematics and statistics. For the sphere, concentration of measure is demonstrated with the Law of Large Numbers. For the ball, Stirling's approximation is used, demonstrating the relationship between the gamma function, associated factorials, π, and e. For the cube, the central limit theorem applies, and surprisingly, we illustrate how statistical simulation can be used to gather empirical evidence, in the case of the cube's boundary, demonstrating how statistics can be applied even to pure geometry.

At the juncture of high dimensional geometry and probability theory, mathematics can be written with one of two separate sets of notation. While this paper uses mainly the terminology of probability theory, the reader is encouraged to visualize the content of the theorems whenever possible. In some instances, it may be beneficial to mentally restate a given result using geometric language. For example, despite its formal probabilistic statement, Theorem 5.1 simply asserts that the volumes of sections orthogonal to the diagonal of a high dimensional cube follow the common Gaussian function of statistics and probability.

The rich interplay between geometry and probability theory leads naturally to generalizations and new conjectures. For instance, while the central limit theorem for convex sets asserts that any high-dimensional convex body is capable of transforming a uniform random variable into a normal random variable via projection, this same outcome is observed for surfaces as well. An exciting research program would involve a search for and classification of all geometric objects with such capabilities, and we needn't project onto only lines. Moreover, combinatorial analysis of concrete objects, e.g. tetrahedrons and their boundaries, could lead to startling analytic results, producing rational sequences that converge via the central limit theorems to interesting products like πe. For such a sequence arising from the cube see [13].

While this paper could generate novel research hypotheses and lead to interesting results, it has also been written for educational purposes. Readers can improve their understanding of convolution with a recreational problem in Section 7. Foundational issues relating to problems defining uniform random variables on arbitrary sets are discussed in Section 4. Formulas for volumes of balls and their surfaces are derived in the appendix. Proofs and mathematical development occur in sections 2, 3, 5, and 6. Section 6 deals also with statistical simulation.

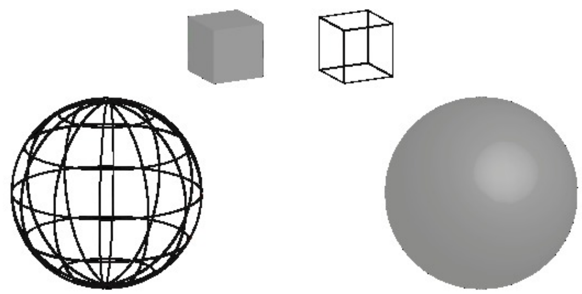

Figure 1: An illustration of the objects in $\mathbb{R}^3$

2 Spheres

We say in this section that a random variable X_r^n is uniformly distributed on a sphere $S_r^n = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 + x_2^2 + ... + x_n^2 = r^2\}$ if X_r^n is distributed according to the unique, rotationally invariant, probability measure on S_r^n. For a proof of uniqueness see [14], Theorem 7.23. Note that we are using n for the ambient dimension, so that S_r^n itself has dimension $n-1$.

With an orthonormal basis for $\mathbb{R}^n$

$$X_r^n = (X_{r,1}^n, X_{r,2}^n, ..., X_{r,n}^n)$$

and the components $\{X_{r,i}^n\}_{i=1}^n$ are identically distributed. Given a fixed $k \in \mathbb{N}$ and a sequence of expanding spheres, $\{S_{\sqrt{n}}^n\}_{n\in\mathbb{N}}$, define the following sequence of projected random variables:

$$\left\{Y^n = \left(X_{\sqrt{n},1}^n, X_{\sqrt{n},2}^n, ..., X_{\sqrt{n},k}^n\right)\right\}_{n\in\mathbb{N}, n>k}.$$

With the symbol $\to^d$ denoting convergence in distribution, and with $\mathbf{N}(\mathbf{0}, \mathbf{I}_k)$ denoting the k-variate standard normal distribution, the projective central limit theorem can be stated as follows.

Theorem 2.1 (Projective Central Limit Theorem for Spheres)**.** *With Y^n as just defined, as $n \to \infty$,*

$$Y^n \to^d \mathbf{N}(\mathbf{0}, \mathbf{I_k}).$$

Proof. It suffices to show that $X_{\sqrt{n},1}^n \to^d \mathrm{N}(0,1)$.

Let $\{Z_1, Z_2, ..., Z_n, ...\}$ be a set of independent, standard normal, random variables, so that each of $\{Z^n = (Z_1^n, Z_2^n, ..., Z_n^n)\}_{k<n<\infty}$ is a standard, multivariate, normal random variable. Each density, namely

$$f_{Z^n}(x_1, x_2, ..., x_n) = \frac{1}{(2\pi)^{n/2}} e^{\sum_{i=1}^n x_i^2/2},$$

defines a rotationally invariant, probability measure on $\mathbb{R}^n$, and thus each $\frac{\sqrt{n}}{|Z^n|}Z^n$ is distributed uniformly on $S^n_{\sqrt{n}}$. By uniqueness of such a uniform measure (see [14], Theorem 7.23),

$$\frac{\sqrt{n}}{|Z^n|}Z^n =^d X^n_{\sqrt{n}}. \tag{1}$$

Furthermore, the strong law of large numbers, applied to $\chi^2(1)$ random variables, ensures that

$$\frac{|Z^n|}{\sqrt{n}} = \left(\frac{((Z_1)^2 + (Z_2)^2 + ... + (Z_n)^2)}{n}\right)^{1/2} \to 1 \ a.s.$$

Therefore, by way of the equality in (1), $X^n_{\sqrt{n},1} \to^d Z_1 =^d N(0,1)$. □

In closing, for curiosity, we note that for all $t \in \mathbb{N}$, $E((\chi^2(n))^t) < \infty$, so for small $\epsilon > 0$,

$$P(||Z^n| - \sqrt{n}| > \epsilon) = O(1/n^t), \tag{2}$$

since $P(||Z^n|^2 - n| > \epsilon) = O(1/n^t)$ (see [3]). Said another way, for any $\epsilon > 0$, as the dimension increases, the standard, normal, multivariate, random variable $\mathbf{N}(\mathbf{0}, \mathrm{I}_k)$ is located within ϵ of $S^n_{\sqrt{n}}$ with probability that approaches one.

3 Balls

We say in this section that a random variable X^n_r is uniformly distributed on a ball $B^n_r = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 + x_2^2 + ... + x_n^2 \leq r^2\}$, if for any measurable subset $E \subseteq B^n_r$, we have with μ denoting measure that

$$P(X^n_r \in E) = \frac{\mu(E)}{\mu(B^n_r)}.$$

Assume henceforth the use of Lebesgue measure and assume also that all ecountered sets are measurable.

With an orthonormal basis for $\mathbb{R}^n$,

$$X^n_r = (X^n_{r,1}, X^n_{r,2}, ..., X^n_{r,n}),$$

and the components $\{X^n_{r,i}\}^n_{i=1}$ are identically distributed. Given a fixed $k \in \mathbb{N}$ and a sequence of expanding balls, $\{B^n_{\sqrt{n}}\}_{n\in\mathbb{N}}$, define the following sequence of projected random variables:

$$\left\{Y^n = \left(X^n_{\sqrt{n},1}, X^n_{\sqrt{n},2}, ..., X^n_{\sqrt{n},k}\right)\right\}_{n\in\mathbb{N}, n>k}.$$

As with spheres, we can conclude the following.

Theorem 3.1 (Projective Central Limit Theorem for Balls). *With Y^n as just defined, as $n \to \infty$,*

$$Y^n \to^d \mathbf{N}(\mathbf{0}, \mathbf{I_k}).$$

Some preparation is required before stating the proof.

Definition 3.2. *For $x > 0$,*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Remark 3.3. $\Gamma(n) = (n-1)!$.

Theorem 3.4.

$$\mu(S_r^n) = \frac{r^{n-1} 2\pi^{n/2}}{\Gamma(n/2)}.$$

Theorem 3.5.

$$\mu(B_r^n) = \frac{r^n 2\pi^{n/2}}{n\Gamma(n/2)}.$$

These formulas are derived in the appendix. See also Figure 3.

Remark 3.6. *In $\mathbb{R}^n$, the measure of the unit sphere is always n times the measure of the unit ball.*

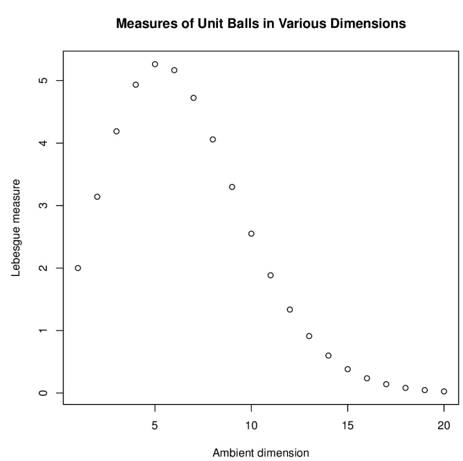

Figure 2: With $r = 1$, as n increases, the measures increase temporarily before decreasing to zero.

In order to prove Theorem 3.1 it suffices to demonstrate that $X^n_{\sqrt{n},1} \to^d$ N$(0,1)$. For $n \geq 2$, the density function for $X^n_{\sqrt{n},1}$, denoted with $f_n(x)$, can be expressed on $\{x : -\sqrt{n} < x < \sqrt{n}\}$ as

$$f_n(x) = \frac{\mu\left(B^{n-1}_{\sqrt{n-x^2}}\right)}{\mu\left(B^n_{\sqrt{n}}\right)} = \frac{\frac{(\sqrt{n-x^2})^{n-1}2\pi^{(n-1)/2}}{(n-1)\Gamma((n-1)/2)}}{\frac{(\sqrt{n})^n 2\pi^{n/2}}{n\Gamma(n/2)}}. \tag{3}$$

Stirling's formula [20], namely

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left(1 + O\left(\frac{1}{x}\right)\right),$$

justifies the following lemma.

Lemma 3.7. *As $n \to \infty$*

$$\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\frac{1}{\sqrt{n-1}} \to \frac{1}{\sqrt{2}}.$$

And the definition $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$ leads to a second lemma.

Lemma 3.8. *For any $x \in \mathbb{R}$, as $n \to \infty$,*

$$\left(1 - \frac{x^2}{n}\right)^{\frac{n-1}{2}} \to e^{-\frac{x^2}{2}}.$$

Together, these lemmas allow for further simplification of (3), with the result being

$$f_n(x) \to \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \tag{4}$$

which is the density for the standard normal random variable.

For any n, $\operatorname*{argmax}_{x\in\mathbb{R}} f_n(x) = 0$, and since $f_n(0)$ is decreasing in n,

$$\max\{f_n(x)\}_{x\in\mathbb{R},n\geq 2} = f_2(0) = \pi^{-1}.$$

Thus, on any interval $[a,b]$, and for any $n \geq 2$, the function $f_n(x)$ is bounded by the function $g(x) = \pi^{-1}$, which is integrable over $[a,b]$. Lebesgue's dominated convergence theorem (see [5], Chapter 8) then justifies

$$P(a \leq X^n_{\sqrt{n},1} \leq b) = \int_a^b f_n(x)dx \to \int_a^b \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = P(a \leq N(0,1) \leq b).$$

This implies that $X^n_{\sqrt{n},1} \to^d N(0,1)$.

4 Interlude

Note that direct computation with density functions, as carried out in Section 3, could have been used to prove the projective central limit theorem of Section 2, with only slight modification. The main difference being that with a sphere, the measures of sections are not proportional to the corresponding values of the density function for the projected, random variable—the curvature of the sphere must be accounted for. The ball, however, provides a more geometric route to the bell curve of the Gaussian function, because in high dimensions, the measures of sections orthogonal to any axis of the ball, are approximately the values of a scaled, Gaussian function. The scaling is necessary because the Lebesgue measure of balls of radius $\sqrt{n}$, or unit balls, is not constant in the dimension n.

No scaling is necessary with unit cubes though. In all dimensions the unit cube has constant Lebesgue measure equal to one. Furthermore, as the dimension increases, the diameter of the unit cube, being $\sqrt{n}$, increases as did the expanding radius required previously of spheres and balls. Perhaps not surprisingly then, we find that the measures of sections orthogonal to the diagonal are approximately a Gaussian function of their positions along the diagonal (see Figure 3). However, this Gaussian function represents a normal random variable with variance $1/12$. If a standard normal limit is desired, the edge lengths for the cubes must not be one, but rather $\sqrt{3}$.

It seems that the *standard* units of Lebesgue measure are not in tune with the *standard* deviations assigned to distributions, and thus our geometric, central limit theorems, as currently formulated, lack a sense of aesthetic beauty. If not balls or cubes, what type of geometric object, definable at once in all dimensions, is required for the bell curve of the *standard* Gaussian function to be realized as the high dimensional limit of the *Lebesgue* measures of sections? Perhaps different measures are required. The curious reader is encouraged to consult Federer's classic text on geometric measure theory [8].

As a final point for consideration, before we proceed with cubes, note that we have defined a uniform random variable on the sphere by specifying that its measure be rotationally invariant. With balls, however, we have worked directly with Lebesgue measures of subsets, defining a uniform random variable as one whose associated measure is proportional to Lebesgue measure. This direct approach could have been used on the sphere as well, assuming a specified differentiable structure in order to define the measure (see [19], [9] or [4] for introductions to differentiable topology). The standard differentiable structure on a sphere, arising through stereographic projection for example, would result in a rotationally invariant measure, and the normal results for projections that we have now become accustomed to would remain unchanged. However, there are exotic spheres, discovered in 1956 by Milnor [16]. An exotic sphere consists of the same set of points as a standard sphere, but with a different differentiable structure. Not all standard spheres have exotic analogues, and some spheres in higher dimensions admit more differentiable structures than others [12]. It would be interesting to define uniform random variables on exotic spheres, and to investigate whether projection still gives rise to normality.

5 Cubes

Given $c > 0$, we say that a random variable X_c^n is uniformly distributed on the cube $C_c^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \forall i \in \{1, 2, ..., n\}, -c/2 \leq x_i \leq c/2\}$, if for any measurable subset $E \subseteq C_c^n$, we have with μ denoting the measure that

$$P(X_c^n \in E) = \frac{\mu(E)}{\mu(C_c^n)}.$$

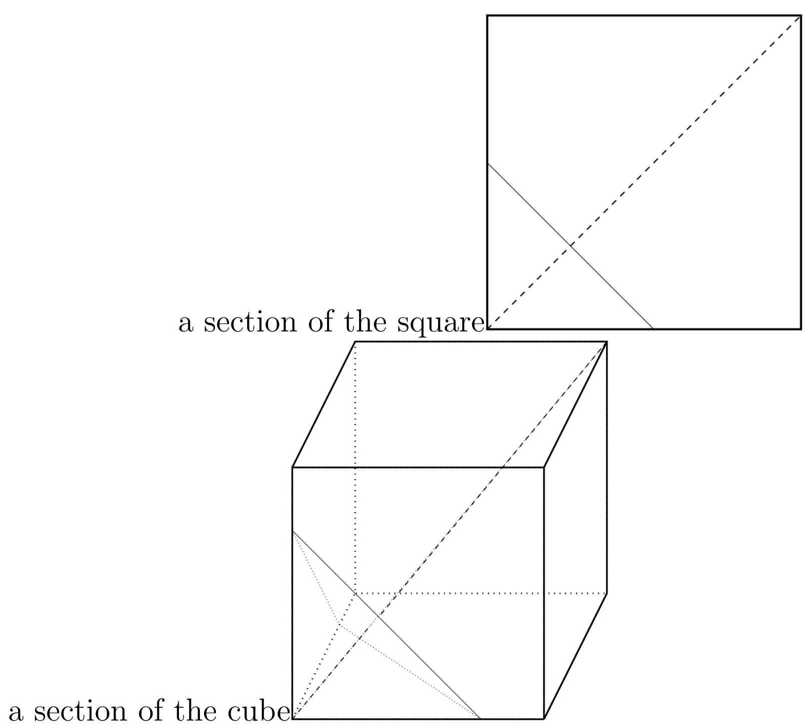

Figure 3: The measure of a section is a function of its position along the diagonal

Let $D^n = \text{span}((1, 1, ..., 1_n)/\sqrt{n})$ represent a canonical line in $\mathbb{R}^n$ that intersects C_c^n along a diagonal. Let $p(X_c^n)$ denote the projection of $X_{\sqrt{3}}^n$ onto D^n. Consider

$$\left\{Y^n = |p(X_{\sqrt{3}}^n)|\right\}_{n \in \mathbb{N}}.$$

Similar to previous results for spheres and balls, we can conclude the following for cubes.

Theorem 5.1 (Projective Central Limit Theorem for Cubes)**.** *With Y^n as just defined, as $n \to \infty$,*

$$Y^n \to^d N(0,1).$$

Proof. Assuming the standard, orthonormal basis for $\mathbb{R}^n$ we have

$$X_c^n = (X_{c,1}^n, X_{c,2}^n, ..., X_{c,n}^n),$$

where each of $\{X_{c_i}^n\}_{i=1}^n$ is independent and uniformly distributed on $[-c/2, c/2]$. Note that $E(X_{c,i}^n) = 0$ and $E((X_{c,i}^n)^2) = 1$. The following lemma is easily verified.

Lemma 5.2. $p(X_c^n) = \frac{\sum_{i=1}^n X_{c,i}^n}{\sqrt{n}}(1,1,...,1_n)/\sqrt{n}$

An immediate consequence of Lemma 5.2 is that $Y^n = \frac{\sum_{i=1}^n X_{c,i}^n}{\sqrt{n}}$ and thus according to our observations, Theorem 5.1 follows directly from the central limit theorem. □

To conclude that the volumes of sections orthogonal to a diagonal of a high dimensional unit cube are approximately a Gaussian function with variance $1/12$, the same reasoning is used, but with $c = \pm 1/2$, and the local version of the central limit theorem, that applies to densities, is required. Both the local central limit theorem and the standard central limit theorem are treated in Petrov's book *Sums of Independent Random Variables* [17]. In the next section we will employ Lyapunov's central limit theorem.

6 Faces of Cubes

For $c > 0$, consider the n-cube,

$$C_c^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \forall i \in \{1,2,...,n\}, -c/2 \le x_i \le c/2\},$$

and then consider its topological boundary, ∂C_c^n. The boundary is a union of $2n$ faces:

$$\partial C_c^n = \cup_{i \in \{1,2,...,n\}, j \in \{+,-\}} F_{i,j},$$

where $F_{i,j} = \{(x_1, x_2, ..., x_n) \in C_c^n : x_i = jc/2\}\}$.

A random variable is is said to be uniformly distributed on ∂C_c^n, if for any measurable subset $E \subseteq \partial C_c^n$, we have

$$P(X_c^n \in E) = \frac{\mu(E)}{\mu(C_c^n)}.$$

Note that we are using $(n-1)$-dimensional Lebesgue measure on the faces. These faces overlap on sets of measure zero.

Let $D^n = \text{span}((1,1,...,1_n)/\sqrt{n})$ represent a canonical line in $\mathbb{R}^n$. Let $p(X^n_{\sqrt{3}})$ denote the projection of $X^n_{\sqrt{3}}$ onto D^n. Consider

$$\left\{Y^n = |p(X^n_{\sqrt{3}})|\right\}_{n\in\mathbb{N}}.$$

As with the sphere, ball, and cube, we can conclude the following for the boundary of the cube.

Theorem 6.1 (Projective Central Limit Theorem for Boundaries of Cubes)**.** *With Y^n as just defined, as $n \to \infty$,*

$$Y^n \to^d N(0,1).$$

Experimental evidence for Theorem 6.1 can be obtained through statistical simulation. We can simulate a random sample of observations of Y^n, by randomly* selecting m points on $\partial C^n_{\sqrt{3}}$, projecting them onto D^n, and then recording their magnitudes.

A randomly selected point of $\partial C^n_{\sqrt{3}}$ can be obtained by randomly selecting a face, $F_{i,j}$, and then upon that face randomly selecting a point. Since the point is to be projected via $(x_1, x_2, ..., x_n) \mapsto \frac{\sum_{i=1}^n x_i}{\sqrt{n}}(1,1,...,1)/\sqrt{n}$, we need not consider all faces, but only $F_{1,-}$ and $F_{1,+}$. Thus a single, simulated observation of Y^n can be obtained through the following two-step procedure. First, randomly select x_1 to be either $-\sqrt{3}/2$ or $\sqrt{3}/2$, and then for $i = 2, 3, ..., n$ randomly select x_i within $[-\sqrt{3}/2, \sqrt{3}/2]$. Second, given the resulting $(x_1, x_2, ..., x_n)$ compute $\frac{\sum_{i=1}^n x_i}{\sqrt{n}}$, which is the simulated observation of Y^n.

In order to take advantage of existing computer software, note that within the procedure just described, x_1 can be thought of as a Bernoulli random variable, while each of $\{x_2, x_3, ..., x_n\}$ can be thought of as continuous, uniform random variables. Thus, within R (see [18]), for example, our simulated sample of m independent observations of Y^n is obtainable through the following commands.

```
> A <- matrix(c(2*sqrt(3)*(rbinom(m,1,.5)-.5),
        runif(n*m-m,-sqrt(3),sqrt(3))),nrow=m)
> apply(A,1,sum)/sqrt(n)
```

In order to avoid sampling error, the simulation should be run with m as large as possible. For sufficiently large, fixed m, as n increases, it is possible

*Throughout this section the adverb "randomly" is to be interpreted as specifying that the selection is to be done according to the uniform distribution, continuous or discrete, on the space in question.

to observe the convergence of Y^n to $N(0,1)$. This convergence in the case of the boundary of the cube is slower than in the case of the cube.

Fortunately, our simulation procedure points the way to a theoretical proof for Theorem 6.1. We have seen that $Y^n = (W + U_2 + U_3 + ... + U_n)/\sqrt{n}$, where the random variables, $\{W, \{U_i\}_{i=2}^n\}$, are independent, W takes the values $-\sqrt{3}$ or $\sqrt{3}$ each with probability $1/2$, and each U_i is uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$. This sets the stage for the Lyapunov central limit theorem (see [6], Theorem 24.4).

Theorem 6.2. (Lyapunov Central Limit Theorem) *Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent random variables, each with a mean of zero and a finite third moment. Define $s_n^2 := \sum_{i=1}^n E(X_i^2)$. If Lyapunov's condition,*

$$\lim_{n\to\infty} \frac{1}{s_n^3} \sum_{i=1}^n E(|X_i|^3) = 0,$$

is satisfied, then

$$\frac{\sum_{i=1}^n X_i}{s_n} \to^d N(0,1).$$

In our case, $E(W^2) = 3$ and $E(U_i^2) = 1$, resulting in $s_n^2 = n + 2$. Also, $E(|W|^3) = 3\sqrt{3}$ and $E(|U_i|^3) = 3\sqrt{3}/4$, resulting in $E(|W|^3) + \sum_{i=2}^n E(|U_i|^3) = 3\sqrt{3}(1 + \frac{n-1}{4})$. Combining these results demonstrates that Lyapunov's condition is satisfied:

$$\lim_{n\to\infty} \frac{3\sqrt{3}(1 + \frac{n-1}{4})}{(n+2)^{3/2}} = 0.$$

Lyapunov's central limit theorem thus applies and implies $\frac{\sqrt{n}}{\sqrt{n+2}} Y^n \to^d N(0,1)$, which in turn implies $Y^n \to^d N(0,1)$.

7 Recreation

For $c > 0$ and $n \in \mathbb{N}$ we have defined $C_c^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \forall i \in \{1, 2, ..., n\}, -c/2 \leq x_i \leq c/2\}$. Let $t \mapsto t(1, 1, ..., 1_n)/\sqrt{n}$ parametrize a line in $\mathbb{R}^n$. Define the orthogonal section of C_c^n at $t \neq 0$ to be $H_{c,t}^n = \{(x_1, x_2, ..., x_n) \in C_c^n : \langle (x_1, x_2, ..., x_n) - t(1, 1, ..., 1_n)/\sqrt{n}, t(1, 1, ..., 1_n)/\sqrt{n} \rangle = 0\}$. For $t \neq 0$, define the family of functions $\{g_c^n(t)\}_{c>0, n\in\mathbb{N}}$, via $g_c^n(t) = \mu(H_{c,t}^n)$. Extend each function by defining $g_c^n(0) = \lim_{t\to 0} g_c^n(t)$.

Proposition 7.1. *$\forall c > 0$, and $\forall n \in N : n > 1$, the function $g_c^n : \mathbb{R} \to \mathbb{R}$ has the following properties:*

(i) $g_c^n(-t) = g_c^n(t)$

(ii) $0 < a < b \implies g(a) > g(b)$.

Corollary 7.2. *$\forall c > 0$, and $\forall n \in N : n > 1$,*

$$\operatorname*{argmax}_t g_c^n(t) = 0.$$

These are statements regarding the geometry of high dimensional cubes. The corollary follows immediately from the proposition. A roundabout proof for the proposition is possible using the theory from Section 5.

It can be assumed that $c = 1$ in all dimensions. When $n = 2$ the truth of the proposition is easily verified. Thus, assuming the proposition to be true for $n = N$, by using the principle of mathematical induction, it remains to show that the proposition is true for $n = N + 1$.

With $\{U_i\}_{i=1}^{\infty}$ a sequence of independent random variables, each uniformly distributed on $[-1/2, 1/2]$, observe that $g_1^N(t)$ is the density function for $\frac{\sum_{i=1}^N U_i}{\sqrt{N}}$ (see Lemma 5.2 of Section 5). Furthermore, $g_1^{N+1}(t)$ is the density function for

$$\frac{\sum_{i=1}^{N+1} U_i}{\sqrt{N+1}} = \left(\frac{\sum_{i=1}^{N} U_i}{\sqrt{N}} + \frac{U_1}{\sqrt{N}} \right) \frac{\sqrt{N}}{\sqrt{N+1}} = \frac{\sqrt{N}}{\sqrt{N+1}} \frac{\sum_{i=1}^{N} U_i}{\sqrt{N}} + \frac{U_1}{\sqrt{N+1}}.$$

Because the density for the sum of two independent random variables is the convolution of their densities (see Section 6.4 of [1]), we thus conclude

$$g_1^{N+1}(t) = \frac{N+1}{\sqrt{N}} \int_{t-2/\sqrt{N}}^{t+2/\sqrt{N}} g_1^N \left(\frac{\sqrt{N}}{\sqrt{N+1}} s \right) ds. \tag{5}$$

The following lemma, when applied to (5), completes the argument.

Lemma 7.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function with the following properties:*

(i) $g(-t) = g(t)$

(ii) $0 < a < b \implies g(a) > g(b)$.

Then for any $k > 0$ the function

$$h(t) = \int_{t-k}^{t+k} g(s) ds$$

has the same properties.

Proof. Since g is an even function, $h(-t) = \int_{-t-k}^{-t+k} g(s)ds = -\int_{-t+k}^{-t-k} g(s)ds = -\int_{t-k}^{t+k} g(-s)ds = \int_{-t-k}^{t+k} g(s)ds = h(t)$. Also since g is strictly decreasing on the positive reals, for $0 < a < b$ we have $|a - k| < |b + k|$, and thus $\int_{a-k}^{a+k} g(s)ds - \int_{b-k}^{b+k} g(s)ds = \int_{a-k}^{b-k} g(s)ds - \int_{a+k}^{b+k} g(s)ds > 0$, which implies $h(a) = \int_{a-k}^{a+k} g(s)ds > \int_{b-k}^{b+k} g(s)ds = h(b)$. $\square$

A Appendix: Measures of Balls and Spheres

Lemma A.1. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2} dx &= \left(\left(\int_{-\infty}^{\infty} e^{-x_1^2} dx_1\right)\left(\int_{-\infty}^{\infty} e^{-x_2^2} dx_2\right)\right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x_1^2} dx_1\right) e^{-x_2^2} dx_2\right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{-x_1^2} e^{-x_2^2}\right) dx_1 dx_2\right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{-(x_1^2+x_2^2}\right) dx_1 dx_2\right)^{\frac{1}{2}} \\
&= \left(\int_{0}^{\infty} e^{-r^2} 2\pi r dr\right)^{\frac{1}{2}} \\
&= \left(\int_{0}^{\infty} e^{-u} \pi du\right)^{\frac{1}{2}} = \left((-e^{-u})|_0^\infty \pi\right)^{\frac{1}{2}} = ((0-(-1))\pi)^{\frac{1}{2}} = \pi^{\frac{1}{2}} = \sqrt{\pi}.
\end{aligned}$$

□

Proposition A.2. $\mu(S_1^n) = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$

Proof.

$$\begin{aligned}
\pi^{n/2} = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^n = \int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} &= \int_0^\infty \mu(S_1^n) r^{n-1} e^{-r^2} dr \\
&= \frac{\mu(S_1^n)}{2}\int_0^\infty u^{\frac{n}{2}-1} e^{-u} du \\
&= \frac{\mu(S_1^n)}{2}\Gamma(n/2).
\end{aligned}$$

□

Remark A.3. $\mu(S_r^n) = \mu(S_1^n) r^{n-1}$.

Proposition A.4. $\mu(B_1^n) = \mu(S_1^n)/n$.

Proof. $\mu(B_1^n) = \int_0^1 \mu(S_1^n) r^{n-1} dr = \mu(S_1^n)\frac{r^n}{n}|_0^1 = \mu(S_1^n)/n$. □

Remark A.5. $\mu(B_r^n) = \mu(B_1^n) r^n$.

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