Some Third-Order Modifications of Newton's Method [*]

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Abstract

In this paper a new family of methods, free from second derivative, is presented. This new family of methods is constructed such that convergence is of order three and requires two require two evaluations of the function and first derivative per iteration. To illustrate the efficiency and performance of the new family of methods, several numerical examples are presented. Further numerical comparisons are made with several other existing third-order methods to show the abilities of the presented family of methods.

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1   Introduction

In this paper, we consider iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$. The design of iterative formulae for solving these equations are very important and interesting tasks in applied mathematics and other disciplines. In recent years, several variants of the methods with free second-derivative have been proposed and analyzed (see [1-8] and the reference therein). These new methods can be considered as alternatives for Newton's method which is a well-known iterative method for finding α by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

That converge quadratic ally in some neighborhood of α.

2   Derivation of Method and Convergence Analysis

To illustrate the idea of the present paper, let us consider Kou et al. third-order method [1] defined by

$$x_{n+1} = x_n - \frac{f(y_n) - f(x_n)}{f'(x_n)}, \tag{1}$$

Where

$$y_n = x_n + \frac{f(x_n)}{f'(x_n)} \tag{2}$$

To develop a new family, let us introduce the following iteration scheme:

$$x_{n+1} = x_n - \frac{Af(x_n) + Bf(y_n)}{Cf(x_n) + Df(y_n)} \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

Where

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, m \neq 1 \tag{4}$$

(this notation will be used throughout), where A, B, C and D are disposal parameters to be determined such that iterative method defined by (3) and (4) have a three order of convergence.

In the following, sufficient conditions of these parameters have been presented:

Theorem 2.1 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}\ \mathbb{R}\ \mathbb{R}\ \Re$ for an open interval, which contains x_0 as initial approximation of. If A, B, C and D, satisfy the conditions:*

$$A = \frac{C(m^2 + m - 1) - D(m - 1)^2}{m^2}, B = \frac{C + D(m^2 - m + 1)}{m^2},$$

Where $C^2 + D^2 \neq 0$; then the family of methods defined by (3) and (4) is of third-order.

Proof. If $\alpha \in I$ be the root and e_n be the error at n th iteration, than $e_n = x_n - \alpha$ using Taylors expansion, we have

$$f(x_n) = f'(\alpha)[e_n + C_2 e_n^2 + C_3 e_n^3 + C_4 e_n^4 + O(e_n^5)], \tag{5}$$

$$f'(x_n) = f'(\alpha)[1 + 2C_2 e_n + 3C_3 e_n^2 + 4C_4 e_n^3 + O(e_n^4)], \tag{6}$$

Where

$$c_k = f^{(k)}(\alpha) / k\ !f'(\alpha), k = 2, 3, ...$$

Using (5), (6) and (2), we have

$$y_n = \alpha + (1 - m)e_n + mC_2 e_n^2 + 2m(C_3 - 2C_1^2)e_n^3 + O(e_n^4). \tag{7}$$

Now again by Taylor s series, we have

$$f(y_n) = f'(\alpha)[(1 - m)e_n + (m^2 - m + 1)C_2 e_n^2 \tag{8}$$
$$- (2m^2 C_2^2 + (m^3 - 3m^2 + m - 1)C_3)e_n^3] + O(e_n^4).$$

Finally, using (5)-(8) and (3), we get

$$x_{n+1} = x_n + K_1 e_n + K_2 e_n + O(e_n^3). \tag{9}$$

Where

$$K_1 = -\frac{A + B(1 - m)}{C + D(1 - m)},$$

And

$$K_2 = \frac{AD(m^2 - m + 1) + BD(1 - m)^2 + BC(-m^2 - m + 1) + AC}{(C + D(1 - m))^2},$$

An easy manipulation shows that $K_1 = -1$ and $K_2 = 0$ when

$$A = \frac{C(m^2 + m - 1) - D(m - 1)^2}{m^2}, B = \frac{C + D(m^2 - m + 1)}{m^2},$$

Thus,

$$e_{n+1} = O(e_n^3).$$

which complete the proof.

By some simplifications in (3) for those parameters that satisfy conditions of the theorem (9) and by introducing $\beta = D / C$, following third-order family of methods will be obtained:

$$x_{n+1} = x_n - \frac{f(x_n)\left[(m^2 + m - 1) - \beta(m - 1)^2\right] + f(y_n)\left[1 + \beta(m^2 - m + 1)\right]}{m^2\left[f(x_n) + \beta f(y_n)\right]} \frac{f(x_n)}{f'(x_n)}, \quad (10)$$

Where

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, m \neq 1$$

Similar to procedure of Theorem 2.1, it can be seen that family (10) has the following error equation:

$$e_{n+1} = [C_2^2(2\beta + 2 + \beta m^2 - 2^2 m) - (m\beta - \beta - 1)(m - 1)C_3]e_n^3 + O(e_n^4).$$

Formula (10) includes, as particular cases, the following ones:

For $m = -1$ and $\beta = 0$, we obtain a new third-order method which were obtained by Kou (1),(2).

For $m = -1$ and $\beta = -1$, we obtain a third-order method, called Newton Steffensen method [2]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_n + f(x_n)/f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

For $m = -1$, $\beta = -1/3$ and $m = -1/2$, $\beta = -4/7$, we obtain the following third-order methods:

$$x_{n+1} = x_n - \frac{f(x_n)}{3f(x_n) - f(x_n + f(x_n)/f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

And

$$x_{n+1} = x_n - \frac{f(x_n)}{7f(x_n) - 4f(x_n + f(x_n)/2f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

Respectively, which were obtained by Kou [3].

For $m = -1$, we obtain a new third-order family of methods:

$$x_{n+1} = x_n - \frac{(-1 - 4\beta)f(x_n) + (1 + 3\beta)f(x_n + f(x_n)/f'(x_n))}{f(x_n) + \beta f(x_n + f(x_n)/f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

For $m = 1/2$, we obtain a new third-order family of methods:

$$x_{n+1} = x_n - \frac{-(1+\beta)f(x_n)+(4+3\beta)f(x_n-f(x_n)/2f'(x_n))}{f(x_n)+\beta f(x_n-f(x_n)/2f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

For $m = -1$, $\beta = -1/4$, we obtain another new third-order method

$$x_{n+1} = x_n - \frac{f(x_n+f(x_n)/f'(x_n))}{4f(x_n)-f(x_n+f(x_n)/f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

For $m = 1/2$, $\beta = -1$, we obtain a new third-order method:

$$x_{n+1} = x_n + \frac{f(x_n-f(x_n)/2f'(x_n))}{f(x_n)-f(x_n-f(x_n)/2f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

Per iteration of these methods require two evaluations of the function and one of its derivatives. If we consider the definition of efficiency index in [4] as $p^{1/d}$, where p is the order of the method and d is the number of functional evaluations, the iteration formula defined by (10) has the efficiency index equal to $3^{1/3} \approx 1.4422$, which is better than that of Newton's method $2^{1/2} \approx 1.4142$.

3   Numerical Examples

In this section, some examples are presented to illustrate the efficiency of the iterative family (10). Table 1 shows the results of comparison between the new methods (12) (NM1) and (13) (NM2), which introduced in this contribution, the Newton method (NeM) and the following well-known third-order methods:

Weerakoon and Fernando [5] (WF):

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)-f'(x_n-f(x_n)/f'(x_n))},$$

the method derived from midpoint rule in [6] (MP):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n-f(x_n)/2f'(x_n))},$$

Homeier method [7] (HM):

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n-f(x_n)/f'(x_n))}\right),$$

In the present contribution, we used the following test functions as in [9]

$$f_1(x) = x^3 + 4x^2 - 10$$
$$f_2(x) = \sin^2(x) - x^2 + 1$$
$$f_3(x) = x^2 - e^x - 3x + 2$$
$$f_4(x) = \cos(x) - x$$
$$f_5(x) = (x-1)^3 - 1.$$

All computations were performed using MAPLE10 and 64 digit floating point arithmetic (Digits: = 64).

In Table 1, the results were obtained by using following stopping criteria:

$$|x_{n+1} - x_n| < 10^{-15}, \quad |f(x_n)| < 10^{-15}.$$

Therefore, as soon as the stopping criteria are satisfied, x_{n+1} is taken as the approximation solution of $f(x) = 0$. In Table 1, (IT) stands for the number of iterations the number of functional evaluations; (NFE) stands for counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative.

Table 1: Comparison of various third-order convergent iterative methods

$f(x_{IT})$	IT	NFE	x_{IT}	$f(x_{IT})$
$f_1, x_0 = 1.27$				
NeM	5	10	1.365230013414096	2.7e-41
WF	4	12	1.365230013414096	0.0e-1.0
MP	4	12	1.365230013414096	3.0e-35
HM	3	9	1.365230013414096	2.0e-16
NM1	3	9	1.365230013414096	1.7e-33
NM2	3	9	1.365230013414096	2.4e-26
$f_2, x_0 = 2.0$				
NeM	6	12	1.404491648215341	1.0e-16
WF	5	15	1.404491648215341	6.0e-42
MP	5	15	1.404491648215341	7.1e-41
HM	4	12	1.404491648215341	1.0e-24
NM1	4	12	1.404491648215341	3.4e-26
NM2	4	12	1.404491648215341	4.2e-36
$f_3, x_0 = 2.5$				
NeM	6	12	0.257530285439860	9.1e-28
WF	5	15	0.257530285439860	1.6e-34
MP	4	12	0.257530285439860	3.9e-24
HM	5	15	0.257530285439860	9.3e-43
NM1	4	12	0.257530285439860	3.3e-39
NM2	3	9	0.257530285439860	2.2e-19

$f_4, x_0 = 1.7$				
NeM	5	10	0.739085133215160	1.8e-16
WF	4	12	0.739085133215160	5.3e-28
MP	4	12	0.739085133215160	6.7e-24
HM	4	12	0.739085133215160	1.1e-24
NM1	4	12	0.739085133215160	2.3e-17
NM2	3	9	0.739085133215160	3.9e-47
$f_5, x_0 = 1.27$				
NeM	7	14	2.0	4.2e-25
WF	5	15	2.0	2.7e-29
MP	5	15	2.0	1.4e-32
HM	4	12	2.0	1.7e-16
NM1	4	12	2.0	3.5e-22
NM2	4	12	2.0	1.6e-33

4   Conclusion

In this work, we have constructed an iterative family of methods for solving nonlinear equations. It has been shown by illustration that the proposed iterative family of order three can be effectively used for solving such equations. One of the members of this obtained family of the methods was also compared, in their performance, with various other iteration methods of the same order, and it was observed that they demonstrate at least equal behavior.

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