Gen. Math. Notes, Vol. 1, No. 2, December 2010, pp. 96-107
ISSN 2219-7184; Copyright ©ICSRS Publication, 2010
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A Unique Common Fixed Point Theorem for Six Maps in D^*- Cone Metric Spaces

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(Received 8.10.2010, Accepted 29.10.2010)

Abstract

In this paper we obtain a unique common fixed point theorem for three pairs of weakly compatible mappings in D^*-cone metric spaces.

Keywords: D^*-cone metric, common fixed points,weakly compatible mappings.

2000 MSC No.: 47H10; 54H25

1 Introduction and Preliminaries

Huang and Zhang [8] generalized the notion of metric spaces, replacing the real numbers by an ordered Bannach space and defined cone metric spaces. Dhage [1,2,3,4] et al. introduced the concept of D-metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims [14] and Naidu et al. [11,12,13] demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect.Alternatively, Mustafa and Sims [15] introduced more appropriate notion of generalized metric space which called a G-metric space, and obtained some topological properties. Later in 2007 Shaban Sedghi et.al [10] modified the D-metric space and defined D^*-metric spaces and then C.T.Aage and

J.N.Salunke [5] generalized the D^*-metric spaces by replacing the real numbers by an ordered Banach space and defined D^*-cone metric spaces and prove the topological properties. In this paper, we obtain a unique common fixed point theorem for three pairs of weakly compatible mappings in D^*-metric spaces. First, we present some known definitions and propositions in D^*- cone metric spaces.

Let E be a real Banach space and P a subset of E. P is called cone if and only if :
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a,b,
(iii) $P \cap (-P) = \{0\}$.
Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in P^0$, where P^0 denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K \|y\|$.

Definition 1.1 ([5]).*Let X be a nonempty set and let $D^* : X \times X \times X \to E$ be a function satisfying the following properties :*
(1)*: $D^*(x, y, z) \geq 0$,*
(2)*: $D^*(x, y, z) = 0$ if and only if $x = y = z$,*
(3)*: $D^*(x, y, z) = D^*(x, z, y) = D^*(y, z, x) = ...$,symmetry in three variables,*
(4)*: $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.*
Then the function D^* is called a D^*-cone metric and the pair (X, D^*) is called a D^*- cone metric space.

Example 1.2 ([5]). *Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $D^* : X \times X \times X \to E$ defined by $D^*(x, y, z) = (|x - y| + |y - z| + |x - z| , \alpha(|x - y| + |y - z| + |x - z|)$,where $\alpha \geq 0$ is a constant. Then (X, D^*) is a D^* - cone metric space.*

Remark 1.3 ([5]). *If (X, D^*) is a D^* - cone metric space, then for all $x, y, z \in X$,we have $D^*(x, x, y) = D^*(x, y, y)$.*

Proposition 1.4 ([7]).*Let P be a cone in a real Banach space E . If $a \in P$ and $a \leq \lambda \ a$ for some $\lambda \in [0, 1)$ then $a = 0$.*

Proposition 1.5 ($Cor.1.4, [9]$).*Let P be a cone in a real Banach space E .*
(i) If $a \leq b$ and $b << c$, then $a << c$
(ii) If $a \in E$ and $a << c$ for all $c \in P^o$, then $a = 0$.

Remark 1.6 ([9]). *$\lambda \ P^o \subseteq P^o$ for $\lambda > 0$ and $P^o + P^o \subseteq P^o$.*

Remark 1.7 ([7]). *If $c \in P^o, 0 \leq a_n$ and $a_n \longrightarrow 0$, then there exists $n_0 \in \mathcal{N}$ such that for all $n > n_0$ we have $a_n << c$.*

Definition 1.8 ([5]).*Let (X, D^*) be a D^*-cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $m, n > N$,$D^*(x_m, x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\{x_n\} \to x$.*

Lemma 1.9 ([5]). *Let (X, D^*) be a D^*-cone metric space then the following are
equivalent.*
(i): $\{x_n\}$ is D^*- convergent to x.
(ii): $D^*(x_n, x_n, x) \to 0$, as $n \to \infty$.
(iii): $D^*(x_n, x, x) \to 0$, as $n \to \infty$.
(iv): $D^*(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Lemma 1.10 *Let $\{x_n\}$ be a sequence in D^*-cone metric space (X, D^*). If $\{x_n\}$ converges to x and y, then $x = y$.*

Proof. For any $c \in E$ with $0 << c$, there is N such that for all $n > N$,
$D^*(x_n, x_n, x) << \frac{c}{2}$ and $D^*(x_n, x_n, y) << \frac{c}{2}$. Now,
$D^*(x, x, y) \leq D^*(x, x, x_n) + D^*(x_n, y, y) << \frac{c}{2} + \frac{c}{2} = c$.
Thus $D^*(x, x, y) << \frac{c}{k}$ for all $k \geq 1$. Hence $\frac{c}{k} - D^*(x, x, y) \in P$ for all $k \geq 1$.
Since P is closed and $\frac{c}{k} \to 0$ as $k \to \infty$, we have $-D^*(x, x, y) \in P$.
But $D^*(x, x, y) \in P$. Hence $D^*(x, x, y) = 0$. Thus $x = y$.

Lemma 1.11 *Let (X, D^*) be a D^*-cone metric space and $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be three sequences in X and $\{x_m\} \to x$, $\{y_n\} \to y$,$\{z_l\} \to z$ then $D^*(x_m, y_n, z_l) \to D^*(x, y, z)$ as $m, n, l \to \infty$.*

Proof. For any $c \in E$ with $0 << c$, there is N such that $D^*(x_m, x, x) << \frac{c}{3}$, $D^*(y_n, y, y) << \frac{c}{3}$ and $D^*(z_m, z, z) << \frac{c}{3}$ for all $m, n, l > N$.
Now for all $m, n, l > N$, we have

$$\begin{aligned}
&D^*(x_m, y_n, z_l) \\
&\leq D^*(x_m, y_n, z) + D^*(z, z_l, z_l) \\
&\leq D^*(x_m, z, y) + D^*(y, y_n, y_n) + D^*(z, z_l, z_l) \\
&\leq D^*(z, y, x) + D^*(x, x_m, x_m) + D^*(y, y_n, y_n) + D^*(z, z_l, z_l)
\end{aligned}$$

Thus

$$\begin{aligned}
D^*(x_m, y_n, z_l) - D^*(x, y, z) &\leq D^*(x, x_m, x_m) + D^*(y, y_n, y_n) + D^*(z, z_l, z_l) \\
&<< \tfrac{c}{3} + \tfrac{c}{3} + \tfrac{c}{3} = c.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&D^*(x,y,z)\\
&\le D^*(x,y,z_l)+D^*(z_l,z,z)\\
&\le D^*(x,z_l,y_n)+D^*(y_n,y,y)+D^*(z_l,z,z)\\
&\le D^*(x_m,,z_l,y_n)+D^*(x_m,x,x)+D^*(y_n,y,y)+D^*(z_l,z,z)
\end{aligned}$$

Thus

$$D^*(x,y,z)-D^*(x_m,y_n,z_l)\le D^*(x_m,x,x)+D^*(y_n,y,y)+D^*(z_l,z,z) \\ << \tfrac{c}{3}+\tfrac{c}{3}+\tfrac{c}{3}=c.$$

Thus for all $k\ge 1$, we have
$D^*(x_m,y_n,z_l)-D^*(x,y,z) << \frac{c}{k}$ and $D^*(x,y,z)-D^*(x_m,y_n,z_l) << \frac{c}{k}$.
Hence $\frac{c}{k}-[D^*(x_m,y_n,z_l)-D^*(x,y,z)]\in P$ and
$\frac{c}{k}-[D^*(x,y,z)-D^*(x_m,y_n,z_l)]\in P$.
Since P is closed and $\frac{c}{k}\to 0$ as $k\to\infty$, we have

$$\lim_{m,n,l\to\infty} D^*(x_m,y_n,z_l)-D^*(x,y,z)\in P, D^*(x,y,z)-\lim_{m,n,l\to\infty} D^*(x_m,y_n,z_l)\in P.$$

Hence $\lim_{m,n,l\to\infty} D^*(x_m,y_n,z_l)=D^*(x,y,z)$.

Aage and Salunke[5] proved the above Lemmas 1.10,1.11, when P is a normal cone (See,Lemma 1.7,Lemma 1.13[5]).

Definition 1.12 ([5]). *Let (X,D^*) be a D^*-cone metric space, $\{x_n\}$ be a sequence in X. if for any $c\in E$ with $0\ll c$, there is N such that for all $m,n,l>N$, $D^*(x_m,x_n,x_l)\ll c$, then $\{x_n\}$ is called Cauchy sequence in X.*

Definition 1.13 ([5]).*Let (X,D^*) be a D^*-cone metric space. if every Cauchy sequence in X is convergent in X , then X is called a complete D^*-cone metric space.*

Now, we give our main Lemma.

Lemma 1.14 *Let X be a D^*-cone metric space ,P be a cone in a real Banach space E and $k_1,k_2,k_3,k_4\ge 0$ and $k>0$. If $x_n\to x, y_n\to y, z_n\to z$ and $p_n\to p$ in X and $a\in P$ and*

(1.14.1) $ka\le k_1D^*(x_n,x_m,x)+k_2D^*(y_n,y_m,y)+k_3D^*(z_n,z_m,z)+k_4D^*(p_n,p_m,p)$

then $a=0$.

Proof. If $k_1=k_2=k_3=k_4=0$,then $ka\le 0$ implies $-ka\in P$.But $ka\in P$. Hence $ka=0$,which implies that $a=0$.
Now assume that atleast one of k_1,k_2,k_3,k_4 is not equal to zero. Since $x_n\to$

$x, y_n \to y, z_n \to z$ and $p_n \to p$, we have for $c \in P^o$, there exists a positive integer N_c such that

$\frac{c}{k_1+k_2+k_3+k_4} - D^*(x_n, x_m, x), \frac{c}{k_1+k_2+k_3+k_4} - D^*(y_n, y_m, y),$

$\frac{c}{k_1+k_2+k_3+k_4} - D^*(z_n, z_m, z), \frac{c}{k_1+k_2+k_3+k_4} - D^*(p_n, p_m, p) \in P^o \ \ \forall \ \ n > N_c.$

From Remark 1.6, we have $\forall \ \ n > N_c$,

$\frac{k_1 c}{k_1+k_2+k_3+k_4} - k_1 D^*(x_n, x_m, x), \frac{k_2 c}{k_1+k_2+k_3+k_4} - k_2 D^*(y_n, y_m, y),$

$\frac{k_3 c}{k_1+k_2+k_3+k_4} - k_3 D^*(z_n, z_m, z), \frac{k_4 c}{k_1+k_2+k_3+k_4} - k_4 D^*(p_n, p_m, p) \in P^o$.

Adding these four and by Remark 1.6, we have $\forall \ \ n > N_c$,

$$c - [k_1 D^*(x_n, x_m, x) + k_2 D^*(y_n, y_m, y) + k_3 D^*(z_n, z_m, z) + k_4 D^*(p_n, p_m, p)] \in P^o.$$

Now from(1.14.1) and Proposition 1.5(i), we have $ka << c \ \ \forall c \in P^o$.
By Proposition 1.5(ii),we have $a = 0$ as $k > 0$.

Definition 1.15 ([6]).*A pair of self mappings is called weakly compatible if they commute at their coincidence points, that is, $Ax = Sx$ implies $ASx = SAx$.*

2 The Main Result

Theorem 2.1 *Let (X, D^*) be a D^*-cone metric space, P be a cone and $S, T, R, f, g, h : X \to X$ be mappings satisfying*
(2.1.1) *$S(X) \subseteq g(X), T(X) \subseteq h(X)$ and $R(X) \subseteq f(X)$,*
(2.1.2) *one of $f(X), g(X)$ and $h(X)$ is a complete subspace of X,*
(2.1.3) *the pairs $(S, f), (T, g)$ and (R, h) are weakly compatible, and*

$$(2.1.4) \ D^*(Sx, Ty, Rz) \leq q \max \left\{ \begin{array}{c} D^*(fx, gy, hz), D^*(fx, Sx, Ty), D^*(gy, Ty, Rz), D^*(hz, Rz, Sx), \\ D^*(fx, Sx, Sx), D^*(gy, Ty, Ty), D^*(hz, Rz, Rz). \end{array} \right\}$$

for all $x, y, z \in X$, where $0 \leq q < 1$.
(2.1.5) *$D^*(x, x, y) \leq D^*(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.*
Then either one of the pairs $(S, f), (T, g)$ and (R, h) has a coincidence point or the maps S, T, R, f, g and h have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an orbitrary point,then from (2.1.1), there exist x_1, x_2, $x_3 \in X$ such that $Sx_0 = gx_1 = y_0, say$, $Tx_1 = hx_2 = y_1, say$ and $Rx_2 = fx_3 = y_2, say$.
Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{3n} = Sx_{3n} = gx_{3n+1}$, $y_{3n+1} = Tx_{3n+1} = hx_{3n+2}$ and $y_{3n+2} = Rx_{3n+2} = fx_{3n+3}$, where $n = 0, 1, 2, ...$

If $y_{3n} = y_{3n+1}$ then x_{3n+1} is a coincidence point of g and T.
If $y_{3n+1} = y_{3n+2}$ then x_{3n+2} is a coincidence point of h and R.
If $y_{3n+2} = y_{3n+3}$ then x_{3n+3} is a coincidence point of f and S.
Now assume that $y_n \neq y_{n+1}$ for all n.
Denote $d_n = D^*(y_n, y_{n+1}, y_{n+2})$.
$d_{3n} = D^*(y_{3n}, y_{3n+1}, y_{3n+2}) =$
$D^*(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2})$

$$\leq q \max \left\{ \begin{array}{c} D^*(y_{3n-1}, y_{3n}, y_{3n+1}), D^*(y_{3n-1}, y_{3n}, y_{3n+1}), \\ D^*(y_{3n}, y_{3n+1}, y_{3n+2}), D^*(y_{3n+1}, y_{3n+2}, y_{3n}), \\ D^*(y_{3n-1}, y_{3n}, y_{3n}), D^*(y_{3n}, y_{3n+1}, y_{3n+1}), \\ D^*(y_{3n+1}, y_{3n+2}, y_{3n+2}). \end{array} \right\}$$

From Remark 1.3 and (2.1.5) we have

$$\leq q \max \left\{ \begin{array}{c} d_{3n-1}, d_{3n-1}, d_{3n}, d_{3n}, \\ d_{3n-1}, d_{3n-1}, d_{3n} \end{array} \right\} \text{..........} \ (1)$$

If $d_{3n} > d_{3n-1}$ then from (1), we have $d_{3n} \leq q\ d_{3n} < d_{3n}$. It is a contradiction.
Hence $d_{3n} \leq d_{3n-1}$. Now from (1), $d_{3n} \leq q\ d_{3n-1}$.........(2)
Similarly, by putting $x = x_{3n+3}, y = x_{3n+1}, z = x_{3n+2}$ and $x = x_{3n+3}$, $y = x_{3n+4}, z = x_{3n+2}$ in (2.1.4), we get $d_{3n+1} \leq q\ d_{3n}$.......(3)
and $d_{3n+2} \leq q\ d_{3n+1}$.......(4) respectively.
Thus from (2),(3) and (4), we have

$$\begin{aligned} D^*(y_n, y_{n+1}, y_{n+2}) &\leq q\ D^*(y_{n-1}, y_n, y_{n+1}) \\ &\leq q^2\ D^*(y_{n-2}, y_{n-1}, y_n) \\ &. \\ &. \\ &. \\ &\leq q^n\ D^*(y_0, y_1, y_2) \qquad ....(5) \end{aligned}$$

From (2.1.5) and (5), we have
$D^*(y_n, y_n, y_{n+1}) \leq D^*(y_n, y_{n+1}, y_{n+2}) \leq q^n\ D^*(y_0, y_1, y_2)$.
Now for $m > n$

$$\begin{aligned} D^*(y_n, y_n, y_m) &\leq D^*(y_n, y_n, y_{n+1}) + D^*(y_{n+1}, y_{n+1}, y_{n+2}) + ... + D^*(y_{m-1}, y_{m-1}, y_m) \\ &\leq q^n\ D^*(y_0, y_1, y_2) + q^{n+1}\ D^*(y_0, y_1, y_2) + ... + q^{m-1}\ D^*(y_0, y_1, y_2) \\ &\leq \tfrac{q^n}{1-q}\ D^*(y_0, y_1, y_2) \to 0 \ as\ n \to \infty. \end{aligned}$$

From Remark 1.7, it follows that for $0 << c$ and large n, we have $\frac{q^n}{1-q}\ D^*(y_0, y_1, y_2) << c$. Now, from Proposition 1.5(i), we have $D^*(y_n, y_n, y_m) << c$ for $m > n$. Hence $\{y_n\}$ is a D^*-Cauchy sequence.
Suppose $f(X)$ is D^*-complete.
Then there exist $p, t \in X$ such that $y_{3n+2} \to p = ft$. Since $\{y_n\}$ is D^*-Cauchy,

it follows that $y_{3n} \to p$ and $y_{3n+1} \to p$ as $n \to \infty$.

$$\begin{aligned}
D^*(St,p,p) &\leq D^*(p,Tx_{3n+1},Tx_{3n+1}) + D^*(Tx_{3n+1},St,p) \\
&\leq D^*(p,Tx_{3n+1},Tx_{3n+1}) + D^*(p,Rx_{3n+2},Rx_{3n+2}) \\
&\quad + D^*(St,Tx_{3n+1},Rx_{3n+2}) \\
&\leq D^*(p,y_{3n+1},y_{3n+1}) + D^*(p,y_{3n+2},y_{3n+2}) \\
&\qquad + q \max \left\{ \begin{array}{c} D^*(p,y_{3n},y_{3n+1}), D^*(p,St,y_{3n+1}), \\ D^*(y_{3n},y_{3n+1},y_{3n+2}), D^*(y_{3n+1},y_{3n+2},St), \\ D^*(p,St,St), D^*(y_{3n},y_{3n+1},y_{3n+1}), \\ D^*(y_{3n+1},y_{3n+2},y_{3n+2}). \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq D^*(p,y_{3n+1},y_{3n+1}) + D^*(p,y_{3n+2},y_{3n+2}) \\
&\qquad + q \max \left\{ \begin{array}{c} D^*(p,y_{3n},y_{3n+1}), D^*(St,p,p) + D^*(p,p,y_{3n+1}), \\ D^*(y_{3n},p,p) + D^*(p,y_{3n+1},y_{3n+2}), \\ D^*(St,p,p) + D^*(p,y_{3n+1},y_{3n+2}), D^*(St,p,p), \\ D^*(y_{3n},p,p) + D^*(p,y_{3n+1},y_{3n+1}), \\ D^*(y_{3n+1},p,p) + D^*(p,y_{3n+2},y_{3n+2}). \end{array} \right\}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&D^*(St,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(p,y_{3n},y_{3n+1}) \quad or \\
&(1-q)D^*(St,p,p) \leq (1+q)D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) \quad or \\
&D^*(St,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(y_{3n},p,p) \\
&\qquad\qquad + qD^*(p,y_{3n+1},y_{3n+2}) \quad or \\
&(1-q)D^*(St,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(p,y_{3n+1},y_{3n+2}) \quad or \\
&(1-q)D^*(St,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) \quad or \\
&D^*(St,p,p) \leq (1+q)D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(y_{3n},p,p) \quad or \\
&D^*(St,p,p) \leq (1+q)D^*(p,p,y_{3n+1}) + (1+q)D^*(p,p,y_{3n+2})
\end{aligned}$$

From Lemma 1.9 and Lemma 1.14 , we have $St = p$. Thus $ft = p = St$. Since the pair (S, f) is weakly compatible, we have $fp = Sp$.

$$\begin{aligned}
Now, D^*(Sp,p,p) &\leq D^*(p,Tx_{3n+1},Tx_{3n+1}) + D^*(Tx_{3n+1},Sp,p) \\
&\leq D^*(p,Tx_{3n+1},Tx_{3n+1}) + D^*(p,Rx_{3n+2},Rx_{3n+2}) + D^*(Sp,Tx_{3n+1},Rx_{3n+2}) \\
&\leq D^*(p,y_{3n+1},y_{3n+1}) + D^*(p,y_{3n+2},y_{3n+2}) \\
&\qquad + q \max \left\{ \begin{array}{c} D^*(fp,y_{3n},y_{3n+1}), D^*(fp,Sp,y_{3n+1}), \\ D^*(y_{3n},y_{3n+1},y_{3n+2}), D^*(y_{3n+1},y_{3n+2},Sp), \\ D^*(fp,Sp,Sp), D^*(y_{3n},y_{3n+1},y_{3n+1}), \\ D^*(y_{3n+1},y_{3n+2},y_{3n+2}). \end{array} \right\}
\end{aligned}$$

$$D^*(Sp,p,p) \leq D^*(p,y_{3n+1},y_{3n+1}) + D^*(p,y_{3n+2},y_{3n+2}) + q \max \left\{ \begin{array}{c} D^*(Sp,p,p)+D^*(p,y_{3n},y_{3n+1}), \\ D^*(Sp,p,p)+D^*(p,y_{3n+1},y_{3n+1}), \\ D^*(y_{3n},p,p)+D^*(p,y_{3n+1},y_{3n+2}), \\ D^*(Sp,p,p)+D^*(p,y_{3n+1},y_{3n+2},),0, \\ D^*(y_{3n},p,p)+D^*(p,y_{3n+1},y_{3n+1}), \\ D^*(y_{3n+1},p,p)+D^*(p,y_{3n+2},y_{3n+2}). \end{array} \right\}$$

Now we have

$$\begin{array}{l} (1-q)D^*(Sp,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(p,y_{3n},y_{3n+1}) \ \ \textit{or} \\ (1-q)D^*(Sp,p,p) \leq (1+q)D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) \ \ \textit{or} \\ D^*(Sp,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(y_{3n},p,p) \\ \qquad\qquad\qquad\qquad\qquad\qquad +qD^*(p,y_{3n+1},y_{3n+2}) \ \ \textit{or} \\ (1-q)D^*(Sp,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(p,y_{3n+1},y_{3n+2}) \ \ \textit{or} \\ D^*(Sp,p,p) \leq D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) \ \ \textit{or} \\ D^*(Sp,p,p) \leq (1+q)D^*(p,p,y_{3n+1}) + D^*(p,p,y_{3n+2}) + qD^*(y_{3n},p,p) \ \ \textit{or} \\ D^*(Sp,p,p) \leq (1+q)D^*(p,p,y_{3n+1}) + (1+q)D^*(p,p,y_{3n+2}) \end{array}$$

From Lemma 1.9 and Lemma 1.14 , we have $Sp = p$.
Hence $fp = Sp = p$. ....(6).
Since $p = Sp \in g(X)$, there exists $u \in X$ such that $p = gu$.
Now, $D^*(p,Tu,p) = D^*(Sp,Tu,p)$

$$\begin{aligned} &\leq D^*(p,Rx_{3n+2},Rx_{3n+2}) + D^*(Sp,Tu,Rx_{3n+2}). \\ &\leq D^*(p,y_{3n+2},y_{3n+2}) + \ q \max \left\{ \begin{array}{c} D^*(fp,gu,y_{3n+1}), D^*(fp,Sp,Tu), \\ D^*(gu,Tu,y_{3n+2}), D^*(y_{3n+1},y_{3n+2},Sp), \\ D^*(fp,Sp,Sp), D^*(gu,Tu,Tu), \\ D^*(y_{3n+1},y_{3n+2},y_{3n+2}). \end{array} \right\} \\ &\leq D^*(p,y_{3n+2},y_{3n+2}) + \ q\max \left\{ \begin{array}{c} D^*(p,p,y_{3n+1}), D^*(p,p,Tu), \\ D^*(Tu,p,p)+D^*(p,p,y_{3n+2}), \\ D^*(p,y_{3n+1},y_{3n+2}),0,D^*(p,p,Tu), \\ D^*(y_{3n+1},p,p)+D^*(p,y_{3n+2},y_{3n+2}). \end{array} \right\} \end{aligned}$$

Now we have
$D^*(p,Tu,p) \leq D^*(p,y_{3n+2},y_{3n+2}) + qD^*(p,p,y_{3n+1})$ or
$(1-q)D^*(p,Tu,p) \leq D^*(p,y_{3n+2},y_{3n+2})$ or
$(1-q)D^*(p,Tu,p) \leq D^*(p,y_{3n+2},y_{3n+2}) + qD^*(p,p,y_{3n+2})$ or
$D^*(p,Tu,p) \leq D^*(p,y_{3n+2},y_{3n+2}) + qD^*(p,y_{3n+1},y_{3n+2})$ or
$D^*(p,Tu,p) \leq D^*(p,y_{3n+2},y_{3n+2})$ or
$(1-q)D^*(p,Tu,p) \leq D^*(p,y_{3n+2},y_{3n+2})$ or
$D^*(p,Tu,p) \leq D^*(p,p,y_{3n+1}) + (1+q)D^*(p,y_{3n+2},y_{3n+2})$.

From Lemma 1.9 and Lemma 1.14 , we have $Tu = p$. Hence $p = Tu = gu$.
Since the pair (T, g) is weakly compatible, we have $Tp = gp$.
Now, $D^*(p, Tp, p) = D^*(Sp, Tp, p)$

$$\leq D^*(p, Rx_{3n+2}, Rx_{3n+2}) + D^*(Sp, Tp, Rx_{3n+2}).$$

$$\leq D^*(p, y_{3n+2}, y_{3n+2}) + q \max \left\{ \begin{array}{c} D^*(fp, gp, hx_{3n+2}), D^*(fp, Sp, Tp), \\ D^*(gp, Tp, Rx_{3n+2}), D^*(hx_{3n+2}, Rx_{3n+2}, Sp), \\ D^*(fp, Sp, Sp), D^*(gp, Tp, Tp), \\ D^*(hx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}). \end{array} \right\}$$

$$= D^*(p, y_{3n+2}, y_{3n+2}) + q \max \left\{ \begin{array}{c} D^*(p, Tp, y_{3n+1}), D^*(p, p, Tp), \\ D^*(Tp, Tp, y_{3n+2}), D^*(p, y_{3n+1}, y_{3n+2}), 0, 0, \\ D^*(y_{3n+1}, y_{3n+2}, y_{3n+2}) \end{array} \right\}$$

$$\leq D^*(p, y_{3n+2}, y_{3n+2}) + q \max \left\{ \begin{array}{c} D^*(Tp, p, p) + D^*(p, p, y_{3n+1}), D^*(p, p, Tp), \\ D^*(p, y_{3n+2}, y_{3n+2}) + D^*(Tp, p, p), \\ D^*(p, y_{3n+1}, y_{3n+2}), 0, 0, \\ D^*(y_{3n+1}, p, p) + D^*(p, y_{3n+2}, y_{3n+2}) \end{array} \right\}$$

Now we have
$(1 - q)D^*(p, Tp, p) \leq D^*(p, y_{3n+2}, y_{3n+2}) + qD^*(p, p, y_{3n+1})$ or
$(1 - q)D^*(p, Tp, p) \leq D^*(p, y_{3n+2}, y_{3n+2})$ or
$(1 - q)D^*(p, Tp, p) \leq (1 + q)D^*(p, y_{3n+2}, y_{3n+2})$ or
$D^*(p, Tp, p) \leq D^*(p, y_{3n+2}, y_{3n+2}) + qD^*(p, y_{3n+1}, y_{3n+2})$ or
$D^*(p, Tp, p) \leq D^*(p, y_{3n+2}, y_{3n+2})$ or
$D^*(p, Tp, p) \leq D^*(p, p, y_{3n+1}) + (1 + q)D^*(p, y_{3n+2}, y_{3n+2})$.
From Lemma 1.9 and Lemma 1.14 , we have $Tp = p$.
Hence $p = Tp = gp$. ......(7).
Since $p = Tp \in h(X)$, there exists $w \in X$ such that $p = hw$.
Now, $D^*(p, p, Rw) = D^*(Sp, Tp, Rw)$

$$\leq q \max \left\{ \begin{array}{c} D^*(fp, gp, hw), D^*(fp, Sp, Tp), D^*(gp, Tp, Rw), \\ D^*(hw, Rw, Sp)D^*(fp, Sp, Sp), D^*(gp, Tp, Tp), \\ D^*(hw, Rw, Rw). \end{array} \right\}$$

$$= q \max \left\{ 0, 0, D^*(p, p, Rw), D^*(p, Rw, p), 0, 0, D^*(p, Rw, Rw) \right\}$$

$$= q\ D^*(p, p, Rw) .$$

From Proposition 1.4 , we have $Rw = p$. Thus $hw = Rw = p$.
Since the pair (R, h) is weakly compatible, we have $Rp = hp$.
Now, $D^*(p, p, Rp) = D^*(Sp, Tp, Rp)$

$$\leq q \max \left\{ \begin{array}{c} D^*(fp, gp, hp), D^*(fp, Sp, Tp), D^*(gp, Tp, Rp), \\ D^*(hp, Rp, Sp)D^*(fp, Sp, Sp), D^*(gp, Tp, Tp), \\ D^*(hp, Rp, Rp). \end{array} \right\}$$

$$= q\ max\,\{D^*(p,p,Rp),0,D^*(p,Rp,p),D^*(p,Rp,Rp),0,0,0\}\,.$$

$$= q\ D^*(p,p,Rp)\ .$$

From Proposition 1.4 ,we have $Rp = p$. Thus $hp = Rp = p$ ....(8).
From (6),(7) and (8), it follows that p is a common fixed point of S,T,R,f,g and h.
Let p' be another common fixed point of S,T,R,f,g and h .
$D^*(p,p,p') = D^*(Sp,Tp,Rp')$
$= q\ max\,\{D^*(p,p,p'),0,D^*(p,p,p'),D^*(p,p,p'),0,0,0\}$.
$= q\ D^*(p,p,p')$.
From Proposition 1.4 ,we have $p = p'$.
Similarly, we can prove the theorem when $g(X)$ or $h(X)$ is D^*- complete .

Corollary 2.2 . *Let (X,D^*) be a complete D^*- cone metric space and $S,T,R : X \to X$ be satisfying*

(2.2.1) $D^*(Sx,Ty,Rz)$

$$\leq q \max\left\{\begin{array}{c} D^*(x,y,z), D^*(x,Sx,Ty), D^*(y,Ty,Rz), D^*(z,Rz,Sx), \\ D^*(x,Sx,Sx), D^*(y,Ty,Ty), D^*(z,Rz,Rz) \end{array}\right\}$$

for all $x,y,z \in X$, where $0 \leq q < 1$.
(2.2.2) $D^*(x,x,y) \leq D^*(x,y,z)$ *for all $x,y,z \in X$ with $y \neq z$.*
Then the maps S,T and R have a unique common fixed point.

Proof. From Theorem 2.1 with $f = g = h = I$(Identity map) , we have either S or T or R has a fixed point in X or the maps S,T and R have a unique common fixed point in X.
Suppose $Sx = x$. Assume that $Tx \neq Rx$.
Then from (2.2.1) and from Remark 1.3 ,

$$\begin{aligned} & D^*(x,Tx,Rx) \\ &= D^*(Sx,Tx,Rx) \\ &\leq q \quad \max\left\{\begin{array}{c} 0, D^*(x,x,Tx), D^*(x,Tx,Rx), D^*(x,Rx,x) \\ 0, D^*(x,Tx,Tx), D^*(x,Rx,Rx) \end{array}\right\} \\ &= q \quad \max\{D^*(x,x,Tx), D^*(x,Tx,Rx), D^*(x,x,Rx)\}\ ...(1) \\ &\leq q \quad D^*(x,Tx,Rx) \quad from(2.2.2) \end{aligned}$$

It is a contradiction. Hence $Tx = Rx$.
Now from(1),$D^*(x,Tx,Tx) \leq qD^*(x,Tx,Tx)$.
Hence from Proposition 1.4,we have $Tx = x$. Hence $Rx = x$.
Thus x is a common fixed point of S,T and R. similarly, if $Tx = x$ or $Rx = x$ then also x is a common fixed point of S,T and R.

An open Problem: Is Theorem 2.1 or Corollary 2.2 holds without the condition "$D^*(x,x,y) \leq D^*(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$" ?.

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