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ROOTS OF UNITY AS A LIE ALGEBRA

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Dedicated to Professor Hvedri Inassaridze

Abstract. This note gives a categorical development arising from a theorem of A. A. Klyachko relating the Lie operad to roots of unity. We examine the "substitude" structure on the groupoid $\mathbf{C}$ whose homsets are the cyclic groups. The roots of unity representations of the cyclic groups form a Lie algebra for a certain oplax monoidal structure on the category of linear representations of $\mathbf{C}$.

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1. Introduction

Given a functor $J : \mathscr{C} \to \mathscr{A}$ into a monoidal category $\mathscr{A}$, there is a substitude structure [5] obtained on $\mathscr{C}$ by restriction along J. If J is fully faithful, the substitude $\mathscr{C}$ is a multicategory in the sense of [9]. While a substitude structure is fairly weak, we can actually define Lie algebras in any additive symmetric substitude. The particular functor $J : \mathbf{C} \to \mathbf{P}$ we wish to consider is the union of the inclusions of the cyclic groups in the symmetric groups; this J is faithful and bijective on objects, but not full.

Now $\mathbf{P}$ is the free symmetric strict monoidal category on a single generating object; the tensor product is denoted by $+$. The category $\mathscr{V}^{\mathbf{P}}$ of linear representations of the symmetric groups is the category of tensorial species in the sense of [7]. Linear symmetric operads are monoids in $\mathscr{V}^{\mathbf{P}}$ equipped with the substitution tensor product. The Lie operad **lie** is a Lie algebra in $\mathscr{V}^{\mathbf{P}}$ equipped with the convolution tensor product coming from $\mathbf{P}$ with $+$. We are interested in a relaxedly associative tensor operation on the category $\mathscr{V}^{\mathbf{C}}$ of linear representations of the cyclic groups, which is derived by convolution and restriction from $+$ on $\mathbf{P}$. The object $\boldsymbol{\omega}$ in $\mathscr{V}^{\mathbf{C}}$ made up of all the roots of unity representations of the cyclic groups turns out to be a Lie algebra for this lax tensor structure. By a theorem of Klyachko [8], the representation of the symmetric groups induced along J by $\boldsymbol{\omega}$ is **lie**. While the monoidal structure on $\mathscr{V}^{\mathbf{P}}$ is traditional, the object **lie** is rather complicated. In contrast, we have only a lax monoidal structure on $\mathscr{V}^{\mathbf{C}}$, yet the object $\boldsymbol{\omega}$ is quite easily understood.

2. The Symmetric and Cyclic Groupoids

Write $\mathbf{P}$ for the *symmetric groupoid*; it is the category whose objects are natural numbers and whose morphisms are permutations; so the homset $\mathbf{P}(m, n)$ is

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empty for $m \neq n$ and the endomorphism monoid $\mathbf{P}(n,n)$ is the permutation (or symmetric) group $\mathbf{P}_n$ on the set $\{1, 2, \ldots, n\}$. Write $\mathbf{C}$ for the *cyclic groupoid*; its objects are natural numbers and the endomorphism monoid $\mathbf{C}(n,n)$ is the cyclic group $\mathbf{C}_n$ of order n. Up to isomorphism there is a unique faithful functor $J : \mathbf{C} \to \mathbf{P}$ which is the identity on objects; to be explicit we choose the J that takes a distinguished generator of $\mathbf{C}_n$ to the permutation $i \mapsto i+1 \pmod n$.

3. Substitudes

Following [5] we use the term "substitude" for a slight weakening of the notion of multicategory. More precisely, a substitude is a category $\mathscr{A}$ together with:

- for each integer $n \geq 0$, a functor

$$P_n : \mathscr{A}^{\mathrm{op}} \times \cdots \times \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathbf{Set}$$

  whose value at $(A_1, \ldots, A_n, A)$ is denoted by $P_n(A_1, \ldots, A_n; A)$;
- for each partition[1] $\xi : m_1 + \cdots + m_n = m$, a natural family of functions

$$\mu_\xi : P_n(A_\bullet; A) \times P_{m_1}(A_{1\bullet}; A_1) \times \cdots \times P_{m_n}(A_{n\bullet}; A_n) \longrightarrow P_\xi(A_{\bullet\bullet}; A)$$

  called *substitution*, where we use the shorthand

$$P_k(X_\bullet; X) = P_k(X_1, \ldots, X_k; X) \text{ and}$$
$$P_\xi(X_{\bullet\bullet}; X) = P_m(X_{11}, \ldots, X_{1m_1}, \ldots, X_{n1}, \ldots, X_{nm_n}; X); \text{ and}$$

- a natural family of functions

$$\eta : \mathscr{A}(A, B) \longrightarrow P_1(A; B)$$

  called the *unit*;

subject to three conditions that can be found in [5].

4. Restriction

As a particular case of Proposition 4.1 of [4], given any functor $J : \mathscr{C} \to \mathscr{A}$, we can restrict the substitude structure on $\mathscr{A}$ to $\mathscr{C}$ by defining

$$P_n(C_1, \ldots, C_n; C) = P_n(JC_1, \ldots, JC_n; JC),$$

by defining the substitution operation to be that of $\mathscr{A}$ restricted to objects which are values of J, and by defining the unit to be the composite

$$\mathscr{C}(C, D) \xrightarrow{J} \mathscr{A}(JC, JD) \xrightarrow{\eta} P_1(JC, JD).$$

In particular, addition of natural numbers gives a monoidal structure on $\mathbf{P}$ which restricts along J to yield a substitude structure on $\mathbf{C}$.

[1]The summands in our partitions are allowed to be zero and in non-monotone order.

5. Lax and Oplax Monoidal Categories

Here we use the term *linear* to mean enrichment (in the sense of [6]) over the base monoidal category $\mathscr{V}$ of vector spaces over an algebraically closed field k of characteristic zero. Recall from [4] that a *lax monoidal* structure on a linear category $\mathscr{X}$ consists of linear functors

$$\underset{n}{\otimes} : \mathscr{X} \otimes \cdots \otimes \mathscr{X} \longrightarrow \mathscr{X}$$

(called *n-fold tensor product*) together with natural *substitutions*

$$\mu_\xi : \underset{n}{\otimes}(\underset{m_1}{\otimes}(X_{11}, \ldots, X_{1m_1}), \ldots, \underset{m_n}{\otimes}(X_{n1}, \ldots, X_{nm_n})) \longrightarrow \underset{m}{\otimes}(X_{11}, \ldots, X_{nm_n})$$

and *unit* $\eta : X \to \otimes_1(X)$, satisfying three axioms. An *oplax monoidal* structure on the linear category $\mathscr{X}$ is a lax monoidal structure on $\mathscr{X}^{\mathrm{op}}$; again we have linear functors

$$\underset{n}{\otimes} : \mathscr{X} \otimes \cdots \otimes \mathscr{X} \longrightarrow \mathscr{X}$$

however, we have *cosubstitutions*

$$\delta_\xi : \underset{m}{\otimes}(X_{11}, \ldots, X_{nm_n}) \longrightarrow \underset{n}{\otimes}(\underset{m_1}{\otimes}(X_{11}, \ldots, X_{1m_1}), \ldots, \underset{m_n}{\otimes}(X_{n1}, \ldots, X_{nm_n}))$$

and *counit* $\varepsilon : \otimes_1(X) \to X$.

6. Standard Convolution

For any small substitude $\mathscr{A}$ and any cocomplete lax monoidal linear category $\mathscr{X}$, the linear category of functors from $\mathscr{A}$ to $\mathscr{X}$, with natural transformations as morphisms, supports the *standard convolution* lax monoidal structure (see [5]) in which the n-fold tensor product is defined by

$$\underset{n}{\otimes}(F_1, \ldots, F_n)(A) = \int^{A_1, \ldots, A_n} P_n(A_1, \ldots, A_n; A) \times \underset{n}{\otimes}(F_1 A_1, \ldots, F_n A_n)$$

where $S \times V$ denotes the coproduct of S copies of the object V and the integral notation denotes the "coend" (see [10]).

7. Symmetry

A *symmetry* on a substitude $\mathscr{A}$ is a natural family of isomorphisms

$$\gamma_\sigma : P_n(A_1, \ldots, A_n; A) \longrightarrow P_n(A_{\sigma(1)}, \ldots, A_{\sigma(n)}; A)$$

for each permutation σ satisfying certain axioms documented in [5]. For example, the symmetry on the monoidal category $\mathbf{P}$ induces a symmetry on the substitude $\mathbf{C}$. A symmetry on a lax or an oplax monoidal linear category $\mathscr{X}$ is a natural family of isomorphisms

$$c_\sigma : \underset{n}{\otimes}(X_1, \ldots, X_n) \longrightarrow \underset{n}{\otimes}(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

for each permutation σ satisfying certain axioms documented in [5]. If $\mathscr{A}$ and $\mathscr{X}$ are symmetric, there is an induced symmetry on the standard convolution lax linear monoidal category $\mathscr{X}^{\mathscr{A}}$.

8. Induced Representations

Let H be a subgroup of the group G. Then the inclusion $i : H \to G$ can be regarded as a functor between one-object categories. Right Kan extension along i defines a linear functor

$$\mathrm{Ind}_H^G : \mathscr{V}^H \longrightarrow \mathscr{V}^G$$

between the categories of linear representations; it is right adjoint to restriction along i which takes the representation V of G to the representation V_H of H with the same vector space and restricted action. For any S we write $k[S]$ for the vector space with basis S; if S is a G-set then $k[S]$ is a linear representation of G. For any representation M of H, we have the induced representation of G given by

$$\mathrm{Ind}_H^G(M) = \mathscr{V}^H(k[G]_H, M).$$

The functor $\mathrm{Ind}_H^G : \mathscr{V}^H \longrightarrow \mathscr{V}^G$ *is faithful*: to see this we need to observe that the counit $\mathrm{Ind}_H^G(M)_H = \mathscr{V}^H(k[G]_H, M)_H \to \mathscr{V}^H(k[H], M)_H \cong M$ is an epimorphism, but this follows from the fact that the inclusion $k[H] \to k[G]_H$ is a split monomorphism in $\mathscr{V}^H$.

9. Duality for Induced Representations

In Part 8, if G is finite then Ind_H^G is also left Kan extension Lan_i along $i : H \to G$. We have

$$\begin{aligned} \mathrm{Ind}_H^G(M) &= \mathscr{V}^H(k[G]_H, M) \cong k[G]^* \otimes_{k[H]} M \\ &\cong k[G^{\mathrm{op}}] \otimes_{k[H]} M \cong k[G] \otimes_{k[H]} M \\ &= \mathrm{Lan}_i(M), \end{aligned}$$

since $k[G]_H$ has a dual in $\mathscr{V}^H$ and $G^{\mathrm{op}} \cong G$ as groups.

10. Kan Extensions along $J : \mathbf{C} \to \mathbf{P}$

The functor $\mathscr{V}^J : \mathscr{V}^{\mathbf{P}} \to \mathscr{V}^{\mathbf{C}}$ defined by restriction along J has a left adjoint Lan_J and a right adjoint Ran_J. There is a natural bijection between natural transformations $\mathrm{Lan}_J(M) \to F$ and natural transformations $M \to FJ$. Also there is a natural bijection between natural transformations $F \to \mathrm{Ran}_J(M)$ and natural transformations $FJ \to M$. The formulas for left and right Kan extension are respectively:

$$\begin{aligned} \mathrm{Lan}_J(M)(p) &= \int^r P(J(r), p) \times M(r) \cong k[P_p]_{C_p} \otimes_{k[C_p]} M(p) \qquad \text{and} \\ \mathrm{Ran}_J(M)(p) &\cong \int_r M(r)^{P(p,J(r))} \cong \mathrm{Hom}_{C_p}(k[P_p]_{C_p}, M(p)), \end{aligned}$$

where V^S denotes the vector space of functions from the set S to the vector space V. It follows for this J, using Parts 8 and 9, that Lan_J and Ran_J are isomorphic faithful functors.

11. Left Kan Extension along a Product

Suppose we have functors $F : \mathscr{A} \to \mathscr{V}$, $F' : \mathscr{A}' \to \mathscr{V}$, $J : \mathscr{A} \to \mathscr{B}$, and $J' : \mathscr{A}' \to \mathscr{B}'$. We write $F \otimes F' : \mathscr{A} \otimes \mathscr{A}' \to \mathscr{V}$ for the functor defined by

$$(F \otimes F')(A, A') = FA \otimes F'A'.$$

An easy calculation shows that the left Kan extension of the functor $F \otimes F'$ along $J \times J' : \mathscr{A} \times \mathscr{A}' \to \mathscr{B} \times \mathscr{B}'$ is given by

$$\mathrm{Lan}_{J \times J'}(F \otimes F') \cong \mathrm{Lan}_J(F) \otimes \mathrm{Lan}_{J'}(F').$$

12. Restriction Followed by Convolution

Given any functor $J : \mathscr{C} \to \mathscr{A}$ into a monoidal category $\mathscr{A}$, as a particular case of Part 4 we obtain a restriction substitude structure on $\mathscr{C}$ defined by

$$P_n(C_1, \ldots, C_n; C) = \mathscr{A}(JC_1 \otimes \cdots \otimes JC_n, JC).$$

Now we can apply standard convolution as in Part 6 to obtain a <u>lax monoidal structure</u> on the linear category $\mathscr{V}^{\mathscr{C}}$; explicitly we have

$$\underset{n}{\otimes}(M_1, \ldots, M_n)(C) = \int^{C_1, \ldots, C_n} \mathscr{A}(JC_1, \ldots, JC_n; JC) \times M_1C_1 \otimes \cdots \otimes M_nC_n.$$

13. Convolution Followed by Restriction

Given our same functor $J : \mathscr{C} \to \mathscr{A}$ into the monoidal category $\mathscr{A}$ as in Part 12, we can restrict the convolution monoidal structure on $\mathscr{V}^{\mathscr{A}}$ to $\mathscr{V}^{\mathscr{C}}$ by means of the left Kan extension functor $K = \mathrm{Lan}_J : \mathscr{V}^{\mathscr{C}} \to \mathscr{V}^{\mathscr{A}}$. This leads to a substitude structure on $\mathscr{V}^{\mathscr{C}}$ defined by

$$P_n(M_1, \ldots, M_n; M) = \mathscr{V}^{\mathscr{A}}(\underset{n}{\otimes}(KM_1, \ldots, KM_n), KM)$$

$$\cong \int_A \mathscr{V}\Big(\int^{A_1, \ldots, A_n} \mathscr{A}(A_1 \otimes \cdots \otimes A_n, A) \times (KM_1)A_1 \otimes \cdots \otimes (KM_n)A_n, (KM)A\Big)$$

$$\cong \int_A \mathscr{V}\Big(\int^{A_1, \ldots, A_n} \mathscr{A}(A_1 \otimes \cdots \otimes A_n, A) \times$$

$$\times \int^{C_1, \ldots, C_n} \mathscr{A}(JC_1, A_1) \times \cdots \times \mathscr{A}(JC_n, A_n) \otimes M_1C_1 \otimes \cdots \otimes M_nC_n, (KM)A\Big)$$

$$\cong \int_{A; C_1, \ldots, C_n} \mathscr{V}(\mathscr{A}(JC_1 \otimes \cdots \otimes JC_n, A) \otimes M_1C_1 \otimes \cdots \otimes M_nC_n, (KM)A)$$

$$\cong \mathscr{V}^{\mathscr{A}}\Big(\int^{C_1, \ldots, C_n} \mathscr{A}(JC_1 \otimes \cdots \otimes JC_n, -) \otimes M_1C_1 \otimes \cdots \otimes M_nC_n, KM\Big).$$

In the case such as in Part 10 where KM is also a right Kan extension of M along J, this last vector space is isomorphic (naturally in all variables) to

$$\mathscr{V}^{\mathscr{C}}(\underset{n}{\otimes}(M_1, \ldots, M_n), M),$$

where $\otimes_n(M_1, \ldots, M_n)$ is defined in Part 12. In other words, the substitude structure on $\mathscr{V}^{\mathscr{C}}$ is representable and so defines an <u>oplax monoidal structure</u>

on $\mathscr{V}^{\mathscr{C}}$. So we have both an oplax and a lax monoidal structure on $\mathscr{V}^{\mathscr{C}}$ in which the multiple tensor product functors agree. In the situation of Part 10, in fact, the cosubstitution operation for the oplax structure is left inverse to the substitution operation for the lax structure and the counit is left inverse to the unit. To reiterate, the cosubstitution and counit of the oplax monoidal structure on $\mathscr{V}^{\mathscr{C}}$ are not invertible (as would occur if we had a true monoidal structure) but they are split epimorphisms for which the splittings are natural and satisfy the axioms for a lax monoidal structure. We are particularly interested in this symmetric oplax monoidal structure on $\mathscr{V}^{\mathbf{C}}$ coming from $J : \mathbf{C} \to \mathbf{P}$.

14. Lie Algebras

In the paper [5], Lie algebras were defined in any braided additive substitude. We now make this definition explicit in the case of a symmetric oplax monoidal linear category $\mathscr{X}$. A *Lie algebra in* $\mathscr{X}$ is an object L together with a morphism

$$\beta : \underset{2}{\otimes}(L, L) \longrightarrow L,$$

called the *bracket*, satisfying the two conditions

$$\beta(1 + c_{\tau_2}) = 0 \text{ and } \lambda(1 + c_{\tau_3} + c_{\tau_3}^2) = 0,$$

where τ_n is the permutation $i \mapsto i + 1(\mod n)$ and λ is the composite

$$\otimes_3(L, L; L) \xrightarrow{\delta_{3=2+1}} \otimes_2(\otimes_2(L, L), \otimes_1(L)) \xrightarrow{\otimes_2(\beta,\varepsilon)} \otimes_2(L, L) \xrightarrow{\beta} L.$$

In particular, a Lie algebra in $\mathscr{V}$, with the usual tensor product of vector spaces, is a Lie algebra over k in the usual sense. The purpose of [1] was to study Lie algebras in a non-standard symmetric linear substitude of representations of a Hopf algebra.

15. The Lie Operad

Let $E : \mathbf{P} \to \mathscr{V}$ denote the functor which takes each object n of $\mathbf{P}$ to the vector space k^n and takes each permutation to the linear function represented by the corresponding permutation matrix. This E is a symmetric strong monoidal (symmetry and tensor-preserving) functor for the $+$ structure on $\mathbf{P}$ and the direct sum structure on $\mathscr{V}$. It follows (see [3]) that left Kan extension

$$T = \mathrm{Lan}_E : \mathscr{V}^{\mathbf{P}} \to \mathscr{V}^{\mathscr{V}}$$

is symmetric strong monoidal with respect to the convolution structures. The convolution structure on $\mathscr{V}^{\mathscr{V}}$ is merely pointwise tensor product. The functor T takes each tensorial species F to its "Taylor series" TF defined by

$$(TF)V = \bigoplus_{n=0}^{\infty} F_n \otimes V^{\otimes n}/\mathbf{P}_n.$$

It is also true that T is strong monoidal with respect to the substitution structure on $\mathscr{V}^{\mathbf{P}}$ and the composition structure on $\mathscr{V}^{\mathscr{V}}$. Each species $F \in \mathscr{V}^{\mathbf{P}}$ gives a representation $F_n = Fn$ of the symmetric group $\mathbf{P}_n$ since F is defined as a functor on permutations of every n. Let $\mathbf{L}V$ denote the free Lie algebra on the vector space V. This gives an object $\mathbf{L}$ of $\mathscr{V}^{\mathscr{V}}$ which is a monoid for the composition structure (that is, $\mathbf{L}$ is a monad on the category $\mathscr{V}$) and a Lie algebra for the pointwise tensor product. By a very general argument, it is shown in [7] that there is a Lie algebra **lie** in the convolution symmetric monoidal linear category (called "une algèbre de Lie tordue") providing the Taylor coefficients for $\mathbf{L}$; that is, $T\mathbf{lie} \cong \mathbf{L}$ as Lie algebras. Less important for our purpose here is that **lie** is also a monoid for the substitution structure on $\mathscr{V}^{\mathbf{P}}$ and so is a symmetric linear operad [11]. The representation $\mathbf{lie}_n$ of $\mathbf{P}_n$ has underlying vector space spanned by those elements of the free Lie algebra $\mathbf{L}k^n$ in which each of the canonical basis vectors $e_1, \ldots, e_n$ of k^n occurs precisely once; the permutations act by applying them to these basis vectors; so **lie** is a subobject of $\mathbf{L}E$. The bracket and substitution operations on **lie** are easily guessed.

16. The Lie Algebra $\boldsymbol{\omega}$

Let $\boldsymbol{\omega}_n$ denote the linear representation of $\mathbf{C}_n$ whose supporting vector space is k and whose action by the generator of $\mathbf{C}_n$ is multiplication by a primitive n-th root of unity; any choice of generator and primitive root gives an isomorphic representation. A theorem of Klyachko [8] (also see [2]) is that the representation induced on $\mathbf{P}_n$ by $\boldsymbol{\omega}_n$ is $\mathbf{lie}_n$; that is,

$$\mathrm{Ind}_{\mathbf{C}_n}^{\mathbf{P}_n}(\boldsymbol{\omega}_n) \cong \mathbf{lie}_n.$$

In other words, the roots of unity representations make up an object $\boldsymbol{\omega} \in \mathscr{V}^{\mathbf{C}}$ satisfying

$$K\boldsymbol{\omega} \cong \mathbf{lie}$$

where $K = \mathrm{Lan}_J \cong \mathrm{Ran}_J$. From Part 13 we have the isomorphisms

$$\mathscr{V}^{\mathbf{C}}(\underset{n}{\otimes}(\boldsymbol{\omega}, \ldots, \boldsymbol{\omega}), \boldsymbol{\omega}) \cong \mathscr{V}^{\mathbf{P}}(\underset{n}{\otimes}(\mathbf{lie}, \ldots, \mathbf{lie}), \mathbf{lie})$$

which are compatible with the substitution and unit operations. So the bracket on **lie** corresponds to a bracket on $\boldsymbol{\omega}$ and we have our result.

Theorem. *The roots of unity representation $\boldsymbol{\omega}$ is a Lie algebra in the symmetric oplax monoidal linear category $\mathscr{V}^{\mathbf{C}}$ obtained by restriction along K of the convolution structure on $\mathscr{V}^{\mathbf{P}}$. There is an isomorphism of Lie algebras $K\boldsymbol{\omega} \cong \mathbf{lie}$.*

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