Georgian Mathematical Journal
Volume 9 (2002), Number 4, 671–682

CENTRAL SERIES FOR GROUPS WITH ACTION AND LEIBNIZ ALGEBRAS

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Abstract. The notion of central series for groups with action on itself is introduced. An analogue of Witt's construction is given for such groups. A certain condition is found for the action and the corresponding category is defined. It is proved that the above construction defines a functor from this category to the category of Lie–Leibniz algebras and in particular to Leibniz algebras; also the restriction of this functor on the category of groups leads us to Lie algebras and gives the result of Witt.

2000 Mathematics Subject Classification: 17B60.
Key words and phrases: Central series, groups, groups with action on itself, Lie algebras, Leibniz algebras, Lie–Leibniz algebras.

Introduction

The well-known construction of Witt defines a functor from the category of groups to the category of Lie algebras [6], [5]. The aim of this paper is to define a category and to give an analogue of Witt's construction for its objects, which will lead us to the category of Leibniz algebras. This problem was stated by J.-L. Loday; later an analogous question for the possibly defined partial Leibniz algebras was proposed, which was inspired by the work of Baues and Conduché [1]. Since the main interest lies in the absolute case, the author decided to begin with this one.

In Section 1 we define the category of groups with action on itself $\mathbb{Gr}^{\bullet}$, the category of abelian groups with action on itself $\mathbb{Ab}^{\bullet}$ and the category of groups with bracket operation $\mathbb{Gr}^{[\,]}$. This kind of groups are Ω-groups in the sense of [2]. We construct adjoint pairs of functors relating categories $\mathbb{Gr}^{\bullet}$, $\mathbb{Ab}^{\bullet}$, $\mathbb{Gr}^{[\,]}$, $\mathbb{Gr}$. In Section 2 we define ideals and commutators for the objects of $\mathbb{Gr}^{\bullet}$ (similarly for $\mathbb{Gr}^{[\,]}$) and show that these notions are equivalent to the special case of the known definitions for Ω-groups [2]. In Section 3 we define central series of groups with action on itself and a category of Lie–Leibniz algebras $\mathbb{LL}$. We consider the category of groups with action on itself satisfying a certain condition $\mathbb{Gr}^{c}$. We give an analogue of Witt's construction [6] and prove that it defines a functor $LL : \mathbb{Gr}^{c} \longrightarrow \mathbb{LL}$, in particular this gives a functor $\mathbb{Gr}^{c} \longrightarrow \mathbb{Leibniz}$. In a similar way one can construct a functor $\mathbb{Ab}^{c} \longrightarrow \mathbb{Leibniz}$, which is actually the restriction of LL on $\mathbb{Ab}^{c}$. The functorial relations with the classical situation ($\mathbb{Gr} \longrightarrow \mathbb{Lie}$) is considered, namely by the restriction of LL on $\mathbb{Gr}$ we obtain the result of Witt [6], [5].

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

1. Groups with Action on Itself

Let G be a group which acts on itself from the right side; i.e. we have a map $\varepsilon : G \times G \longrightarrow G$ with

$$\begin{aligned}
&\varepsilon(g, g' + g'') = \varepsilon(\varepsilon(g, g'), g''),\\
&\varepsilon(g, 0) = g,\\
&\varepsilon(g' + g'', g) = \varepsilon(g', g) + \varepsilon(g'', g),\\
&\varepsilon(0, g) = 0,
\end{aligned} \tag{1.1}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$. We denote the group operation additively, nevertheless the group is not commutative in general. If (G', ε') is another group with action, then a homomorphism $(G, \varepsilon) \longrightarrow (G', \varepsilon')$ is a group homomorphism $\varphi : G \longrightarrow G'$ for which the diagram

$$\begin{tikzcd}
G \times G \arrow[r, "\varepsilon"] \arrow[d, "(\varphi,\varphi)"'] & G \arrow[d, "\varphi"] \\
G' \times G' \arrow[r, "\varepsilon'"] & G'
\end{tikzcd}$$

commutes. In other words, we have

$$\varphi(g^h) = \varphi(g)^{\varphi(h)}, \quad g, h \in G.$$

If we consider an action as a group homomorphism $G \xrightarrow{\nu} \operatorname{Aut} G$, then a homomorphism between two groups with action means the commutativity of the diagram

$$\begin{tikzcd}
G \arrow[rr, "\nu"] \arrow[dd, "\varphi"'] & & \operatorname{Aut} G \subset \operatorname{Hom}(G, G) \arrow[d, "\operatorname{Hom}(G,\varphi)"] \\
& & \operatorname{Hom}(G, G') \\
G' \arrow[rr, "\nu'"] & & \operatorname{Aut} G' \subset \operatorname{Hom}(G', G') \arrow[u, "\operatorname{Hom}(\varphi,G')"']
\end{tikzcd}$$

so that $\varphi \cdot (\nu(h)) = \nu'(\varphi(h)) \cdot \varphi$, $h \in G$.

Recall [2] that an Ω-group is a group with a system of n-ary algebraic operations Ω $(n \geq 1)$, which satisfies the condition

$$00 \cdots 0\omega = 0, \tag{1.2}$$

where 0 is the identity element of G, and 0 on the left side occurs n times if ω is an n-ary operation. In special cases Ω-groups give groups, rings and groups with action on itself. In the latter case Ω consists of one binary operation, an action; or Ω consists of only unary operations, elements of G, and this operation is an action again. In both cases the condition (1.2) is satisfied. We shall denote the category of groups with action on itself by $\mathbb{Gr}^\bullet$. Let $\mathbb{Ab}^\bullet$ denote the category of abelian groups with action on itself. We have functors

$$\mathbb{Ab}^{\bullet} \underset{A}{\overset{E}{\rightleftarrows}} \mathbb{Gr}^{\bullet} \begin{array}{c} \xrightarrow{Q_1} \\ \xleftarrow{T} \\ \xrightarrow{Q_2} \\ \xleftarrow{C} \end{array} \mathbb{Gr},$$

where $Q_1(G)$, for $G \in \mathbb{Gr}^{\bullet}$, is the greatest quotient group of G which makes the action trivial; $Q_2(G)$ is a quotient of G by the equivalence relation generated by the relation $g^h \sim -h + g + h$, $g, h \in G$; A is the abelianization functor, thus $A(G) = G/(G,G)$, where (G,G) is the ideal of G generated by the commutator normal subgroup of G (for the definition of an ideal see Section 2). $A(G)$ has the induced operation of action on itself. Each group can be considered as a group with the trivial action or with the action by conjugation, these give functors T and C, respectively. Every object of $\mathbb{Ab}^{\bullet}$ can be considered as an object of $\mathbb{Gr}^{\bullet}$; this functor is denoted by E. It is easy to see that the functors Q_1, Q_2 and A are left adjoints to the functors T, C and E respectively. Let $G \in \mathbb{Gr}^{\bullet}$. Define the operation of square brackets $[\,,\,] : G \times G \longrightarrow G$ on G by

$$[g,h] = -g + g^h, \quad g,h \in G.$$

Proposition 1.1. *For the operation* $[\,,\,]$ *we have the following identities:*
(i) $[g, h_1 + h_2] = [g, h_1] + [g + [g, h_1], h_2]$;
(ii) $[g + g', h] = -g' + [g, h] + g' + [g', h]$;
(iii) $[g, 0] = [0, g] = 0$.

Proof. These identities follow directly from (1.1). □

Corollary 1.2. *For* $g, h \in G$

$$[g^h, -h] = -[g,h];$$
$$[-g, h] = g - [g,h] - g.$$

Denote by $\mathbb{Gr}^{[\,]}$ the category of groups with an additional bracket operation $[\,,\,]$ satisfying the conditions (i)–(iii) of Proposition 1.1; morphisms of $\mathbb{Gr}^{[\,]}$ are group homomorphisms preserving the bracket operation. We shall denote the objects of $\mathbb{Gr}^{[\,]}$ by $G^{[\,]}$.

Conversely, if $G^{[\,]} \in \mathbb{Gr}^{[\,]}$, we can define an action of $G^{[\,]}$ on itself due to the bracket operation by

$$g^h = g + [g,h], \quad g, h \in G^{[\,]}.$$

It is easy to prove that these two procedures are converse to each other and actually we have an isomorphism of categories

$$\mathbb{Gr}^{\bullet} \approx \mathbb{Gr}^{[\,]}.$$

2. Ideals and Commutators in $\mathbb{Gr}^{\bullet}$

Let $G \in \mathbb{Gr}^{\bullet}$.

Definition 2.1. A nonempty subset A of G is called an ideal of G if it satisfies the following conditions:
1. A is a normal subgroup of G as a group;
2. $a^g \in A$, for $a \in A$, $g \in G$;

3. $-g + g^a \in A$, for $a \in A$ and $g \in G$.

Definition 2.2 (Kurosh [2])**.** A nonempty subset A of an Ω-group G is called an ideal if

(a) A is an additive normal subgroup of G;

(b) For any n-any operation ω from Ω, any element $a \in A$ and elements $x_1, x_2, \ldots, x_n \in G$

$$-(x_1 \cdots x_n\omega) + x_1 \cdots x_{i-1}(a + x_i)x_{i+1} \cdots x_n\omega \in A,$$

for $i = 1, 2, \ldots, n$.

This definition in the case of groups is the definition of a normal subgroup of a group, and in the case of rings is the definition of a two-sided ideal of a ring.

Proposition 2.3. *For a group $G \in \mathbb{Gr}^\bullet$ considered as an Ω-group, where Ω consists of one binary operation of action, Definitions* 2.1 *and* 2.2 *are equivalent.*

Proof. The condition (b) of Definition 2.2 has the forms:

$$-x_1^{x_2} + (a + x_1)^{x_2} \in A, \quad \text{for} \quad i = 1; \tag{2.1}$$

$$-x_1^{x_2} + x_1^{a+x_2} \in A, \quad \text{for} \quad i = 2. \tag{2.2}$$

Taking $x_1 = 0$ in (2.1), we obtain $a^{x_2} \in A$, which is condition 2 of Definition 2.1. Taking $x_2 = 0$ in (2.2), we have $-x_1 + x_1^a \in A$, which is condition 3 of Definition 2.1.

Conversely, we shall show that conditions 2 and 3 of Definition 2.1 imply conditions (2.1) and (2.2). From condition 2 we have $a^{x_2} \in A$; also

$$-x_1^{x_2} + (a + x_1)^{x_2} = -x_1^{x_2} + a^{x_2} + a_1^{x_2},$$

and it is an element of A since A is a normal subgroup of G. By condition 3 of Definition 2.1, $-x_1 + x_1^a \in A$. We have $-x_1^{x_2} + x_1^{a+x_2} = (-x_1 + x_1^a)^{x_2}$ and this is an element of A due to condition 2, which ends the proof. □

Thus an ideal of G is a subobject of G in $\mathbb{Gr}^\bullet$. It is clear that G itself and the trivial subobject of G are ideals of G. An intersection of any system of ideals of G is an ideal, and therefore we conclude that there exists the ideal generated by a system of elements of G.

Proposition 2.4. *Let A be an ideal of G. For $a_1, a_2 \in A$, $g_1, g_2 \in G$ we have*

$$(a_1 + g_1)^{a_2+g_2} \in g_1^{g_2} + A.$$

Proof. Since A is an ideal of G there exist $a_1', a_2' \in A$, such that $a_1 + g_1 = g_1 + a_1'$, $a_2 + g_2 = g_2 + a_2'$. Therefore

$$\begin{aligned}(a_1 + g_1)^{a_2+g_2} &= (g_1 + a_1')^{g_2+a_2'} = (g_1^{g_2})^{a_2'} + {a_1'}^{g_2+a_2'} \\ &= g_1^{g_2} - g_1^{g_2} + (g_1^{g_2})^{a_2'} + {a_1'}^{g_2+a_2'} \in g_1^{g_2} + A;\end{aligned}$$

here we apply $-g_1^{g_2} + (g_1^{g_2})^{a_2'} \in A$. □

Let A and B be subobjects of G. Denote by $\{A, B\}$ the subobject of G generated by A and B, and let $A + B$ denote the subset of G

$$A + B = \{a + b \,|\, a \in A,\ b \in B\}.$$

Proposition 2.5. *If A is an ideal of G and B is a subobject of G, then*

$$\{A, B\} = A + B.$$

Proof. It is obvious that $A + B \subset \{A, B\}$. Since A is an ideal, it follows that $A + B$ is a subgroup of G. By Proposition 2.4, $(a_1 + b_1)^{a_2+b_2} \in b_1^{b_2} + A$. Since B is a subobject, $b_1^{b_2} \in B$, and since A is an ideal, $b_1^{b_2} + A = A + b_1^{b_2} \in A + B$ which ends the proof. $\square$

For Ω-groups see Propositions 2.4 and 2.5 in [2].

Proposition 2.6. *If A and B are ideals of G, then $A + B$ is also an ideal.*

Proof. For $g \in G$, $a \in A$ and $b \in B$ we have

$$g + (a + b) = (a' + g) + b = a' + b' + g \in A + B + g,$$

for certain $a' \in A$ and $b' \in B$. Thus $g+(A+B) \subset (A+B)+g$. In the same way we show that $(A+B)+g \subset g+(A+B)$ and thus $g+(A+B) = (A+B)+g$. It is obvious that $(a+b)^g \in A+B$. Now we have to show that $-g+g^{a+b} \in A+B$. We have

$$-g + g^{a+b} = -g + g^a - g^a + (g^a)^b \in A + B$$

since $-g + g^a \in A$, $-g^a + (g^a)^b \in B$. $\square$

It is easy to verify that the ideal generated by a system of ideals of G coincides with the additive subgroup of G generated by these ideals. For Ω-groups see [2].

Definition 2.1′. Let $G^{[\,]} \in \mathbb{Gr}^{[\,]}$ and A be a nonempty subset of $G^{[\,]}$. A is called an ideal of $G^{[\,]}$ if

1′. A is a normal subgroup of $G^{[\,]}$ as of an additive group;
2′. $[a, g] \in A$, for $a \in A$, $g \in G^{[\,]}$;
3′. $[g, a] \in A$, for $a \in A$, $g \in G^{[\,]}$.

It is easy to see that the isomorphism of categories $\mathbb{Gr}^\bullet \approx \mathbb{Gr}^{[\,]}$ carries ideals to ideals.

Proposition 2.7. *If A is an ideal of G, then the quotient group G/A with the induced action on itself is an object of $\mathbb{Gr}^\bullet$.*

Proof. Straightforward verification. $\square$

In what follows, for $G \in \mathbb{Gr}^\bullet$ and $g, g' \in G$, $[g, g']$ will indicate the element $-g + g^{g'}$ of G and (g, g') the commutator $-g - g' + g + g'$. Let A and B be subobjects of G.

Definition 2.8. A commutator $[A, B]$ of G generated by A and B is the ideal of $\{A, B\}$ generated by the elements

$$\{[a, b], [b, a], (a, b) \mid a \in A,\ b \in B\}.$$

Definition 2.9 ([2])**.** Let G be an Ω-group, A,B be Ω-subgroups of G and $\{A,B\}_\Omega$ be the Ω-subgroup of G generated by A and B. The commutator $[A,B]_\Omega$ is the ideal of $\{A,B\}_\Omega$ generated by elements of the form

$$(a,b)=-a-b+a+b,\quad a\in A,\quad b\in B,$$

and

$$\begin{aligned}[a_1,\dots,a_n;b_1,\dots,b_n;\omega]&=-a_1a_2\cdots a_n\omega-b_1b_2\cdots b_n\omega\\&\quad+(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\omega,\end{aligned}\tag{2.3}$$

where ω is an n-any operation from Ω, $a_1,\dots,a_n\in A$ and $b_1,\dots,b_n\in B$.

If G is a group with the trivial action on itself or with the action by conjugation, then $[A,B]$ in Definition 2.8 is the normal subgroup of G generated in $\{A,B\}$ by commutators (a,b), $a\in A$, $b\in B$, i. e. the usual commutator for the case of groups. The same is true for Definition 2.9; if an Ω-group is a group without multioperations, then the commutator $[A,B]_\Omega$ is the usual commutator (A,B) of a group [2].

Proposition 2.10. *In the case of groups with action on itself Definitions* 2.8 *and* 2.9 *are equivalent.*

Proof. For groups with action (2.3) has the form

$$-a^{a_2}-b_1^{b_2}+(a_1+b_1)^{a_2+b_2}.\tag{2.4}$$

Take $a_1=a$, $a_2=b_1=0$, $b_2=b$, then $-a+a^b\in[A,B]_\Omega$. Take in (2.4) $a_1=b_2=0$, $a_2=a$, $b_1=b$, then we obtain

$$-b+b^a\in[A,B]_\Omega.$$

Thus we have shown that $[A,B]\subset[A,B]_\Omega$. Conversely, for $x=-a_1^{a_2}-b_1^{b_2}+(a_1+b_1)^{a_2+b_2}\in[A,B]_\Omega$ we have $x=-a_1^{a_2}-b_1^{b_2}+(a_1^{a_2})^{b_2}+(b_1^{a_2})^{b_2}\in\{A,B\}$. Let $\overline{\{A,B\}}=\{A,B\}/[A,B]$ and let $\overline{g}$ be the class of the element $g\in\{A,B\}$ in $\overline{\{A,B\}}$. We have $\overline{a^b}=\overline{a}$, $\overline{b^a}=\overline{b}$ in $\overline{\{A,B\}}$. Thus

$$\begin{aligned}\overline{x}&=\overline{-a_1^{a_2}-b_1^{b_2}+(a_1^{a_2})^{b_2}+(b_1^{a_2})^{b_2}}=\overline{a_1^{a_2}}-\overline{b_1^{b_2}}+\overline{a_1^{a_2}}+\overline{b_1^{a_2}}^{\overline{b_2}}\\&=-\overline{a_1^{a_2}}-\overline{b_1^{b_2}}+\overline{a_1^{a_2}}+\overline{b_1^{b_2}}=\overline{-a_1^{a_2}-b_1^{b_2}+a_1^{a_2}+b_1^{b_2}}=0,\end{aligned}$$

which means that $x\in[A,B]$. $\square$

Below we formulate without proofs two statements for Ω-groups from [2], which in the case of groups with action give the corresponding results.

Proposition 2.11. *For any Ω-subgroups A and B in G we have*

$$[A,B]_\Omega=[B,A]_\Omega.$$

Proposition 2.12. *An Ω-subgroup A is an ideal of G if and only if*

$$[A,G]_\Omega\subseteq A.$$

Corollary 2.13. *Any Ω-subgroup A of an Ω-group G which contains the commutator $[G,G]_\Omega$ is an ideal of G.*

Proof. It follows from the inclusions

$$[A, G]_\Omega \subset [G, G]_\Omega \subset A. \qquad \square$$

3. Central Series in $\mathbb{Gr}^\bullet$ and the Main Result

Let $G \in \mathbb{Gr}^\bullet$.

Definition 3.1. The (lower) central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$$

of the object G is defined inductively by

$$G_n = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots + [G_{n-1}, G_1].$$

By definition, we have $[G_n, G_m] \subset G_{n+m}$.

Proposition 3.2. *For each $n \geq 1$, G_{n+1} is an ideal of G_n.*

Proof. We have $G_2 = [G_1, G_1]$, which is an ideal of G_1, by definition. $G_3 = [G_1, G_2] + [G_2, G_1]$. By Proposition 2.11, $[G_1, G_2] = [G_2, G_1]$. We have

$$[G_1, G_2] \subset [G_1, G_1] = G_2 \subset \{G_1, G_2\}$$

and $[G_1, G_2]$ is an ideal of $\{G_1, G_2\}$; from this it follows that $[G_1, G_2]$ is an ideal of G_2 and therefore, by Proposition 2.6, G_3 is an ideal of G_2. We have

$$G_{n+1} = [G_1, G_n] + [G_2, G_{n-1}] + \ldots [G_{n-1}, G_2] + [G_n, G_1].$$

For $1 \leq k \leq n$, $[G_k, G_{n-k+1}]$ is an ideal of $\{G_k, G_{n-k+1}\}$; $G_n \subseteq G_k$ from which it follows that $G_n \subseteq \{G_k, G_{n-k+1}\}$. At the same time

$$[G_k, G_{n-k+1}] \subset [G_k, G_{n-k}] \subset G_n.$$

Therefore $[G_k, G_{n-k+1}]$ is an ideal of G_n for each $1 \leq k \leq n$. Thus each summand of G_{n+1} is an ideal of G_n. By Propositions 2.6 and 2.11 we conclude that G_{n+1} is an ideal of G_n. $\square$

Since $(G_i, G_i) \subset G_{2i} \subset G_{i+1}$, each G_i/G_{i+1} has an abelian group structure. Let

$$LL_G = G_1/G_2 \oplus G_2/G_3 \oplus \cdots \oplus G_n/G_{n+1} \oplus \cdots, \tag{3.1}$$

where $\oplus$ denotes the direct sum of abelian groups.

Let k be a commutative ring with the unit, and A a k-module. We recall the definitions of Lie and Leibniz algebras.

Definition 3.3. A Lie algebra $(A, (\ ,\ ))$ over k is given by a k-module A and a k-module homomorphism $(\ ,\ ) : A \otimes_k A \longrightarrow A$ called a round bracket such that the equation

$$(x, x) = 0$$

and the Jacobi identity

$$(x, (y, z)) + (y, (z, x)) + (z, (x, y)) = 0 \tag{3.2}$$

hold for $x, y, z \in A$.

Let $\mathbb{Lie}$ be the category of Lie algebras. Morphisms in $\mathbb{Lie}$ are k-module homomorphisms φ with

$$\varphi(x, y) = (\varphi(x), \varphi(y)).$$

Definition 3.4 ([3])**.** A Leibniz algebra A over k is a k-module A equipped with a k-module homomorphism called a square bracket

$$[\,,\,] \,:\, A \otimes_k A \longrightarrow A,$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \tag{3.3}$$

for $x, y, z \in A$.

This is in fact a right Leibniz algebra. The dual notion of a left Leibniz algebra is made out of the dual relation

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

for $x, y, z \in A$.

A morphism of Leibniz algebras is a k-module homomorphism $f : A \longrightarrow A'$ with $\varphi[x, y] = [\varphi(x), \varphi(y)]$.

In this paper we deal with right Leibniz algebras. Denote this category by $\mathbb{Leibniz}$.

Definition 3.5. A Lie–Leibniz algebra is a k-module A together with two k-module homomorphisms

$$(\,,\,), [\,,\,] : \, A \otimes_k A \longrightarrow A$$

called round and square brackets, respectively, such that $(x, x) = 0$ for $x \in A$ and both Jacobi and Leibniz identities ((3.2) and (3.3)) hold.

A morphism of Lie–Leibniz algebras is a k-module homomorphism $\varphi : A \longrightarrow A'$ with

$$\varphi(x, y) = (\varphi(x), \varphi(y)),$$
$$\varphi[x, y] = [\varphi(x), \varphi(y)].$$

We denote the corresponding category by $\mathbb{LL}$.

Condition 1. For each $x, y, z \in G$, $G \in \mathbb{Gr}^\bullet$

$$x - x^{(z^x)} + x^{y+z^x} - x + x^z - x^{z+y^z} = 0.$$

It is straightforward to verify that if G satisfies Condition 1, then the group $G^{[\,]}$, which corresponds to G (i.e. $[\,,\,]$ is defined by $[g, h] = -g + g^h$, $g, h \in G$) satisfies the following condition.

Condition 1′.

$$[x^y, [y, z]] = [[x, y], z^x] + [-[x, z], y^z], \quad x, y, z \in G^{[\,]}.$$

Let G be a group. Consider G as a group with the (right) action by conjugation, i.e. $g^{g'} = -g' + g + g'$. Then G satisfies Condition 1 and in this case Condition 1′ is equivalent to the Witt–Hall identity for groups. Each group with the trivial action on itself (i.e. $g^{g'} = g$, $g, g' \in G$) also satisfies Condition 1. For an arbitrary set X let $\mathscr{F}_X$ be the free group with action on itself generated by X. The quotient $\mathscr{F}_X/_\sim$ of $\mathscr{F}_X$ by the equivalence relation generated by the relation corresponding to Condition 1 is obviously a group which satisfies Condition 1. See also an example at the end of the proof of Theorem 3.6.

Denote by $\mathbb{Gr}^{\mathrm{c}}$ a category of groups with action on itself satisfying Condition 1. In an analogous way we define the category $\mathbb{Ab}^{\mathrm{c}}$. It is easy to see that the functors E, A, T, C, Q_1, Q_2, defined in Section 1, give the functors between categories $\mathbb{Ab}^{\mathrm{c}}$, $\mathbb{Gr}^{\mathrm{c}}$ and $\mathbb{Gr}$. We shall denote below these functors by the same letters. $\mathscr{F}_X/_\sim$ is a free object in $\mathbb{Gr}^{\mathrm{c}}$ and consequently the action in it is neither the trivial one nor the conjugation.

Let $G \in \mathbb{Gr}^c$. Denote $\overline{G}_m = G_m/G_{m+1}$, then $LL_G = \sum\limits_{m \geq 1} \overline{G}_m$.

Consider maps $(\ ,\ )_{mn}, [\ ,\ ]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ defined by round and square brackets in G, respectively:

$$\begin{aligned} x, y &\longmapsto (x, y), \\ x, y &\longmapsto [x, y]. \end{aligned}$$

By the definition of G_i, it is clear that if $x \in G_m$, $y \in G_n$, then $(x, y), [x, y] \in G_{m+n}$. For $x \in G_m$, denote by $\overline{x}$ the corresponding class in $\overline{G}_m$.

Theorem 3.6. *Let G be a group with action on itself satisfying Condition* 1. *Then we have:*

(a) $\overline{x^y} = \overline{x}$, $\overline{-y + x + y} = \overline{x}$, *for each* $x \in G_m$, $y \in G_n$;

(b) *The maps* $(\ ,\ )_{mn}$ *and* $[\ ,\ ]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ *induce bilinear maps* $\alpha_{mn}, \beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$;

(c) *The maps* α_{mn}, β_{mn}, $m, n \geq 1$ *define bilinear maps* $(\ ,\ ), [\ ,\ ] : LL_G \times LL_G \longrightarrow LL_G$, *which give a Lie–Leibniz structure on* LL_G.

Proof. (a) Let $x \in G_m$, $y \in G_n$, $m, n \geq 1$. Then $[x, y] = -x + x^y \in G_{m+n} \subset G_m$ and since $x \in G_m$ we obtain that $x^y \in G_m$. In $\overline{G}_m$ we have $\overline{[x, y]} = -\overline{x} + \overline{x^y}$, but since $[x, y] \subset G_{m+n} \subset G_{m+1}$ we have $\overline{[x, y]} = 0$ in $\overline{G}_m$ and thus in $\overline{G}_m$ we have $\overline{x} = \overline{x^y}$. In the same way we show for the action with conjugation that $\overline{-y + x + y} = \overline{x}$. (see also [5]).

(b) We shall check this condition for a square bracket; for a round bracket the proof is similar [5]. First we shall show that the map $\beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$ is defined correctly. Let $\overline{x} \in \overline{G}_m$, $\overline{y} \in \overline{G}_n$, where $x \in G_m$, $y \in G_n$. By definition, $\beta_{mn}(\overline{x}, \overline{y}) = [\overline{x}, \overline{y}] = \overline{[x, y]}$, where $[x, y] \in G_{n+m}$. Let $\overline{x} = \overline{x'}$ for $x' \in G_m$, thus $x - x' \in G_{m+1}$. For simplicity, suppose that $x - x' \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$ (a more general case is treated similarly). Then $x = [a, b] + x'$, where $a \in G_{i+1}$, $b \in G_{m-i}$. From this we have in $\overline{G}_{m+n}$:

$$\begin{aligned} \overline{[x, y]} = \overline{[[a, b] + x', y]} &= \overline{-x' + [[a, b], y] + x' + [x', y]} \\ &= \overline{-x' + [[a, b], y] + x'} + \overline{[x', y]}. \end{aligned} \tag{3.4}$$

$[[a,b],y] \in G_{m+n+1} \subset G_{m+n}$. Applying the condition (a), we obtain

$$\overline{-x' + [[a,b],y] + x'} = \overline{[[a,b],y]} = 0 \quad \text{in} \quad \overline{G}_{m+n}.$$

Thus from (3.4) we have $\overline{[x,y]} = \overline{[x',y]}$. If $x - x' = (a,b) \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$, then by the same argument we have

$$\begin{aligned} \overline{[x,y]} &= \overline{[x' + (a,b), y]} = \overline{-x' + [(a,b),y] + x'} + \overline{[x',y]} \\ &= \overline{[(a,b),y]} + \overline{[x',y]} = \overline{[x',y]}, \quad \text{since} \quad \overline{[(a,b),y]} = 0 \ \text{in} \ \overline{G}_{m+n}. \end{aligned}$$

The correctness of β_{mn} for the second argument is proved in an analogous way.

Now we shall show that the maps β_{mn} are bilinear. Let $\overline{x}_1, \overline{x}_2 \in \overline{G}_m$ and $\overline{y} \in \overline{G}_n$. We have in $\overline{G}_{m+n}$

$$\begin{aligned} [\overline{x}_1 + \overline{x}_2, \overline{y}] = \overline{[x_1 + x_2, y]} &= \overline{-x_2 + [x_1,y] + x_2} \\ &+ \overline{[x_2,y]} = \overline{[x_1,y]} + \overline{[x_2,y]}; \end{aligned}$$

here we again apply the condition (a). Let $\overline{x} \in \overline{G}_m$ and $\overline{y}_1, \overline{y}_2 \in \overline{G}_n$. We have in $\overline{G}_{m+n}$

$$\begin{aligned} [\overline{x}, \overline{y}_1 + \overline{y}_2] &= \overline{[x, y_1 + y_2]} = \overline{[x,y_1] + [x^{y_1}, y_2]} \\ = \overline{[x,y_1]} + \overline{[x^{y_1}, y_2]} &= [\overline{x}, \overline{y}_1] + [\overline{x^{y_1}}, \overline{y}_2] = [\overline{x}, \overline{y}_1] + [\overline{x}, \overline{y}_2], \end{aligned}$$

since, by the condition (a) $\overline{x^{y_1}} = \overline{x}$. This proves that maps β_{mn} are bilinear.

(c) The maps α_{mn}, β_{mn} can be continued linearly in a natural way up to the bilinear maps $(\ ,\ ), [\ ,\ ] : LL_G \times LL_G \longrightarrow LL_G$. The proof of the fact that $(\ ,\ )$ satisfies the condition (3.2) and $(l,l) = 0$ for any $l \in LL_G$ is similar to the proof of the corresponding statement in Witt's theorem (see [5], Proposition 2.3; [6]). It remains to show that the square bracket operation $[\ ,\ ]$ satisfies the Leibniz identity (3.3).

The object G satisfies Condition 1, therefore we have Condition 1$'$ for the square bracket in G. Since the square bracket operation in LL_G is linear for both arguments, we can limit ourself to the case where $\overline{x} \in G_m$, $\overline{y} \in \overline{G}_n$, $\overline{z} \in \overline{G}_t$. Applying the conditions (a) and (b) of the theorem we have

$$\begin{aligned} [\overline{x}, [\overline{y}, \overline{z}]] &= [\overline{x^y}, [\overline{y}, \overline{z}]] = \overline{[x^y, [y,z]]}; \\ [[\overline{x}, \overline{y}], \overline{z}] &= [[\overline{x}, \overline{y}], \overline{z^x}] = \overline{[[x,y], z^x]}; \\ -[[\overline{x}, \overline{z}], \overline{y}] &= [-[\overline{x}, \overline{z}], \overline{y^z}] = \overline{[-[x,z], y^z]}. \end{aligned}$$

By Condition 1$'$ we obtain

$$[\overline{x}, [\overline{y}, \overline{z}]] = [[\overline{x}, \overline{y}], \overline{z}] - [[\overline{x}, \overline{z}], \overline{y}] \ \text{in} \ \overline{G_{m+n+t}},$$

which completes the proof of the theorem. $\square$

The following example is due to the referee.

Example. Let G be the abelian group of integers $\mathbb{Z}^\bullet$, which acts on itself in the following way: $x^y = (-1)^y x$. We have $[x,y] = 0$ for y even, $[x,y] = -2x$ for y odd and $G_n = 2^{n-1}\mathbb{Z}^\bullet$. It is easy to see that $\mathbb{Z}^\bullet \in \mathbb{G}\mathrm{r}^{\mathrm{c}}$ and $LL_{\mathbb{Z}^\bullet}$ is a free

Leibniz algebra generated by a single element over a two element field (see also [4]).

It is easy to see that by Theorem 3.6 we have actually constructed the functor $LL : \mathbb{Gr}^{\mathrm{c}} \longrightarrow \mathbb{LL}$. In an analogous way one can construct the functor $L : \mathbb{Ab}^{\mathrm{c}} \longrightarrow \mathbb{Leibniz}$. For $A \in \mathbb{LL}$ let $S_1(A)$ denote the greatest quotient algebra of A which makes square bracket in A trivial. Then $S_1(A) \in \mathbb{Lie}$ and we have a functor $S_1 : \mathbb{LL} \longrightarrow \mathbb{Lie}$. Similarly, we construct the functor $S_2 : \mathbb{LL} \longrightarrow \mathbb{Leibniz}$. S_1 and S_2 are left adjoints to the embedding functors E_1 and E_2 respectively. Denote by $W : \mathbb{Gr} \longrightarrow \mathbb{Lie}$ the functor defined by Witt's theorem [6], [5]. Thus we have the following functors between the well defined categories:

$$\begin{tikzcd}
\mathbb{Ab}^{\mathrm{c}} \arrow[r, shift left, "E"] \arrow[d, "L"'] & \mathbb{Gr}^{\mathrm{c}} \arrow[l, shift left, "A"] \arrow[r, shift left=3, "Q_1"] \arrow[r, shift right=1, "Q_2"] \arrow[d, "LL"] & \mathbb{Gr} \arrow[l, shift left=1, "T"'] \arrow[l, shift right=3, "C"] \arrow[d, "W"] \\
\mathbb{Leibniz} \arrow[r, shift left, "E_2"] & \mathbb{LL} \arrow[l, shift left, "S_2"] \arrow[r, shift right, "S_1"'] & \mathbb{Lie}, \arrow[l, shift right, "E_1"']
\end{tikzcd}$$

where $LL \circ C = E_1 \circ W$, $E_2 \circ L = LL \circ E$. A more detailed account of this diagram will be given in the forthcoming paper, where free objects in $\mathbb{Gr}^{\bullet}$ and free Leibniz algebras are studied.

Acknowledgment

I would like to express my gratitude to J.-L. Loday who suggested the consideration of this problem. The work was partially supported by the grants: PICS CNRS Franco-Georgien de Mathematiques 542, TMR-Network "K-Theory and algebraic groups" ERB FMRX CT-97-0107 and INTAS-97-31961.

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(Received 4.06.2002)

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