

CENTRAL SERIES FOR GROUPS WITH ACTION AND LEIBNIZ ALGEBRAS

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Abstract. The notion of central series for groups with action on itself is introduced. An analogue of Witt's construction is given for such groups. A certain condition is found for the action and the corresponding category is defined. It is proved that the above construction defines a functor from this category to the category of Lie–Leibniz algebras and in particular to Leibniz algebras; also the restriction of this functor on the category of groups leads us to Lie algebras and gives the result of Witt.

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INTRODUCTION

The well-known construction of Witt defines a functor from the category of groups to the category of Lie algebras [6], [5]. The aim of this paper is to define a category and to give an analogue of Witt's construction for its objects, which will lead us to the category of Leibniz algebras. This problem was stated by J.-L. Loday; later an analogous question for the possibly defined partial Leibniz algebras was proposed, which was inspired by the work of Baues and Conduché [1]. Since the main interest lies in the absolute case, the author decided to begin with this one.

In Section 1 we define the category of groups with action on itself Gr^\bullet , the category of abelian groups with action on itself Ab^\bullet and the category of groups with bracket operation $\text{Gr}^{[1]}$. This kind of groups are Ω -groups in the sense of [2]. We construct adjoint pairs of functors relating categories Gr^\bullet , Ab^\bullet , $\text{Gr}^{[1]}$, Gr . In Section 2 we define ideals and commutators for the objects of Gr^\bullet (similarly for $\text{Gr}^{[1]}$) and show that these notions are equivalent to the special case of the known definitions for Ω -groups [2]. In Section 3 we define central series of groups with action on itself and a category of Lie–Leibniz algebras LL . We consider the category of groups with action on itself satisfying a certain condition Gr^c . We give an analogue of Witt's construction [6] and prove that it defines a functor $LL : \text{Gr}^c \rightarrow \text{LL}$, in particular this gives a functor $\text{Gr}^c \rightarrow \text{Leibniz}$. In a similar way one can construct a functor $\text{Ab}^c \rightarrow \text{Leibniz}$, which is actually the restriction of LL on Ab^c . The functorial relations with the classical situation ($\text{Gr} \rightarrow \text{Lie}$) is considered, namely by the restriction of LL on Gr we obtain the result of Witt [6], [5].

1. GROUPS WITH ACTION ON ITSELF

Let G be a group which acts on itself from the right side; i.e. we have a map $\varepsilon : G \times G \rightarrow G$ with

$$\begin{aligned} \varepsilon(g, g' + g'') &= \varepsilon(\varepsilon(g, g'), g''), \\ \varepsilon(g, 0) &= g, \\ \varepsilon(g' + g'', g) &= \varepsilon(g', g) + \varepsilon(g'', g), \\ \varepsilon(0, g) &= 0, \end{aligned} \tag{1.1}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$. We denote the group operation additively, nevertheless the group is not commutative in general. If (G', ε') is another group with action, then a homomorphism $(G, \varepsilon) \rightarrow (G', \varepsilon')$ is a group homomorphism $\varphi : G \rightarrow G'$ for which the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\varepsilon} & G \\ (\varphi, \varphi) \downarrow & & \downarrow \varphi \\ G' \times G' & \xrightarrow{\varepsilon'} & G' \end{array}$$

commutes. In other words, we have

$$\varphi(g^h) = \varphi(g)^{\varphi(h)}, \quad g, h \in G.$$

If we consider an action as a group homomorphism $G \xrightarrow{\nu} \text{Aut } G$, then a homomorphism between two groups with action means the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\nu} & \text{Aut } G \subset \text{Hom}(G, G) \\ \varphi \downarrow & & \downarrow \text{Hom}(G, \varphi) \\ & & \text{Hom}(G, G') \\ & & \uparrow \text{Hom}(\varphi, G') \\ G' & \xrightarrow{\nu'} & \text{Aut } G' \subset \text{Hom}(G', G') \end{array}$$

so that $\varphi \cdot (\nu(h)) = \nu'(\varphi(h)) \cdot \varphi$, $h \in G$.

Recall [2] that an Ω -group is a group with a system of n -ary algebraic operations Ω ($n \geq 1$), which satisfies the condition

$$00 \cdots 0\omega = 0, \tag{1.2}$$

where 0 is the identity element of G , and 0 on the left side occurs n times if ω is an n -ary operation. In special cases Ω -groups give groups, rings and groups with action on itself. In the latter case Ω consists of one binary operation, an action; or Ω consists of only unary operations, elements of G , and this operation is an action again. In both cases the condition (1.2) is satisfied. We shall denote the category of groups with action on itself by Gr^\bullet . Let Ab^\bullet denote the category

of abelian groups with action on itself. We have functors

$$\text{Ab}^\bullet \begin{array}{c} \xrightarrow{E} \\ \xleftarrow{A} \end{array} \text{Gr}^\bullet \begin{array}{c} \xrightarrow{Q_1} \\ \xleftarrow{T} \\ \xrightarrow{Q_2} \\ \xleftarrow{C} \end{array} \text{Gr},$$

where $Q_1(G)$, for $G \in \text{Gr}^\bullet$, is the greatest quotient group of G which makes the action trivial; $Q_2(G)$ is a quotient of G by the equivalence relation generated by the relation $g^h \sim -h + g + h$, $g, h \in G$; A is the abelianization functor, thus $A(G) = G/(G, G)$, where (G, G) is the ideal of G generated by the commutator normal subgroup of G (for the definition of an ideal see Section 2). $A(G)$ has the induced operation of action on itself. Each group can be considered as a group with the trivial action or with the action by conjugation, these give functors T and C , respectively. Every object of Ab^\bullet can be considered as an object of Gr^\bullet ; this functor is denoted by E . It is easy to see that the functors Q_1, Q_2 and A are left adjoints to the functors T, C and E respectively. Let $G \in \text{Gr}^\bullet$. Define the operation of square brackets $[,] : G \times G \rightarrow G$ on G by

$$[g, h] = -g + g^h, \quad g, h \in G.$$

Proposition 1.1. *For the operation $[,]$ we have the following identities:*

- (i) $[g, h_1 + h_2] = [g, h_1] + [g, h_2]$;
- (ii) $[g + g', h] = -g' + [g, h] + g' + [g', h]$;
- (iii) $[g, 0] = [0, g] = 0$.

Proof. These identities follow directly from (1.1). □

Corollary 1.2. *For $g, h \in G$*

$$\begin{aligned} [g^h, -h] &= -[g, h]; \\ [-g, h] &= g - [g, h] - g. \end{aligned}$$

Denote by $\text{Gr}^{[]}$ the category of groups with an additional bracket operation $[,]$ satisfying the conditions (i)–(iii) of Proposition 1.1; morphisms of $\text{Gr}^{[]}$ are group homomorphisms preserving the bracket operation. We shall denote the objects of $\text{Gr}^{[]}$ by $G^{[]}$.

Conversely, if $G^{[]} \in \text{Gr}^{[]}$, we can define an action of $G^{[]}$ on itself due to the bracket operation by

$$g^h = g + [g, h], \quad g, h \in G^{[]}.$$

It is easy to prove that these two procedures are converse to each other and actually we have an isomorphism of categories

$$\text{Gr}^\bullet \approx \text{Gr}^{[]}.$$

2. IDEALS AND COMMUTATORS IN Gr^\bullet

Let $G \in \text{Gr}^\bullet$.

Definition 2.1. A nonempty subset A of G is called an ideal of G if it satisfies the following conditions:

1. A is a normal subgroup of G as a group;
2. $a^g \in A$, for $a \in A, g \in G$;

3. $-g + g^a \in A$, for $a \in A$ and $g \in G$.

Definition 2.2 (Kurosh [2]). A nonempty subset A of an Ω -group G is called an ideal if

- (a) A is an additive normal subgroup of G ;
- (b) For any n -any operation ω from Ω , any element $a \in A$ and elements $x_1, x_2, \dots, x_n \in G$

$$-(x_1 \cdots x_n \omega) + x_1 \cdots x_{i-1}(a + x_i)x_{i+1} \cdots x_n \omega \in A,$$

for $i = 1, 2, \dots, n$.

This definition in the case of groups is the definition of a normal subgroup of a group, and in the case of rings is the definition of a two-sided ideal of a ring.

Proposition 2.3. For a group $G \in \mathbb{G}\mathbb{R}^\bullet$ considered as an Ω -group, where Ω consists of one binary operation of action, Definitions 2.1 and 2.2 are equivalent.

Proof. The condition (b) of Definition 2.2 has the forms:

$$-x_1^{x_2} + (a + x_1)^{x_2} \in A, \quad \text{for } i = 1; \quad (2.1)$$

$$-x_1^{x_2} + x_1^{a+x_2} \in A, \quad \text{for } i = 2. \quad (2.2)$$

Taking $x_1 = 0$ in (2.1), we obtain $a^{x_2} \in A$, which is condition 2 of Definition 2.1. Taking $x_2 = 0$ in (2.2), we have $-x_1 + x_1^a \in A$, which is condition 3 of Definition 2.1.

Conversely, we shall show that conditions 2 and 3 of Definition 2.1 imply conditions (2.1) and (2.2). From condition 2 we have $a^{x_2} \in A$; also

$$-x_1^{x_2} + (a + x_1)^{x_2} = -x_1^{x_2} + a^{x_2} + a_1^{x_2},$$

and it is an element of A since A is a normal subgroup of G . By condition 3 of Definition 2.1, $-x_1 + x_1^a \in A$. We have $-x_1^{x_2} + x_1^{a+x_2} = (-x_1 + x_1^a)^{x_2}$ and this is an element of A due to condition 2, which ends the proof. \square

Thus an ideal of G is a subobject of G in $\mathbb{G}\mathbb{R}^\bullet$. It is clear that G itself and the trivial subobject of G are ideals of G . An intersection of any system of ideals of G is an ideal, and therefore we conclude that there exists the ideal generated by a system of elements of G .

Proposition 2.4. Let A be an ideal of G . For $a_1, a_2 \in A$, $g_1, g_2 \in G$ we have

$$(a_1 + g_1)^{a_2+g_2} \in g_1^{g_2} + A.$$

Proof. Since A is an ideal of G there exist $a'_1, a'_2 \in A$, such that $a_1 + g_1 = g_1 + a'_1$, $a_2 + g_2 = g_2 + a'_2$. Therefore

$$\begin{aligned} (a_1 + g_1)^{a_2+g_2} &= (g_1 + a'_1)^{g_2+a'_2} = (g_1^{g_2})^{a'_2} + a'_1^{g_2+a'_2} \\ &= g_1^{g_2} - g_1^{g_2} + (g_1^{g_2})^{a'_2} + a'_1^{g_2+a'_2} \in g_1^{g_2} + A; \end{aligned}$$

here we apply $-g_1^{g_2} + (g_1^{g_2})^{a'_2} \in A$. \square

Let A and B be subobjects of G . Denote by $\{A, B\}$ the subobject of G generated by A and B , and let $A + B$ denote the subset of G

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Proposition 2.5. *If A is an ideal of G and B is a subobject of G , then*

$$\{A, B\} = A + B.$$

Proof. It is obvious that $A + B \subset \{A, B\}$. Since A is an ideal, it follows that $A + B$ is a subgroup of G . By Proposition 2.4, $(a_1 + b_1)^{a_2 + b_2} \in b_1^{b_2} + A$. Since B is a subobject, $b_1^{b_2} \in B$, and since A is an ideal, $b_1^{b_2} + A = A + b_1^{b_2} \in A + B$ which ends the proof. \square

For Ω -groups see Propositions 2.4 and 2.5 in [2].

Proposition 2.6. *If A and B are ideals of G , then $A + B$ is also an ideal.*

Proof. For $g \in G$, $a \in A$ and $b \in B$ we have

$$g + (a + b) = (a' + g) + b = a' + b' + g \in A + B + g,$$

for certain $a' \in A$ and $b' \in B$. Thus $g + (A + B) \subset (A + B) + g$. In the same way we show that $(A + B) + g \subset g + (A + B)$ and thus $g + (A + B) = (A + B) + g$. It is obvious that $(a + b)^g \in A + B$. Now we have to show that $-g + g^{a+b} \in A + B$. We have

$$-g + g^{a+b} = -g + g^a - g^a + (g^a)^b \in A + B$$

since $-g + g^a \in A$, $-g^a + (g^a)^b \in B$. \square

It is easy to verify that the ideal generated by a system of ideals of G coincides with the additive subgroup of G generated by these ideals. For Ω -groups see [2].

Definition 2.1'. Let $G^{[1]} \in \mathbb{Gr}^{[1]}$ and A be a nonempty subset of $G^{[1]}$. A is called an ideal of $G^{[1]}$ if

- 1'. A is a normal subgroup of $G^{[1]}$ as of an additive group;
- 2'. $[a, g] \in A$, for $a \in A$, $g \in G^{[1]}$;
- 3'. $[g, a] \in A$, for $a \in A$, $g \in G^{[1]}$.

It is easy to see that the isomorphism of categories $\mathbb{Gr}^\bullet \approx \mathbb{Gr}^{[1]}$ carries ideals to ideals.

Proposition 2.7. *If A is an ideal of G , then the quotient group G/A with the induced action on itself is an object of \mathbb{Gr}^\bullet .*

Proof. Straightforward verification. \square

In what follows, for $G \in \mathbb{Gr}^\bullet$ and $g, g' \in G$, $[g, g']$ will indicate the element $-g + g^{g'}$ of G and (g, g') the commutator $-g - g' + g + g'$. Let A and B be subobjects of G .

Definition 2.8. A commutator $[A, B]$ of G generated by A and B is the ideal of $\{A, B\}$ generated by the elements

$$\{[a, b], [b, a], (a, b) \mid a \in A, b \in B\}.$$

Definition 2.9 ([2]). Let G be an Ω -group, A, B be Ω -subgroups of G and $\{A, B\}_\Omega$ be the Ω -subgroup of G generated by A and B . The commutator $[A, B]_\Omega$ is the ideal of $\{A, B\}_\Omega$ generated by elements of the form

$$(a, b) = -a - b + a + b, \quad a \in A, \quad b \in B,$$

and

$$\begin{aligned} [a_1, \dots, a_n; b_1, \dots, b_n; \omega] &= -a_1 a_2 \cdots a_n \omega - b_1 b_2 \cdots b_n \omega \\ &+ (a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) \omega, \end{aligned} \quad (2.3)$$

where ω is an n -any operation from Ω , $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$.

If G is a group with the trivial action on itself or with the action by conjugation, then $[A, B]$ in Definition 2.8 is the normal subgroup of G generated in $\{A, B\}$ by commutators (a, b) , $a \in A$, $b \in B$, i. e. the usual commutator for the case of groups. The same is true for Definition 2.9; if an Ω -group is a group without multioperations, then the commutator $[A, B]_\Omega$ is the usual commutator (A, B) of a group [2].

Proposition 2.10. *In the case of groups with action on itself Definitions 2.8 and 2.9 are equivalent.*

Proof. For groups with action (2.3) has the form

$$-a^{a_2} - b_1^{b_2} + (a_1 + b_1)^{a_2 + b_2}. \quad (2.4)$$

Take $a_1 = a$, $a_2 = b_1 = 0$, $b_2 = b$, then $-a + a^b \in [A, B]_\Omega$. Take in (2.4) $a_1 = b_2 = 0$, $a_2 = a$, $b_1 = b$, then we obtain

$$-b + b^a \in [A, B]_\Omega.$$

Thus we have shown that $[A, B] \subset [A, B]_\Omega$. Conversely, for $x = -a_1^{a_2} - b_1^{b_2} + (a_1 + b_1)^{a_2 + b_2} \in [A, B]_\Omega$ we have $x = -a_1^{a_2} - b_1^{b_2} + (a_1^{a_2})^{b_2} + (b_1^{a_2})^{b_2} \in \{A, B\}$. Let $\overline{\{A, B\}} = \{A, B\} / [A, B]$ and let \bar{g} be the class of the element $g \in \{A, B\}$ in $\overline{\{A, B\}}$. We have $\overline{a^b} = \bar{a}$, $\overline{b^a} = \bar{b}$ in $\overline{\{A, B\}}$. Thus

$$\begin{aligned} \bar{x} &= \overline{-a_1^{a_2} - b_1^{b_2} + (a_1^{a_2})^{b_2} + (b_1^{a_2})^{b_2}} = \overline{-a_1^{a_2}} - \overline{b_1^{b_2}} + \overline{a_1^{a_2}} + \overline{b_1^{a_2}} \\ &= \overline{-a_1^{a_2}} - \overline{b_1^{b_2}} + \overline{a_1^{a_2}} + \overline{b_1^{b_2}} = \overline{-a_1^{a_2} - b_1^{b_2} + a_1^{a_2} + b_1^{b_2}} = 0, \end{aligned}$$

which means that $x \in [A, B]$. □

Below we formulate without proofs two statements for Ω -groups from [2], which in the case of groups with action give the corresponding results.

Proposition 2.11. *For any Ω -subgroups A and B in G we have*

$$[A, B]_\Omega = [B, A]_\Omega.$$

Proposition 2.12. *An Ω -subgroup A is an ideal of G if and only if*

$$[A, G]_\Omega \subseteq A.$$

Corollary 2.13. *Any Ω -subgroup A of an Ω -group G which contains the commutator $[G, G]_\Omega$ is an ideal of G .*

Proof. It follows from the inclusions

$$[A, G]_{\Omega} \subset [G, G]_{\Omega} \subset A. \quad \square$$

3. CENTRAL SERIES IN $\mathbb{G}\mathbb{r}^{\bullet}$ AND THE MAIN RESULT

Let $G \in \mathbb{G}\mathbb{r}^{\bullet}$.

Definition 3.1. The (lower) central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$$

of the object G is defined inductively by

$$G_n = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots + [G_{n-1}, G_1].$$

By definition, we have $[G_n, G_m] \subset G_{n+m}$.

Proposition 3.2. For each $n \geq 1$, G_{n+1} is an ideal of G_n .

Proof. We have $G_2 = [G_1, G_1]$, which is an ideal of G_1 , by definition. $G_3 = [G_1, G_2] + [G_2, G_1]$. By Proposition 2.11, $[G_1, G_2] = [G_2, G_1]$. We have

$$[G_1, G_2] \subset [G_1, G_1] = G_2 \subset \{G_1, G_2\}$$

and $[G_1, G_2]$ is an ideal of $\{G_1, G_2\}$; from this it follows that $[G_1, G_2]$ is an ideal of G_2 and therefore, by Proposition 2.6, G_3 is an ideal of G_2 . We have

$$G_{n+1} = [G_1, G_n] + [G_2, G_{n-1}] + \cdots + [G_{n-1}, G_2] + [G_n, G_1].$$

For $1 \leq k \leq n$, $[G_k, G_{n-k+1}]$ is an ideal of $\{G_k, G_{n-k+1}\}$; $G_n \subseteq G_k$ from which it follows that $G_n \subseteq \{G_k, G_{n-k+1}\}$. At the same time

$$[G_k, G_{n-k+1}] \subset [G_k, G_{n-k}] \subset G_n.$$

Therefore $[G_k, G_{n-k+1}]$ is an ideal of G_n for each $1 \leq k \leq n$. Thus each summand of G_{n+1} is an ideal of G_n . By Propositions 2.6 and 2.11 we conclude that G_{n+1} is an ideal of G_n . □

Since $(G_i, G_i) \subset G_{2i} \subset G_{i+1}$, each G_i/G_{i+1} has an abelian group structure. Let

$$LL_G = G_1/G_2 \oplus G_2/G_3 \oplus \cdots \oplus G_n/G_{n+1} \oplus \cdots, \tag{3.1}$$

where \oplus denotes the direct sum of abelian groups.

Let k be a commutative ring with the unit, and A a k -module. We recall the definitions of Lie and Leibniz algebras.

Definition 3.3. A Lie algebra $(A, (,))$ over k is given by a k -module A and a k -module homomorphism $(,) : A \otimes_k A \rightarrow A$ called a round bracket such that the equation

$$(x, x) = 0$$

and the Jacobi identity

$$(x, (y, z)) + (y, (z, x)) + (z, (x, y)) = 0 \tag{3.2}$$

hold for $x, y, z \in A$.

Let $\mathbb{L}ie$ be the category of Lie algebras. Morphisms in $\mathbb{L}ie$ are k -module homomorphisms φ with

$$\varphi(x, y) = (\varphi(x), \varphi(y)).$$

Definition 3.4 ([3]). A Leibniz algebra A over k is a k -module A equipped with a k -module homomorphism called a square bracket

$$[,] : A \otimes_k A \longrightarrow A,$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad (3.3)$$

for $x, y, z \in A$.

This is in fact a right Leibniz algebra. The dual notion of a left Leibniz algebra is made out of the dual relation

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

for $x, y, z \in A$.

A morphism of Leibniz algebras is a k -module homomorphism $f : A \longrightarrow A'$ with $\varphi[x, y] = [\varphi(x), \varphi(y)]$.

In this paper we deal with right Leibniz algebras. Denote this category by $\mathbb{L}eibniz$.

Definition 3.5. A Lie–Leibniz algebra is a k -module A together with two k -module homomorphisms

$$(\ , \), [,] : A \otimes_k A \longrightarrow A$$

called round and square brackets, respectively, such that $(x, x) = 0$ for $x \in A$ and both Jacobi and Leibniz identities ((3.2) and (3.3)) hold.

A morphism of Lie–Leibniz algebras is a k -module homomorphism $\varphi : A \longrightarrow A'$ with

$$\begin{aligned} \varphi(x, y) &= (\varphi(x), \varphi(y)), \\ \varphi[x, y] &= [\varphi(x), \varphi(y)]. \end{aligned}$$

We denote the corresponding category by $\mathbb{L}L$.

Condition 1. For each $x, y, z \in G$, $G \in \mathbb{G}r^\bullet$

$$x - x^{(z^x)} + x^{y+z^x} - x + x^z - x^{z+y^z} = 0.$$

It is straightforward to verify that if G satisfies Condition 1, then the group $G^{[1]}$, which corresponds to G (i.e. $[\ ,]$ is defined by $[g, h] = -g + g^h$, $g, h \in G$) satisfies the following condition.

Condition 1'.

$$[x^y, [y, z]] = [[x, y], z^x] + [-[x, z], y^z], \quad x, y, z \in G^{[1]}.$$

Let G be a group. Consider G as a group with the (right) action by conjugation, i.e. $g^{g'} = -g' + g + g'$. Then G satisfies Condition 1 and in this case Condition 1' is equivalent to the Witt–Hall identity for groups. Each group with

the trivial action on itself (i.e. $g^{g'} = g, g, g' \in G$) also satisfies Condition 1. For an arbitrary set X let \mathcal{F}_X be the free group with action on itself generated by X . The quotient \mathcal{F}_X/\sim of \mathcal{F}_X by the equivalence relation generated by the relation corresponding to Condition 1 is obviously a group which satisfies Condition 1. See also an example at the end of the proof of Theorem 3.6.

Denote by $\mathbb{G}\mathbb{r}^c$ a category of groups with action on itself satisfying Condition 1. In an analogous way we define the category $\mathbb{A}\mathbb{b}^c$. It is easy to see that the functors E, A, T, C, Q_1, Q_2 , defined in Section 1, give the functors between categories $\mathbb{A}\mathbb{b}^c, \mathbb{G}\mathbb{r}^c$ and $\mathbb{G}\mathbb{r}$. We shall denote below these functors by the same letters. \mathcal{F}_X/\sim is a free object in $\mathbb{G}\mathbb{r}^c$ and consequently the action in it is neither the trivial one nor the conjugation.

Let $G \in \mathbb{G}\mathbb{r}^c$. Denote $\overline{G}_m = G_m/G_{m+1}$, then $LL_G = \sum_{m \geq 1} \overline{G}_m$.

Consider maps $(,)_{mn}, [,]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ defined by round and square brackets in G , respectively:

$$\begin{aligned} x, y &\longmapsto (x, y), \\ x, y &\longmapsto [x, y]. \end{aligned}$$

By the definition of G_i , it is clear that if $x \in G_m, y \in G_n$, then $(x, y), [x, y] \in G_{m+n}$. For $x \in G_m$, denote by \overline{x} the corresponding class in \overline{G}_m .

Theorem 3.6. *Let G be a group with action on itself satisfying Condition 1. Then we have:*

- (a) $\overline{x^y} = \overline{x}, \overline{-y + x + y} = \overline{x}$, for each $x \in G_m, y \in G_n$;
- (b) The maps $(,)_{mn}$ and $[,]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ induce bilinear maps $\alpha_{mn}, \beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$;
- (c) The maps $\alpha_{mn}, \beta_{mn}, m, n \geq 1$ define bilinear maps $(,), [,] : LL_G \times LL_G \longrightarrow LL_G$, which give a Lie-Leibniz structure on LL_G .

Proof. (a) Let $x \in G_m, y \in G_n, m, n \geq 1$. Then $[x, y] = -x + x^y \in G_{m+n} \subset G_m$ and since $x \in G_m$ we obtain that $x^y \in G_m$. In \overline{G}_m we have $\overline{[x, y]} = \overline{-x + x^y}$, but since $[x, y] \in G_{m+n} \subset G_{m+1}$ we have $\overline{[x, y]} = 0$ in \overline{G}_m and thus in \overline{G}_m we have $\overline{x} = \overline{x^y}$. In the same way we show for the action with conjugation that $\overline{-y + x + y} = \overline{x}$. (see also [5]).

(b) We shall check this condition for a square bracket; for a round bracket the proof is similar [5]. First we shall show that the map $\beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$ is defined correctly. Let $\overline{x} \in \overline{G}_m, \overline{y} \in \overline{G}_n$, where $x \in G_m, y \in G_n$. By definition, $\beta_{mn}(\overline{x}, \overline{y}) = \overline{[x, y]}$, where $[x, y] \in G_{m+n}$. Let $\overline{x} = \overline{x'}$ for $x' \in G_m$, thus $x - x' \in G_{m+1}$. For simplicity, suppose that $x - x' \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$ (a more general case is treated similarly). Then $x = [a, b] + x'$, where $a \in G_{i+1}, b \in G_{m-i}$. From this we have in \overline{G}_{m+n} :

$$\begin{aligned} \overline{[x, y]} &= \overline{[[a, b] + x', y]} = \overline{-x' + [[a, b], y] + x' + [x', y]} \\ &= \overline{-x' + [[a, b], y] + x' + [x', y]}. \end{aligned} \tag{3.4}$$

$[[a, b], y] \in G_{m+n+1} \subset G_{m+n}$. Applying the condition (a), we obtain

$$\overline{-x' + [[a, b], y] + x'} = \overline{[[a, b], y]} = 0 \text{ in } \overline{G_{m+n}}.$$

Thus from (3.4) we have $\overline{[x, y]} = \overline{[x', y]}$. If $x - x' = (a, b) \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$, then by the same argument we have

$$\begin{aligned} \overline{[x, y]} &= \overline{[x' + (a, b), y]} = \overline{-x' + [(a, b), y] + x' + [x', y]} \\ &= \overline{[(a, b), y]} + \overline{[x', y]} = \overline{[x', y]}, \text{ since } \overline{[(a, b), y]} = 0 \text{ in } \overline{G_{m+n}}. \end{aligned}$$

The correctness of β_{mn} for the second argument is proved in an analogous way.

Now we shall show that the maps β_{mn} are bilinear. Let $\bar{x}_1, \bar{x}_2 \in \overline{G}_m$ and $\bar{y} \in \overline{G}_n$. We have in \overline{G}_{m+n}

$$\begin{aligned} \overline{[\bar{x}_1 + \bar{x}_2, \bar{y}]} &= \overline{[x_1 + x_2, y]} = \overline{-x_2 + [x_1, y] + x_2} \\ &\quad + \overline{[x_2, y]} = \overline{[x_1, y]} + \overline{[x_2, y]}, \end{aligned}$$

here we again apply the condition (a). Let $\bar{x} \in \overline{G}_m$ and $\bar{y}_1, \bar{y}_2 \in \overline{G}_n$. We have in \overline{G}_{m+n}

$$\begin{aligned} \overline{[\bar{x}, \bar{y}_1 + \bar{y}_2]} &= \overline{[x, y_1 + y_2]} = \overline{[x, y_1]} + \overline{[x^{y_1}, y_2]} \\ &= \overline{[x, y_1]} + \overline{[x^{y_1}, y_2]} = \overline{[\bar{x}, \bar{y}_1]} + \overline{[x^{y_1}, \bar{y}_2]} = \overline{[\bar{x}, \bar{y}_1]} + \overline{[\bar{x}, \bar{y}_2]}, \end{aligned}$$

since, by the condition (a) $\overline{x^{y_1}} = \bar{x}$. This proves that maps β_{mn} are bilinear.

(c) The maps α_{mn}, β_{mn} can be continued linearly in a natural way up to the bilinear maps $(,), [,] : LL_G \times LL_G \rightarrow LL_G$. The proof of the fact that $(,)$ satisfies the condition (3.2) and $(l, l) = 0$ for any $l \in LL_G$ is similar to the proof of the corresponding statement in Witt's theorem (see [5], Proposition 2.3; [6]). It remains to show that the square bracket operation $[,]$ satisfies the Leibniz identity (3.3).

The object G satisfies Condition 1, therefore we have Condition 1' for the square bracket in G . Since the square bracket operation in LL_G is linear for both arguments, we can limit ourself to the case where $\bar{x} \in G_m, \bar{y} \in \overline{G}_n, \bar{z} \in \overline{G}_t$. Applying the conditions (a) and (b) of the theorem we have

$$\begin{aligned} \overline{[\bar{x}, [\bar{y}, \bar{z}]]} &= \overline{[x^{\bar{y}}, [\bar{y}, \bar{z}]]} = \overline{[x^{\bar{y}}, [y, z]]}; \\ \overline{[[\bar{x}, \bar{y}], \bar{z}]} &= \overline{[[\bar{x}, \bar{y}], z^{\bar{x}}]} = \overline{[[x, y], z^x]}; \\ -\overline{[[\bar{x}, \bar{z}], \bar{y}]} &= \overline{[-[\bar{x}, \bar{z}], \bar{y}^{\bar{z}}]} = \overline{[-[x, z], y^z]}. \end{aligned}$$

By Condition 1' we obtain

$$\overline{[\bar{x}, [\bar{y}, \bar{z}]]} = \overline{[[\bar{x}, \bar{y}], \bar{z}]} - \overline{[[\bar{x}, \bar{z}], \bar{y}]} \text{ in } \overline{G_{m+n+t}},$$

which completes the proof of the theorem. □

The following example is due to the referee.

Example. Let G be the abelian group of integers \mathbb{Z}^\bullet , which acts on itself in the following way: $x^y = (-1)^y x$. We have $[x, y] = 0$ for y even, $[x, y] = -2x$ for y odd and $G_n = 2^{n-1} \mathbb{Z}^\bullet$. It is easy to see that $\mathbb{Z}^\bullet \in \text{Gr}^c$ and $LL_{\mathbb{Z}^\bullet}$ is a free

Leibniz algebra generated by a single element over a two element field (see also [4]).

It is easy to see that by Theorem 3.6 we have actually constructed the functor $LL : \text{Gr}^c \rightarrow \text{LL}$. In an analogous way one can construct the functor $L : \text{Ab}^c \rightarrow \text{Leibniz}$. For $A \in \text{LL}$ let $S_1(A)$ denote the greatest quotient algebra of A which makes square bracket in A trivial. Then $S_1(A) \in \text{Lie}$ and we have a functor $S_1 : \text{LL} \rightarrow \text{Lie}$. Similarly, we construct the functor $S_2 : \text{LL} \rightarrow \text{Leibniz}$. S_1 and S_2 are left adjoints to the embedding functors E_1 and E_2 respectively. Denote by $W : \text{Gr} \rightarrow \text{Lie}$ the functor defined by Witt's theorem [6], [5]. Thus we have the following functors between the well defined categories:

$$\begin{array}{ccccc}
 \text{Ab}^c & \begin{array}{c} \xrightarrow{E} \\ \xleftarrow{A} \end{array} & \text{Gr}^c & \begin{array}{c} \xrightarrow{\frac{Q_1}{T}} \\ \xleftarrow{\frac{Q_2}{C}} \end{array} & \text{Gr} \\
 L \downarrow & & \downarrow LL & & \downarrow W \\
 \text{Leibniz} & \begin{array}{c} \xrightarrow{E_2} \\ \xleftarrow{S_2} \end{array} & \text{LL} & \begin{array}{c} \xrightarrow{E_1} \\ \xleftarrow{S_1} \end{array} & \text{Lie},
 \end{array}$$

where $LL \circ C = E_1 \circ W$, $E_2 \circ L = LL \circ E$. A more detailed account of this diagram will be given in the forthcoming paper, where free objects in Gr^\bullet and free Leibniz algebras are studied.

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REFERENCES

1. H. J. BAUES and D. CONDUCHÉ, The central series for Peiffer commutators in groups with operators. *J. Algebra* **133**(1990), No. 1, 1–34.
2. A. G. KUROSH, Lectures in general algebra. (Russian) *Nauka, Moscow*, 1973.
3. J.-L. LODAY, Cyclic homology. *Grundlehren der mathematischen Wissenschaften*, 301, Springer, 1992.
4. J.-L. LODAY and T. PIRASHVILI, Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.* **296**(1993), 139–158.
5. J.-P. SERRE, Lie algebras and Lie groups. *W. A. Benjamin, INC. London, Amsterdam, Tokyo*, 1965.
6. E. WITT, Treue Darstellung Liescher Ringe. *J. Reine Angew. Math.* **1**(1937), 152–160.

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