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STEM EXTENSIONS AND STEM COVERS OF LEIBNIZ ALGEBRAS

J. M. CASAS AND M. LADRA

Dedicated to Hvedri Inassaridze on his 70th birthday

Abstract. Some properties on stem extensions and stem covers of Leibniz algebras are studied by means of exact sequences associated to a central extension of Leibniz algebras. We obtain particular results in case of perfect Leibniz algebras.

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1. Introduction

Leibniz algebras, introduced by Loday [12, 13], are a non skew-symmetric generalization of Lie algebras. In brief, a (right) Leibniz algebra over a field K is a K-vector space $\mathfrak{g}$ equipped with a bilinear map called the Leibniz bracket, $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the (right) Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, for all $x,y,z \in \mathfrak{g}$. If we assume that $[x,x] = 0$ for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is a K-Lie algebra and the Leibniz identity is the Jacobi identity.

Let $(\mathfrak{g}) : 0 \to \mathfrak{n} \xrightarrow{\chi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ be an extension of Leibniz algebras. If $(\mathfrak{g})$ is a central extension, that is, $[n,g] = 0 = [g,n], n \in \mathfrak{n}, g \in \mathfrak{g}$, then there exists the following natural exact sequence in Leibniz homology [13, 14] associated to $(\mathfrak{g})$:

$$\begin{aligned} HL_3(\mathfrak{g}) \to HL_3(\mathfrak{q}) \to \operatorname{Coker}(\tau) \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q}) \to \mathfrak{n} \\ \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{q}) \to 0, \end{aligned} \tag{1}$$

where the map $\tau : \mathfrak{n} \otimes \mathfrak{n} \to HL_1(\mathfrak{g}) \otimes \mathfrak{n} \oplus \mathfrak{n} \otimes HL_1(\mathfrak{g})$ is given by $\tau(a \otimes b) = (d(\chi(a)) \otimes b, -a \otimes d(\chi(b)))$ and $d : \mathfrak{g} \to HL_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the natural projection [4]. On the other hand, we have the natural isomorphism $\theta_\star : HL^2(\mathfrak{q},\mathfrak{n}) \xrightarrow{\sim} \operatorname{Hom}(HL_2(\mathfrak{q}),\mathfrak{n})$ obtained from Theorem 5.3 in [3] when $\mathfrak{n}$ is a trivial $\mathfrak{q}$-module.

If the morphism $\theta_\star(\mathfrak{g}) : HL_2(\mathfrak{q}) \to \mathfrak{n}$ in the sequence (1) is an epimorphism or an isomorphism, then $(\mathfrak{g})$ is called a stem extension (i.e., $\mathfrak{n} \subseteq [\mathfrak{g},\mathfrak{g}]$) and a stem cover (i.e., $HL_1(\mathfrak{g}) \xrightarrow{\sim} HL_1(\mathfrak{q})$ and $\pi_\star : HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q})$ is the zero map), respectively (see [2] for the characterization of these central extensions). These extensions in the category of groups were studied in [6, 9] and in the category of Lie algebras in [1, 8].

The goal of this paper is to use the sequence (1) and the isomorphism $\theta_\star$ together with the sequence (2) (see below) in order to obtain some properties of stem extensions and stem covers of Leibniz algebras. In the final section we

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consider the particular case of central extensions $(\mathfrak{g})$ in which $\mathfrak{q}$ is a perfect Leibniz algebra (i.e., $HL_1(\mathfrak{q}) = 0$).

2. HL^2 and Extensions

Let $(\mathfrak{g})$: $0 \to \mathfrak{n} \xrightarrow{\chi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ be an abelian extension of Leibniz algebras [3], then $(\mathfrak{g})$ induces a $\mathfrak{q}$-module (representation [13, 14]) structure on $\mathfrak{n}$ [3, 13, 14]. When $\mathfrak{n}$ is a (right) $\mathfrak{q}$-module, then it is said that $(\mathfrak{g})$ is a $\mathfrak{q}$-extension of $\mathfrak{n}$ if the $\mathfrak{q}$-module structure induced by $(\mathfrak{g})$ coincides with the previous one [3]. Two extensions $(\mathfrak{g})$ and $(\mathfrak{g}')$ of $\mathfrak{q}$ by $\mathfrak{n}$ are congruent if there exists an isomorphism of Leibniz algebras $f : \mathfrak{g} \to \mathfrak{g}'$ which induces identities on $\mathfrak{n}$ and $\mathfrak{q}$. We denote by $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ the set of congruence classes $(\mathfrak{g})$ of $\mathfrak{q}$-extensions of $\mathfrak{n}$.

We know that $HL^2(\mathfrak{q}, \mathfrak{n}) \cong \mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ [14] and that associated to the $\mathfrak{q}$-extension $(\mathfrak{g})$ there exists the following natural exact sequence [3]:

$$0 \to \mathrm{Der}(\mathfrak{q}, \mathfrak{n}) \xrightarrow{\mathrm{Der}(\pi)} \mathrm{Der}(\mathfrak{g}, \mathfrak{n}) \xrightarrow{\rho} \mathrm{Hom}_{\mathfrak{q}}(\mathfrak{n}, \mathfrak{n}) \xrightarrow{\theta^\star(\mathfrak{g})} HL^2(\mathfrak{q}, \mathfrak{n}) \xrightarrow{\pi^\star} HL^2(\mathfrak{g}, \mathfrak{n}). \quad (2)$$

We define an application $\Delta : \mathrm{Ext}(\mathfrak{q}, \mathfrak{n}) \to HL^2(\mathfrak{q}, \mathfrak{n}), \Delta[\mathfrak{g}] = \xi = \theta^\star(\mathfrak{g})(1_{\mathfrak{n}})$, which is well defined by naturality of the sequence (2). Moreover we can see that Δ is bijective (it suffices to use the proof of Theorem 3.3, p. 207 in [10]). If $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ is endowed with the structure of a K-vector space by means of the Baer sum [3], then Δ is a linear map; so Δ is a linear isomorphism between $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ and $HL^2(\mathfrak{q}, \mathfrak{n})$ and we can see that the natural structure of $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ induced by Δ from the vector space structure of $HL^2(\mathfrak{q}, \mathfrak{n})$ is precisely the structure given in [3]. It is also known that $\mathrm{Ext}(\mathfrak{q}, -)$ is a covariant functor from the category of (right) $\mathfrak{q}$-modules to the category of K-vector spaces. Obviously, Δ maps the neutral element of $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$, which is the class of split extensions, to the neutral element of $HL^2(\mathfrak{q}, \mathfrak{n})$. If $\mathfrak{n}$ is a trivial $\mathfrak{q}$-module, then $\mathrm{Ext}(\mathfrak{q}, \mathfrak{n})$ is the set of congruence classes of central extensions of $\mathfrak{q}$ by $\mathfrak{n}$.

Finally, by Lemma 5.2 in [3], the following identities hold:

$$\theta_\star \Delta[\mathfrak{g}] = \theta_\star \theta^\star(\mathfrak{g})(1_{\mathfrak{n}}) = \theta_\star(\mathfrak{g}). \quad (3)$$

3. Morphisms of Extensions with Abelian Kernel

In this section we develop some technical results on a morphism of extensions with abelian kernel which are used in the next sections.

Let $(\mathfrak{g})$: $0 \to \mathfrak{n} \xrightarrow{\chi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ be a $\mathfrak{q}$-extension of $\mathfrak{n}$ and let $\alpha : \mathfrak{n} \to \mathfrak{n}'$ be a homomorphism of $\mathfrak{q}$-modules, that is, $\alpha[q, n] = [q, \alpha(n)], \alpha[n, q] = [\alpha(n), q], n \in \mathfrak{n}, q \in \mathfrak{q}$, and let $(\mathfrak{g}')$: $0 \to \mathfrak{n}' \xrightarrow{\chi'} \mathfrak{g}' \xrightarrow{\pi'} \mathfrak{q} \to 0$ be a $\mathfrak{q}$-extension with $\Delta[\mathfrak{g}'] = \xi' \in HL^2(\mathfrak{q}, \mathfrak{n}')$.

Proposition 1. *There exists a homomorphism of Leibniz algebras $f : \mathfrak{g} \to \mathfrak{g}'$ such that the diagram*

$$\begin{tikzcd}
(\mathfrak{g}) : 0 \arrow[r] & \mathfrak{n} \arrow[r, "\chi"] \arrow[d, "\alpha"'] & \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "f"'] & \mathfrak{q} \arrow[r] \arrow[d, equal] & 0 \\
(\mathfrak{g}') : 0 \arrow[r] & \mathfrak{n}' \arrow[r, "\chi'"'] & \mathfrak{g}' \arrow[r, "\pi'"'] & \mathfrak{q} \arrow[r] & 0
\end{tikzcd}$$

is commutative if and only if $\alpha_\star(\xi) = \xi' \in HL^2(\mathfrak{q}, \mathfrak{n}')$.

Proof. The naturality in (2) implies $\alpha_\star(\xi) = \alpha_\star\theta^\star(\mathfrak{g})(1_\mathfrak{n}) = \theta^\star(\mathfrak{g})\alpha_\star 1_\mathfrak{n} = \theta^\star(\mathfrak{g})\alpha = \theta^\star(\mathfrak{g})\alpha^\star(1_{\mathfrak{n}'}) = \theta^\star(\mathfrak{g}')(1_{\mathfrak{n}'}) = \xi'$.

Conversely, for the $\mathfrak{q}$-extension $(\mathfrak{g})$ we construct the *forward induced extension* $({}^\alpha\mathfrak{g})$ [3], obtaining the morphism of extensions $(\alpha, f_\alpha, 1) : (\mathfrak{g}) \to ({}^\alpha\mathfrak{g})$. Thus $\alpha_\star(\xi) = \alpha_\star\theta^\star(\mathfrak{g})(1_\mathfrak{n}) = \alpha_\star\Delta[\mathfrak{g}] = \Delta[{}^\alpha\mathfrak{g}]$. Consequently $\Delta[{}^\alpha\mathfrak{g}] = \alpha_\star(\xi) = \xi' = \Delta[\mathfrak{g}']$ and then $({}^\alpha\mathfrak{g}) \equiv (\mathfrak{g}')$; thus $f'f_\alpha : \mathfrak{g} \to {}^\alpha\mathfrak{g} \to \mathfrak{g}'$ verifies the Proposition. □

We note that the composition $f'f_\alpha$ is not unique; in fact there are many maps $f'f_\alpha$ with the required properties since the previous result is a particular case of Theorem 3.1 in [3], where it is confirmed that there are as many homomorphisms $f'f_\alpha$ as derivations from $\mathfrak{q}$ to $\mathfrak{n}'$.

Let $\gamma : \bar{\mathfrak{q}} \to \mathfrak{q}$ be a homomorphism and let $(\bar{\mathfrak{g}})$: $0 \to \mathfrak{n} \xrightarrow{\overline{\chi}} \bar{\mathfrak{g}} \xrightarrow{\overline{\pi}} \bar{\mathfrak{q}} \to 0$ be a $\bar{\mathfrak{q}}$-extension with $\Delta[\bar{\mathfrak{g}}] = \bar{\xi} \in HL^2(\bar{\mathfrak{q}}, \mathfrak{n})$.

Proposition 2. *There exists* $\bar{f} : \bar{\mathfrak{g}} \to \mathfrak{g}$ *such that the diagram*

$$\begin{tikzcd}
(\bar{\mathfrak{g}}) : 0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d, equal] & \bar{\mathfrak{g}} \arrow[r] \arrow[d, "\bar{f}"'] & \bar{\mathfrak{q}} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
(\mathfrak{g}) : 0 \arrow[r] & \mathfrak{n} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{q} \arrow[r] & 0
\end{tikzcd}$$

is commutative if and only if $HL^2(\gamma, \mathfrak{n})(\xi) = \gamma^\star(\xi) = \bar{\xi}$.

Proof. If there exists $\bar{f}$, then naturality of (2) implies that $\bar{\xi} = \theta^\star(\bar{\mathfrak{g}})(1_\mathfrak{n}) = HL^2(\gamma, \mathfrak{n})\theta^\star(\mathfrak{g})(1_\mathfrak{n}) = \gamma^\star(\xi)$.

Conversely, from the $\mathfrak{q}$-extension $(\mathfrak{g})$ we construct the *backward induced extension* $(\mathfrak{g}_\gamma)$ [3] and so we have the morphism of extensions $(1, \bar{\gamma}, \gamma) : (\mathfrak{g}_\gamma) \to (\mathfrak{g})$. From here, $\Delta[\mathfrak{g}_\gamma] = \gamma^\star\Delta[\mathfrak{g}] = \gamma^\star(\xi) = \bar{\xi} = \Delta[\bar{\mathfrak{g}}]$; consequently $(\mathfrak{g}_\gamma) \equiv (\bar{\mathfrak{g}})$ and then $\bar{f} : \bar{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}_\gamma \to \mathfrak{g}$ satisfies the Proposition. □

Proposition 3. *Given the following diagram*

$$\begin{tikzcd}
(\bar{\mathfrak{g}}) : 0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d, "\alpha"'] & \bar{\mathfrak{g}} \arrow[r] & \bar{\mathfrak{q}} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
(\mathfrak{g}') : 0 \arrow[r] & \mathfrak{n}' \arrow[r] & \mathfrak{g}' \arrow[r] & \mathfrak{q} \arrow[r] & 0
\end{tikzcd}$$

there exists $f : \bar{\mathfrak{g}} \to \mathfrak{g}'$ *making the diagram commutative if and only if* $\alpha_\star(\Delta[\bar{\mathfrak{g}}]) = \gamma^\star(\Delta[\mathfrak{g}'])$. *Moreover, the solution set is in the one-to-one correspondence with the derivation set of* $\bar{\mathfrak{q}}$ *to* $\mathfrak{n}'$.

Proof. If f exists, we consider the composition $(1, \bar{\gamma}, \gamma)(\alpha, \sigma, 1) : (\bar{\mathfrak{g}}) \to (\mathfrak{g}'_\gamma) \to (\mathfrak{g}')$, then Propositions 1 and 2 imply that $\alpha_\star(\Delta[\bar{\mathfrak{g}}]) = \Delta[\mathfrak{g}'_\gamma] = \gamma^\star(\Delta[\mathfrak{g}'])$.

Conversely, we consider the composition $(1, \bar{\gamma}, \gamma)(1, -, 1)(\alpha, \sigma, 1) : (\bar{\mathfrak{g}}) \to ({}^\alpha\bar{\mathfrak{g}}) \to (\mathfrak{g}'_\gamma) \to (\mathfrak{g}')$ and, applying Propositions 1 and 2, we have that $\Delta[{}^\alpha\bar{\mathfrak{g}}] =$

$\alpha_\star(\Delta[\bar{\mathfrak{g}}]) = \gamma^\star(\Delta[\mathfrak{g}']) = \Delta[\mathfrak{g}'_\gamma]$; consequently, $({}^\alpha\bar{\mathfrak{g}})$ and $(\mathfrak{g}'_\gamma)$ are congruent and $(1,-,1) = (1,\phi,1)$ is the wanted morphism.

A proof of the second statement in this Proposition can be seen in Theorem 3.1 in [3]. □

4. Stem Extensions and Stem Covers

Definition 1 ([2]). A central extension of Leibniz algebras $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ is called a stem extension if $\mathfrak{n} \subseteq [\mathfrak{g},\mathfrak{g}]$. It is said that $(\mathfrak{g})$ is a stem cover if $\mathfrak{g}_{ab} \cong \mathfrak{q}_{ab}$ and $HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q})$ is the zero map.

Proposition 4. *Every central extension class of a K-vector space (trivial $\mathfrak{q}$-module) $\mathfrak{n}$ by a Leibniz algebra $\mathfrak{q}$ is forward induced from a stem extension.*

Proof. Choose any central extension class $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$, then $\theta_\star(\mathfrak{g}) : HL_2(\mathfrak{q}) \to \mathfrak{n}$ factors as $i\tau : HL_2(\mathfrak{q}) \to \mathfrak{n} \cap [\mathfrak{g},\mathfrak{g}] = \mathfrak{n}_1 \to \mathfrak{n}$. As $\mathfrak{n}_1$ is a trivial $\mathfrak{q}$-module, then, given τ, there exists a central extension $(\mathfrak{g}_1) \in HL^2(\mathfrak{q},\mathfrak{n}_1)$ such that $\theta_\star(\mathfrak{g}_1) = \tau$. Moreover, $(\mathfrak{g}_1)$ is a stem extension. By naturality of the sequence (1) on the forward construction $(\mathfrak{g}_1) \to i_\star(\mathfrak{g}_1)$ [3], we have that $\theta_\star i_\star(\mathfrak{g}_1) = i\tau = \theta_\star(\mathfrak{g})$, i.e., $i_\star(\mathfrak{g}_1) = (\mathfrak{g})$, and so $(\mathfrak{g})$ is forward induced by $(\mathfrak{g}_1)$, which is a stem extension. □

Proposition 5. *Let $\mathfrak{q}$ be a Leibniz algebra and let $\mathfrak{u}$ be a subspace of $HL_2(\mathfrak{q})$, then there exists a stem extension $(\mathfrak{g})$ with $\mathfrak{u} = \operatorname{Ker}\theta_\star\Delta[\mathfrak{g}]$.*

Proof. We consider the quotient vector space $\mathfrak{n} = HL_2(\mathfrak{q})/\mathfrak{u}$ as a trivial $\mathfrak{q}$-module. We consider the central extension $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0 \in HL^2(\mathfrak{q},\mathfrak{n})$. Thus $\theta_\star\Delta[\mathfrak{g}] = \theta_\star(\mathfrak{g}) \in \operatorname{Hom}(HL_2(\mathfrak{q}),\mathfrak{n})$. If $\theta_\star(\mathfrak{g}) : HL_2(\mathfrak{q}) \to \mathfrak{n} = HL_2(\mathfrak{q})/\mathfrak{u}$ is the canonical projection; then there exists a central extension $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ such that $\theta_\star\Delta[\mathfrak{g}] = \theta_\star(\mathfrak{g})$ is the canonical projection. Associated to $(\mathfrak{g})$, we have the exact sequence (1), in which $\mathfrak{u} = \operatorname{Ker}\theta_\star(\mathfrak{g}) = \operatorname{Ker}\theta_\star\Delta[\mathfrak{g}]$. Moreover, $(\mathfrak{g})$ is a stem extension since $\theta_\star\Delta[\mathfrak{g}] = \theta_\star(\mathfrak{g})$ is an epimorphism. □

Remark 1. A stem extension is a stem cover if and only if $\mathfrak{u} = 0$.

Remark 2. Any stem cover $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ is isomorphic to a stem cover $(\mathfrak{g}') : 0 \to HL_2(\mathfrak{q}) \to \mathfrak{g}' \to \mathfrak{q} \to 0$ with $\theta_\star\Delta[\mathfrak{g}'] = 1_{HL_2(\mathfrak{q})}$. Indeed, there always exists $\mathfrak{g}'$ and thus it suffices to take $\mathfrak{u} = 0$ in Proposition 5; if $\varphi : \mathfrak{n} \to HL_2(\mathfrak{q})$ is the inverse of $\theta_\star\Delta[\mathfrak{g}]$, then the naturality of the isomorphism $\theta_\star : HL^2(\mathfrak{q},\mathfrak{n}) \xrightarrow{\sim} \operatorname{Hom}(HL_2(\mathfrak{q}),\mathfrak{n})$ implies $\theta_\star\varphi_\star(\Delta[\mathfrak{g}]) = \varphi_\star\theta_\star(\Delta[\mathfrak{g}]) = \varphi\theta_\star(\Delta[\mathfrak{g}]) = 1_{HL_2(\mathfrak{q})}$ so that we can choose $(\mathfrak{g}')$ such that $\Delta[\mathfrak{g}'] = \varphi_\star\Delta[\mathfrak{g}]$. By Proposition 1 there exists $f : \mathfrak{g} \to \mathfrak{g}'$ making the following diagram commutative:

$$\begin{tikzcd}
(\mathfrak{g}) : 0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d, "\varphi"'] & \mathfrak{g} \arrow[r] \arrow[d, "f"'] & \mathfrak{q} \arrow[r] \arrow[d, equal] & 0 \\
(\mathfrak{g}') : 0 \arrow[r] & HL_2(\mathfrak{q}) \arrow[r] & \mathfrak{g}' \arrow[r] & \mathfrak{q} \arrow[r] & 0
\end{tikzcd}$$

The name of stem cover is motivated by the following result.

Proposition 6. *Every stem extension of* $\mathfrak{q}$ *is an epimorphic image of some stem cover.*

Proof. Let $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ be a stem extension characterized by $\Delta[\mathfrak{g}] = \xi \in HL^2(\mathfrak{q}, \mathfrak{n})$; then $\varphi = \theta_\star(\xi) = \theta_\star(\Delta[\mathfrak{g}]) = \theta_\star(\mathfrak{g}) : HL_2(\mathfrak{q}) \to \mathfrak{n}$ is an epimorphism. In order to prove the Proposition, we must find $\eta \in HL^2(\mathfrak{q}, HL_2(\mathfrak{q}))$ with $\varphi_\star(\eta) = \xi$ and $\theta_\star(\eta) = 1_{HL_2(\mathfrak{q})}$ (i.e., η is an element in the second cohomology K-vector space which characterizes a stem cover), where $\varphi_\star$ is the induced morphism by naturality of the isomorphism $\theta_\star$ on φ.

Let $\eta \in HL^2(\mathfrak{q}, HL_2(\mathfrak{q}))$ be such that $\theta_\star(\eta) = 1_{HL_2(\mathfrak{q})}$; then $\theta_\star(\xi - \varphi_\star(\eta)) = \varphi - \varphi_\star\theta_\star(\eta) = 0$; consequently $\varphi_\star(\eta) = \xi$. Obviously, η verifies the required conditions. Now, let $(\mathfrak{g}') : 0 \to HL_2(\mathfrak{q}) \to \mathfrak{g}' \to \mathfrak{q} \to 0 \in HL^2(\mathfrak{q}, HL_2(\mathfrak{q}))$ be such that $\Delta[\mathfrak{g}'] = \eta$; by Proposition 1 there exists $f : \mathfrak{g}' \to \mathfrak{g}$ such that $(\varphi, f, 1) : (\mathfrak{g}') \to (\mathfrak{g})$ is an epimorphism. $\square$

Proposition 7. *There exists a unique isomorphism class of stem covers of* $\mathfrak{q}$.

Proof. By Remark 2, stem covers are of the form $(\mathfrak{g}) : 0 \to HL_2(\mathfrak{q}) \to \mathfrak{g} \to \mathfrak{q} \to 0$ such that $\theta_\star\Delta[\mathfrak{g}] = 1_{HL_2(\mathfrak{q})}$. Fix a stem cover $(\mathfrak{g})$; for another stem cover $(\mathfrak{g}')$ we have that $\theta_\star\Delta[\mathfrak{g}] = \theta_\star\Delta[\mathfrak{g}'] = 1_{HL_2(\mathfrak{q})}$; then $[\mathfrak{g}] = [\mathfrak{g}']$. $\square$

In this case we shall speak of the stem cover of $\mathfrak{q}$.

Proposition 8. *Let* $(\bar{\mathfrak{g}}) : 0 \to HL_2(\bar{\mathfrak{q}}) \to \bar{\mathfrak{g}} \to \bar{\mathfrak{q}} \to 0$ *be a stem cover and let* $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ *be a central extension. Then every homomorphism* $f : \bar{\mathfrak{q}} \to \mathfrak{q}$ *can be lifted to a map* $f' : \bar{\mathfrak{g}} \to \mathfrak{g}$.

Proof. Let $\Delta[\mathfrak{g}] = \xi \in HL^2(\mathfrak{q}, \mathfrak{n})$. We define $\varphi = f^\star\theta_\star(\xi) : HL_2(\bar{\mathfrak{q}}) \to \mathfrak{n}$. Since $\eta = \Delta[\bar{\mathfrak{g}}] \in HL^2(\bar{\mathfrak{q}}, HL_2(\bar{\mathfrak{q}}))$ is a stem cover with $\theta_\star(\eta) = \theta_\star\Delta[\bar{\mathfrak{g}}] = \theta_\star(\bar{\mathfrak{g}}) = 1_{HL_2(\bar{\mathfrak{q}})}$, we have $\theta_\star\varphi_\star(\eta) = \varphi_\star\theta_\star(\eta) = \varphi = f^\star\theta_\star(\xi) = \theta_\star f^\star(\xi)$ (by the naturality in the isomorphism $\theta_\star$), and so $\varphi_\star(\eta) = f^\star(\xi)$, i.e., $\varphi_\star(\Delta[\mathfrak{g}]) = f^\star(\Delta[\mathfrak{g}])$; Proposition 3 ends the proof. $\square$

Proposition 9. *If* $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ *is a stem extension, then the following sequence is exact:*

$$HL_3(\mathfrak{g}) \to HL_3(\mathfrak{q}) \to HL_1(\mathfrak{g}) \otimes \mathfrak{n} \oplus \mathfrak{n} \otimes HL_1(\mathfrak{g}) \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q}) \to \mathfrak{n} \to 0$$

Proof. $HL_2(\mathfrak{q}) \to \mathfrak{n}$ is an epimorphism and see Remark 1 in [4]. $\square$

Corollary 1. *Let* $(\mathfrak{g}) : 0 \to HL_2(\mathfrak{q}) \to \mathfrak{g} \to \mathfrak{q} \to 0$ *be a stem cover associated to* $\mathfrak{q}$. *Then the following sequence is exact:*

$$HL_3(\mathfrak{g}) \to HL_3(\mathfrak{q}) \to HL_1(\mathfrak{g}) \otimes HL_2(\mathfrak{q}) \oplus HL_2(\mathfrak{q}) \otimes HL_1(\mathfrak{g}) \to HL_2(\mathfrak{g}) \to 0.$$

5. Central Extensions of Perfect Leibniz Algebras

Proposition 10. *Let* $(\mathfrak{g}) : 0 \to \mathfrak{n} \xrightarrow{\chi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ *and* $(\mathfrak{g}') : 0 \to \mathfrak{n}' \xrightarrow{\chi'} \mathfrak{g}' \xrightarrow{\pi'} \mathfrak{q}' \to 0$ *be central extensions. Let* $\rho : \mathfrak{n} \to \mathfrak{n}'$ *and* $\sigma : \mathfrak{q} \to \mathfrak{q}'$ *be homomorphisms of Leibniz algebras. Then:*

i) *There exists* $\tau : \mathfrak{g} \to \mathfrak{g}'$ *inducing* ρ *and* σ *if and only if the following diagram is commutative:*

$$\begin{tikzcd} HL_2(\mathfrak{q}) \arrow[r, "\theta_\star(\mathfrak{g})"] \arrow[d, "\sigma_\star"'] & \mathfrak{n} \arrow[d, "\rho"] \\ HL_2(\mathfrak{q}') \arrow[r, "\theta_\star(\mathfrak{g}')"'] & \mathfrak{n}' \end{tikzcd}$$

ii) *If* τ *exists, it is unique if and only if* $\operatorname{Hom}(HL_1(\mathfrak{q}), \mathfrak{n}') = 0$.

Proof. i) If τ exists, the commutativity of the square follows from naturality of the sequence (1).

Conversely, we consider the following diagram produced by the naturality of the isomorphism $\theta_\star$:

$$\begin{tikzcd} \theta_\star : HL^2(\mathfrak{q}, \mathfrak{n}) \arrow[r, "\sim"] \arrow[d, "\rho_\star"'] & \operatorname{Hom}(HL_2(\mathfrak{q}), \mathfrak{n}) \arrow[d, "\rho_\star"'] \\ \theta''_\star : HL^2(\mathfrak{q}, \mathfrak{n}') \arrow[r, "\sim"] & \operatorname{Hom}(HL_2(\mathfrak{q}), \mathfrak{n}') \\ \theta'_\star : HL^2(\mathfrak{q}', \mathfrak{n}') \arrow[r, "\sim"] \arrow[u, "\sigma^\star"] & \operatorname{Hom}(HL_2(\mathfrak{q}'), \mathfrak{n}') \arrow[u, "\sigma^\star"] \end{tikzcd}$$

Let $\xi = \Delta[\mathfrak{g}]$ and $\xi' = \Delta[\mathfrak{g}']$, then $\rho_\star\theta_\star(\xi) = \rho_\star\theta_\star(\Delta[\mathfrak{g}]) = \rho_\star\theta_\star(\mathfrak{g}) = \rho\theta_\star(\mathfrak{g}) = \theta_\star(\mathfrak{g}')\sigma_\star = \sigma^\star\theta_\star(\mathfrak{g}') = \sigma^\star\theta'_\star(\Delta[\mathfrak{g}']) = \sigma^\star\theta_\star(\xi')$. So $\theta''_\star\rho_\star(\xi) = \rho_\star\theta_\star(\xi) = \sigma^\star\theta'_\star(\xi') = \theta''_\star\sigma^\star(\xi')$, and consequently $\rho_\star(\xi) = \sigma^\star(\xi')$. Now Proposition 3 provides $\tau : \mathfrak{g} \to \mathfrak{g}'$ inducing ρ and σ.

ii) Suppose that there exists $\tau : \mathfrak{g} \to \mathfrak{g}'$ inducing ρ and σ and let $\tau' : \mathfrak{g} \to \mathfrak{g}'$ be another homomorphism inducing ρ and σ, then there are unique homomorphisms $f : \mathfrak{g} \to \mathfrak{n}'$ such that $\tau' - \tau = \chi' f$ and $\varphi : \mathfrak{q} \to \mathfrak{n}'$ such that $\varphi\pi = f$; consequently, $\tau' = \tau + \chi'\varphi\pi$; that is, for another homomorphism $\tau' : \mathfrak{g} \to \mathfrak{g}'$ there exists a unique homomorphism $\varphi : \mathfrak{q} \to \mathfrak{n}'$ such that $\tau' = \tau + \chi'\varphi\pi$. Conversely, if $\varphi : \mathfrak{q} \to \mathfrak{n}'$ is a homomorphism, then $\tau' = \tau + \chi'\varphi\pi$ induces ρ and σ. τ is unique if and only if $\tau - \tau' = 0$, that is, $\chi'\varphi\pi = 0$, which is equivalent to $\varphi \in \operatorname{Hom}_{Leib}(\mathfrak{q}, \mathfrak{n}') = 0$ and then $\operatorname{Hom}(HL_1(\mathfrak{q}), \mathfrak{n}') = 0$. □

Corollary 2. *Under the hypothesis of Proposition 10, if the map* $\tau : \mathfrak{g} \to \mathfrak{g}'$ *exists, then it is unique when* $\mathfrak{q}$ *is a perfect Leibniz algebra* ($\mathfrak{q}_{ab} = 0$).

Proof. If $\mathfrak{q}$ is a perfect Leibniz algebra, then $\operatorname{Hom}(HL_1(\mathfrak{q}), \mathfrak{n}') = 0$. □

Proposition 11. *The isomorphism classes of stem extensions of* $\mathfrak{q}$ *are in the one-to-one correspondence with the subspaces of* $HL_2(\mathfrak{q})$. *Moreover, if* $\mathfrak{u}$ *and* $\mathfrak{v}$ *are two subspaces of* $HL_2(\mathfrak{q})$, *then* $\mathfrak{u} \subseteq \mathfrak{v}$ *if and only if there is a map (necessarily surjective) from the stem extension corresponding to* $\mathfrak{u}$ *to the stem extension corresponding to* $\mathfrak{v}$.

Proof. Let $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ be a stem extension. Associate to $(\mathfrak{g})$ the subspace $\mathfrak{u} = \operatorname{Ker} \theta_\star(\mathfrak{g}) = \operatorname{Ker}(\theta_\star(\Delta[\mathfrak{g}]) : HL_2(\mathfrak{q}) \to \mathfrak{n})$. It is clear that the isomorphic stem extension yields the same subspace of $HL_2(\mathfrak{q})$.

Conversely, let $\mathfrak{u} \subseteq HL_2(\mathfrak{q})$ be given, $\mathfrak{n} = HL_2(\mathfrak{q})/\mathfrak{u}$ and consider the canonical projection $\tau : HL_2(\mathfrak{q}) \to \mathfrak{n}$, then there exists an element $\Delta[\mathfrak{g}] \in HL^2(\mathfrak{q}, \mathfrak{n})$ such that $\theta_\star \Delta[\mathfrak{g}] = \tau$. Obviously, $(\mathfrak{g})$ is unique, $\theta_\star \Delta[\mathfrak{g}]$ is an epimorphism and then $(\mathfrak{g})$ is a stem extension.

Finally, if $(\mathfrak{g}') : 0 \to \mathfrak{n}' \to \mathfrak{g}' \to \mathfrak{q} \to 0$ is another stem extension associated to $\mathfrak{u}$, then there exists an isomorphism $r : \mathfrak{n} \to \mathfrak{n}'$ such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{u} \arrow[r] \arrow[d, equal] & HL_2(\mathfrak{q}) \arrow[r, "\theta_\star(\mathfrak{g})"] \arrow[d, equal] & \mathfrak{n} \arrow[r] \arrow[d, "r"'] & 0 \\ 0 \arrow[r] & \mathfrak{u} \arrow[r] & HL_2(\mathfrak{q}) \arrow[r, "\theta_\star(\mathfrak{g}')"'] & \mathfrak{n}' \arrow[r] & 0 \end{tikzcd}$$

By Proposition 10 i) there exists $\sigma : \mathfrak{g} \to \mathfrak{g}'$ inducing $r : \mathfrak{n} \to \mathfrak{n}'$ and $1_\mathfrak{q} : \mathfrak{q} \to \mathfrak{q}$ such that $(r, \sigma, 1_\mathfrak{q}) : (\mathfrak{g}) \to (\mathfrak{g}')$ is a morphism of extensions; moreover, σ is an isomorphism and then $(\mathfrak{g})$ and $(\mathfrak{g}')$ are in the same isomorphism class.

For the second statement, we consider a morphism of stem extensions $(r, t, 1) : (\mathfrak{g}) \to (\mathfrak{g}')$. Naturality of the sequence (1) implies $\mathfrak{u} = \operatorname{Ker} \theta_\star(\mathfrak{g}) \subseteq \operatorname{Ker} \theta_\star(\mathfrak{g}') = \mathfrak{v}$.

Conversely, we first recall that every stem extension is isomorphic to an extension $(\mathfrak{g})$ with the canonical projection $\theta_\star \Delta[\mathfrak{g}]$. It is thus enough to consider those. Let $\mathfrak{u} \subseteq \mathfrak{v} \subseteq HL_2(\mathfrak{q}), \mathfrak{n} = HL_2(\mathfrak{q})/\mathfrak{u}$ and $\mathfrak{n}' = HL_2(\mathfrak{q})/\mathfrak{v}$. There exists an epimorphism $r : \mathfrak{n} \to \mathfrak{n}'$ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{u} \arrow[r] \arrow[d] & HL_2(\mathfrak{q}) \arrow[r, "\tau"] \arrow[d, equal] & \mathfrak{n} \arrow[r] \arrow[d, "r"'] & 0 \\ 0 \arrow[r] & \mathfrak{v} \arrow[r] & HL_2(\mathfrak{q}) \arrow[r, "\sigma"'] & \mathfrak{n}' \arrow[r] & 0 \end{tikzcd}$$

commutes. Now, if $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ is an extension with $\theta_\star \Delta[\mathfrak{g}] = \theta_\star(\mathfrak{g}) = \tau$ and $(\mathfrak{g}') : 0 \to \mathfrak{n}' \to \mathfrak{g}' \to \mathfrak{q} \to 0$ is another extension with $\theta_\star \Delta[\mathfrak{g}'] = \theta_\star(\mathfrak{g}') = \sigma$, then by Proposition 10 i) there exists $t : \mathfrak{g} \to \mathfrak{g}'$ inducing r and 1; moreover, t is surjective. $\square$

We recall that if $\mathfrak{q}$ is perfect, then Proposition 10 implies that t is uniquely determined by r.

Proposition 12. *Let $\mathfrak{q}$ be a perfect Leibniz algebra and let $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ be a stem extension, then the sequence*

$$0 \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{q}) \stackrel{\theta_\star(\mathfrak{g})}{\to} \mathfrak{n} \to 0$$

is exact and $\pi_\star : HL_3(\mathfrak{g}) \to HL_3(\mathfrak{q})$ *is an epimorphism.*

Proof. Follows from Proposition 9 and keeping in mind that the hypothesis involves $HL_1(\mathfrak{g}) = 0$. $\square$

Note that from Propositions 11 and 12 it follows that if $\mathfrak{q}$ is a perfect Leibniz algebra, then the second Leibniz homology K-spaces with trivial coefficients of the stem extension of $\mathfrak{q}$ are precisely the subspaces of $HL_2(\mathfrak{q})$.

Corollary 3. *Let $\mathfrak{q}$ be a perfect Leibniz algebra and let $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ be a central extension. Then $(\mathfrak{g})$ is a stem cover if and only if $HL_1(\mathfrak{g}) = HL_2(\mathfrak{g}) = 0$.*

Proof. From the exact sequence (1) associated to $(\mathfrak{g})$, with $HL_1(\mathfrak{g}) = HL_2(\mathfrak{g}) = 0$, we easily derive that $\theta_\star(\mathfrak{g})$ is an isomorphism. $\square$

Definition 2. Let $\mathfrak{h}$ and $\mathfrak{n}$ be two-sided ideals of the Leibniz algebra $\mathfrak{g}$. The commutator subalgebra of $\mathfrak{h}$ and $\mathfrak{n}$ is defined as the Leibniz subalgebra of $\mathfrak{g}$ spanned by the brackets $[h, n]$ and $[n, h]$, for all $h \in \mathfrak{h}, n \in \mathfrak{n}$, that is,

$$[\mathfrak{h}, \mathfrak{n}] = \langle \{[h, n], [n, h] \mid h \in \mathfrak{h}, n \in \mathfrak{n}\} \rangle.$$

It is easy to verify that $[\mathfrak{h}, \mathfrak{n}]$ is a two-sided ideal of $\mathfrak{g}$ and that $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{h} \cap \mathfrak{n}$.

Remark 3. Let $(\mathfrak{g}) : 0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$ be a stem extension with $HL_2(\mathfrak{g}) = 0$, then $(\mathfrak{g})$ is a stem cover. Corollary 3 shows that the converse is true if, in addition, $\mathfrak{q}$ is a perfect Leibniz algebra. In general, however, there are stem covers with $HL_2(\mathfrak{g})$ different from 0. For example, let $\mathfrak{f}$ be a non-abelian or non-nilpotent free Leibniz algebra and let us consider the lower central series

$$\mathfrak{f}_0 = \mathfrak{f}, \mathfrak{f}_{n+1} = [\mathfrak{f}, \mathfrak{f}_n].$$

Then the sequence $0 \to \mathfrak{f}_n/\mathfrak{f}_{n+1} \to \mathfrak{f}/\mathfrak{f}_{n+1} \to \mathfrak{f}/\mathfrak{f}_n \to 0$ is central for $n \geq 2$; moreover, it is a stem cover since $(\mathfrak{f}/\mathfrak{f}_{n+1})_{ab} \cong \mathfrak{f}_{ab} \cong (\mathfrak{f}/\mathfrak{f}_n)_{ab}$ and, on the other hand, the map $HL_2(\mathfrak{f}/\mathfrak{f}_{n+1}) = \mathfrak{f}_{n+1}/\mathfrak{f}_{n+2} \to HL_2(\mathfrak{f}/\mathfrak{f}_n) = \mathfrak{f}_n/\mathfrak{f}_{n+1}$ is trivial. Moreover, $HL_2(\mathfrak{f}/\mathfrak{f}_{n+1}) = \mathfrak{f}_{n+1}/\mathfrak{f}_{n+2}$ is different from 0.

The following Proposition gives, for a perfect Leibniz algebra $\mathfrak{q}$, a description of the stem covers of $\mathfrak{q}$ in terms of free presentations of $\mathfrak{q}$.

Proposition 13. *Let $\mathfrak{q}$ be a perfect Leibniz algebra and let $0 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{f} \mathfrak{q} \to 0$ be a free presentation. Then*

$$0 \to HL_2(\mathfrak{q}) \to \frac{[\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{f}, \mathfrak{r}]} \xrightarrow{\varphi} \mathfrak{q} \to 0$$

is a stem cover of $\mathfrak{q}$, where φ is induced by f.

Proof.

$$0 \to HL_2(\mathfrak{q}) \to \frac{[\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{f}, \mathfrak{r}]} \xrightarrow{\varphi} \mathfrak{q} \to 0$$

is the universal central extension of $\mathfrak{q}$. Moreover, it is a stem cover since $HL_2([\mathfrak{f}, \mathfrak{f}]/[\mathfrak{f}, \mathfrak{r}]) = HL_1([\mathfrak{f}, \mathfrak{f}]/[\mathfrak{f}, \mathfrak{r}]) = 0$ (see Proposition 4.2 in [14]). $\square$

From Proposition 7, when $\mathfrak{q}$ is a perfect Leibniz algebra, we have that any stem cover is isomorphic to $0 \to HL_2(\mathfrak{q}) \to [\mathfrak{f},\mathfrak{f}]/[\mathfrak{f},\mathfrak{r}] \to \mathfrak{q} \to 0$.

Proposition 14. *Let* $(\mathfrak{g}) : 0 \to \mathfrak{n} \xrightarrow{\chi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{q} \to 0$ *be a central extension and let* $f : \mathfrak{x} \to \mathfrak{q}$ *be a homomorphism of Leibniz algebras, where* $\mathfrak{x}$ *is a perfect Leibniz algebra. Then there exists* $\varphi : \mathfrak{x} \to \mathfrak{g}$ *such that* $\pi\varphi = f$ *if and only if* $f_\star(HL_2(\mathfrak{x})) \subseteq \pi_\star(HL_2(\mathfrak{g}))$*. If* φ *exists, it is uniquely determined.*

Proof. If φ exists, then the functor $HL_2(-)$ preserves the composition so that $f_\star(HL_2(\mathfrak{x})) = \pi_\star\varphi_\star(HL_2(\mathfrak{x})) \subseteq \pi_\star(HL_2(\mathfrak{g}))$.

Conversely, let $\mathfrak{q}' = Im\ f \subseteq \mathfrak{q}$ and $\mathfrak{s} = \mathrm{Ker}\ f$, then the exact sequence $0 \to \mathfrak{s} \to \mathfrak{x} \to \mathfrak{q}' \to 0$ induces the exact sequence $0 \to \mathfrak{s}/[\mathfrak{s},\mathfrak{x}] = \mathfrak{s}' \to \mathfrak{x}/[\mathfrak{s},\mathfrak{x}] = \mathfrak{x}' \to \mathfrak{q}' \to 0$, where $f' : \mathfrak{x}' \to \mathfrak{q}'$ is induced by f. Now the sequence (1) implies that $f'_\star(HL_2(\mathfrak{x}')) = f_\star(HL_2(\mathfrak{x})) \subseteq \pi_\star(HL_2(\mathfrak{g})) \subseteq HL_2(\mathfrak{q})$.

To end the proof, we need to construct $\varphi' : \mathfrak{x}' \to \mathfrak{g}$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{s}' \arrow[r] \arrow[d] & \mathfrak{x}' \arrow[r, "f'"] \arrow[d, "\varphi'"'] & \mathfrak{q}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] & \mathfrak{g} \arrow[r, "\pi"'] & \mathfrak{q} \arrow[r] & 0
\end{tikzcd}$$

By naturality of the sequence (1) and since $f'_\star(HL_2(\mathfrak{x}')) \subseteq \pi_\star(HL_2(\mathfrak{g}))$, there exists an injective map $\beta : Im\ f'_\star \to Im\ \pi_\star$ which induces $\tau' : \mathfrak{s}' \to \mathfrak{n}$. From Proposition 10 it follows that there exists $\varphi' : \mathfrak{x}' \to \mathfrak{g}$; moreover φ' is unique if and only if $\mathrm{Hom}(HL_2(\mathfrak{q}'), \mathfrak{n}) = 0$, which is obvious. Now $\varphi : \mathfrak{x} \to \mathfrak{g}$ is obtained by the composition $\varphi'.nat : \mathfrak{x} \to \mathfrak{x}' \to \mathfrak{g}$. □

Examples. Let $\mathfrak{q}$ be a perfect Leibniz algebra, then the universal central extension of $\mathfrak{q}$ is $0 \to HL_2(\mathfrak{q}) \to \mathfrak{q}\star\mathfrak{q} \to \mathfrak{q} \to 0$, where $\star$ denotes the non-abelian tensor product of Leibniz algebras introduced in [11]. Since a universal central extension is a stem cover (see Corollary 3 in [5]), we deduce from Corollary 1 that the map $HL_3(\mathfrak{q} \star \mathfrak{q}) \to HL_3(\mathfrak{q})$ is an epimorphism and from Corollary 3 that $HL_2(\mathfrak{q} \star \mathfrak{q}) = HL_1(\mathfrak{q} \star \mathfrak{q}) = 0$.

We apply now this fact to three examples of universal central extensions:

1. Let A be an associative and unital algebra over a field K. Let $stl_n(A)$ be the non commutative Steinberg algebra, which is a Leibniz algebra for $n \geq 3$ [14]. Let $sl_n(A)$ be the Lie algebra of matrices with entries in A whose trace in $A/[A, A]$ is zero and let $\varphi : stl_n(A) \to sl_n(A)$ be the map defined by $\varphi(v_{ij}(x)) = E_{ij}(x)$ where $E_{ij}(x)$ is the matrix with only the non-zero element x in place of (i, j). For $n \geq 5$ the sequence

$$0 \to \mathrm{HH}_1(A) \to stl_n(A) \xrightarrow{\varphi} sl_n(A) \to 0 \tag{4}$$

is the universal central extension of $sl_n(A)$ in the category of Leibniz algebras [14], where $HH_1(A)$ denotes the Hochschild homology group of A with coefficients in A. Applying the previous remarks to the case of the stem cover (4), we can derive the following consequences: $stl_n(A) \cong sl_n(A) \star sl_n(A)$; the map $HL_3(stl_n(A)) \to HL_3(sl_n(A))$ is an epimorphism; $HL_1(stl_n(A)) = HL_2(stl_n(A)) = 0$; the sequence $0 \to HL_2(stl_n(A)) \to HL_2(sl_n(A)) \to$

$HH_1(A) \to 0$ is exact and therefore $HL_2(sl_n(A)) \cong HH_1(A)$ (see Corollary 4.5 in [14]).

2. The universal central extension of the Lie algebra $\mathfrak{q} = \mathrm{Der}(\mathbf{C}[z, z^{-1}])$ of derivations of Laurent polynomials in the category of Leibniz algebras is the Virasoro algebra denoted by Vir. Similarly to Example 1, we can derive now that Virasoro algebra is isomorphic to $\mathrm{Der}(\mathbf{C}[z, z^{-1}]) \star \mathrm{Der}(\mathbf{C}[z, z^{-1}])$ and that the map $HL_3(Vir) \to HL_3(\mathrm{Der}(\mathbf{C}[z, z^{-1}]))$ is an epimorphism.

3. Let g(A) be a Kac–Moody Lie algebra. It is well-known that g(A) is a perfect Lie algebra, so it has a universal central extension in the category of Leibniz algebras. Since $HL_2(\mathrm{g(A)}) = 0$ for any non-affine Kac-Moody Lie algebra g(A) (see [7]), it is its own universal central extension and thus $0 \to 0 \to \mathrm{g(A)} \to \mathrm{g(A)} \to 0$ is a stem cover in the category of Leibniz algebras; hence we conclude that $\mathrm{g(A)} \cong \mathrm{g(A)} \star \mathrm{g(A)}$.

On the other hand, since $HL_2(\mathrm{g(A)}) \neq 0$ for any affine Kac–Moody Lie algebra g(A), the universal central extension in the category of Leibniz algebras (see [7]) is

$$0 \to \sum_{i \in \mathbf{Z} - \{0\}} kt^{ir-1} \to \mathrm{g(A)} \oplus \sum_{i \in \mathbf{Z} - \{0\}} kt^{ir-1} \to \mathrm{g(A)} \to 0,$$

where A is of affine type $X_n^{(r)}$.

Similarly to Example 1, we can derive

$$\mathrm{g(A)} \oplus \sum_{i \in \mathbf{Z} - \{0\}} kt^{ir-1} \cong \mathrm{g(A)} \star \mathrm{g(A)}.$$

Moreover, the map

$$HL_3 \left(\mathrm{g(A)} \oplus \sum_{i \in \mathbf{Z} - \{0\}} kt^{ir-1} \right) \to HL_3\left(\mathrm{g(A)}\right)$$

is an epimorphism.

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Authors' addresses:

J. M. Casas
Dpto. Matemática Aplicada
Universidad de Vigo
E.U.I.T. Forestal, Campus Universitario A Xunqueira
36005 Pontevedra
Spain
E-mail: jmcasas@uvigo.es

M. Ladra
Dpto. de Algebra
Universidad de Santiago
15782 Santiago de Compostela
Spain
E-mail: ladra@usc.es