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BOOLEAN GALOIS THEORIES

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Abstract. We develop a general approach to adjunctions satisfying the admissibility condition useful for Boolean Galois Theories, i. e. for Galois Theories whose Galois (pre)groupoids are profinite. Various examples and applications are briefly described.

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Introduction

According to [10], any adjunction

$$(I, H, \eta, \epsilon) : \mathbb{C} \longrightarrow \mathbb{X}$$

between categories with pullbacks determines a *Galois Theory* in the category $\mathbb{C}$. The fundamental theorem of Galois Theory is expressed as an equivalence of categories, which, under an appropriate *admissibility condition*, can be written as

$$\mathrm{Spl}(E, p) \simeq \mathbb{X}^{\mathrm{Gal}_I(E,p)},$$

where

- $\mathrm{Spl}(E, p)$ is the category of objects (A, α) in $(\mathbb{C} \downarrow B)$ split over a fixed effective descent morphism $p : E \longrightarrow B$ in $\mathbb{C}$;
- $\mathbb{X}^{\mathrm{Gal}_I(E,p)}$ is the category of internal actions of the *Galois pregroupoid* $\mathrm{Gal}_I(E, p)$ of (E, p) (see section 4);

This "Galois Theory" contains Grothendieck's one [5] as a special case. Recall from [8] that E is admissible whenever the right adjoint

$$H^E : (\mathbb{X} \downarrow I(E)) \longrightarrow (\mathbb{C} \downarrow E)$$

of the functor $I^E : (\mathbb{C} \downarrow E) \longrightarrow (\mathbb{X} \downarrow I(E))$ induced by I is full and faithful.

In order to obtain the classical Galois Theory of *finite* separable field extensions, the theory of covering spaces over a locally connected topological space and many other examples, one should take $\mathbb{X}$ to be the category Set of sets or $\mathrm{Set}_{\mathrm{fin}}$ of finite sets, and then the admissibility condition usually holds for all objects in $\mathbb{C}$. A general approach to such adjunctions with Set (or with $\mathrm{Set}_{\mathrm{fin}}$) is to start with a category $\mathbb{A}$ with a terminal object and then take $\mathbb{C}$ to be the category $\mathbb{C} = \mathrm{Fam}(\mathbb{A})$ of all families of objects of $\mathbb{A}$, or $\mathbb{C} = \mathrm{Fam}_{\mathrm{fin}}(\mathbb{A})$ of finite

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families of objects of $\mathbb{A}$. Then, since Fam and $\mathrm{Fam}_{\mathrm{fin}}$ are 2-functors from CAT to CAT, we obtain adjoint pairs

$$\mathbb{C} = \mathrm{Fam}(\mathbb{A}) \rightleftarrows \mathrm{Fam}(\mathbf{1}) \ ,$$

$$\mathbb{C} = \mathrm{Fam}_{\mathrm{fin}}(\mathbb{A}) \rightleftarrows \mathrm{Fam}_{\mathrm{fin}}(\mathbf{1})$$

induced by $\mathbb{A} \rightleftarrows \mathbf{1}$, where $\mathbf{1}$ is the terminal object in CAT. Since $\mathrm{Fam}(\mathbf{1}) \simeq$ Set and $\mathrm{Fam}_{\mathrm{fin}}(\mathbf{1}) \simeq \mathrm{Set}_{\mathrm{fin}}$, this gives the desired adjunctions, and the admissibility of every object of $\mathbb{C}$ can be easily proved. Recall that this construction does work for fields and spaces because:

- the dual category of finitely dimensional commutative algebras over a field is equivalent to $\mathrm{Fam}_{\mathrm{fin}}(\mathbb{A})$, where $\mathbb{A}$ is the category of connected (i. e. with no non-trivial idempotents) algebras of the same type;
- the category of locally connected topological spaces is equivalent to the category $\mathrm{Fam}(\mathbb{A})$, where $\mathbb{A}$ is the category of connected locally connected topological spaces.

However this construction does not work for "infinite" - or, better to say, "Boolean" - Galois Theories such as A. R. Magid's Galois Theory of commutative rings [12], where the adjoint pair should be

$$(\text{Commutative Rings})^{\mathrm{op}} \rightleftarrows \text{Stone Spaces} \ ,$$

and the admissibility again holds for every object, but the proof uses Pierce's theory, and hence cannot be considered as "trivial" (see [8] for details).

The category of Stone spaces (= profinite spaces = compact totally disconnected spaces) is needed even for the classical Galois Theory of *infinite* separable field extensions, where the Krull topology on the Galois group makes it a *profinite group*, i. e. a group in the category of Stone spaces. All this suggests the following:

Question: Is there a general approach to *Boolean Galois Theories*, where for a given category $\mathbb{C}$ one constructs an adjunction between $\mathbb{C}$ and the category of Stone spaces, with "enough" admissible objects?

In this paper we develop such an approach, essentially based on the following two observations:

- in any *lextensive* category $\mathbb{C}$ the complemented subobjects of any object form a Boolean algebra, and hence there is always the functor $I : \mathbb{C} \longrightarrow$ (Stone spaces) sending any object of $\mathbb{C}$ to the Stone space of the Boolean algebra of its complemented subobjects. When $\mathbb{C}$ is the dual category of commutative rings, it is the same as the Boolean spectrum functor used in the adjunction which gives A. R. Magid's Galois Theory as a special case of the Galois Theory in categories (see [7]–[10]);
- if $\mathbb{C}$ satisfies a stronger condition ("co-locally-indecomposable"), then the same functor was used by Y. Diers [4], formally for different purposes, but many of his results are closely related to what is needed in Galois

Theory, and in fact a certain (different) abstract Galois Theory was developed there.

Other than the present introduction, the paper has four sections:

1. The functor H;
2. The functor I;
3. Admissible objects;
4. Boolean Galois Theories.

In the first section we describe the minimal conditions on a category $\mathbb{C}$ under which we can construct a full and faithful functor

$$H : \mathbb{X} = (\mathrm{Bool})^{\mathrm{op}} \longrightarrow \mathbb{C}$$

(later we will identify the dual category of Boolean algebras with the category of Stone spaces).

In the second section we describe the left adjoint $I : \mathbb{C} \longrightarrow \mathbb{X}$ of the functor $H : \mathbb{X} \longrightarrow \mathbb{C}$.

In the third section we recall the admissibility condition and prove it under various additional assumptions. It looks like the most appropriate level of generality (for "Boolean Galois Theories") is the level of *geometric categories*, which we will introduce there (Definition 3.4).

As we will see in the fourth section, where the main examples and applications are briefly described, the categories such as the dual of commutative rings and the one of topological spaces, are geometric, which motivates the term "geometric". Since the dual of any locally indecomposable category in the sense of Diers is also geometric, our results show that in the context of [4], the categorical Galois Theory applies just as well as in the context of commutative rings.

1. The Functor H

Let $\mathbb{A}$ be a category with small hom-sets and let

$$Y_{\mathbb{A}} : \mathbb{A} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathrm{Set}] \tag{1}$$

be the Yoneda embedding

$$Y_{\mathbb{A}}(A) = \hom_{\mathbb{A}}(-, A)\,. \tag{2}$$

For a given glass $\mathscr{K}$ of small categories, by a $\mathscr{K}$-colimit we will mean a colimit of a functor $S \longrightarrow \mathbb{A}$ with S in $\mathscr{K}$; the smallest subcategory of the functor category $[\mathbb{A}^{\mathrm{op}}, \mathrm{Set}]$ containing all the representables and closed under $\mathscr{K}$-colimits will be denoted by $\mathbb{A}_{\mathscr{K}}$ and

$$Y_{\mathbb{A},\mathscr{K}} : \mathbb{A} \longrightarrow \mathbb{A}_{\mathscr{K}} \tag{3}$$

will denote the functor induced by $Y_{\mathbb{A}}$.

For simplicity, we will assume that the class $\mathscr{K}$ is *closed*, i. e. that for every object A in $\mathbb{A}_{\mathscr{K}}$, there exist a functor $F : S \longrightarrow \mathbb{A}$ with S in $\mathscr{K}$ and an isomorphism $A \simeq \mathrm{colim}(Y_{\mathbb{A},\mathscr{K}} F)$. The following universal property of $Y_{\mathbb{A},\mathscr{K}}$ is well known:

Proposition 1.1. *For every category $\mathbb{C}$ with $\mathscr{K}$-colimits, composition with $Y_{\mathbb{A},\mathscr{K}}$ induces an equivalence between the category of all functors from $\mathbb{A}$ to $\mathbb{C}$ and the category of all $\mathscr{K}$-colimit preserving functors from $\mathbb{A}_{\mathscr{K}}$ to $\mathbb{C}$. In particular, for every functor $T : \mathbb{A} \longrightarrow \mathbb{C}$ there exist an essentially unique functor $\overline{T} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathbb{C}$ with a natural isomorphism*

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad T \quad} & \mathbb{C} \\ & {\scriptstyle Y_{\mathbb{A},\mathscr{K}}} \searrow \quad \cong \quad \nearrow {\scriptstyle \overline{T}} & \\ & \mathbb{A}_{\mathscr{K}} & \end{array} \tag{4}$$

The next proposition is also well known (see, e.g., [6]), but we recall the proof.

Proposition 1.2. *Under the previous assumptions, the functor $\overline{T}$ is full and faithful provided T satisfies the following conditions:*

1. *T is full and faithful;*
2. *each hom-functor $hom_{\mathbb{C}}(T(A),\ \ ) : \mathbb{C} \longrightarrow \mathrm{Set}$ preserves $\mathscr{K}$-colimits.*

Proof. Let R be the functor $R : \mathbb{C} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathrm{Set}]$

$$R(C) = \hom_{\mathbb{C}}(T(\ \ ), C)\,. \tag{5}$$

Since colimits in the functor category $[\mathbb{A}^{\mathrm{op}}, \mathrm{Set}]$ are argument-wise, condition 2 says that R preserves $\mathscr{K}$-colimits. On the other hand, $RT \cong Y_{\mathbb{A}}$ since T is full and faithful, and then by applying the universal property of $Y_{\mathbb{A},K}$ to

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad T \quad} & [\mathbb{A}^{\mathrm{op}}, \mathrm{Set}] \\ & {\scriptstyle Y_{\mathbb{A},\mathscr{K}}} \searrow \quad \cong \quad \nearrow {\scriptstyle R\overline{T}} & \\ & \mathbb{A}_{\mathscr{K}} & \end{array} \tag{6}$$

we conclude that $R\overline{T}$ is naturally isomorphic to the inclusion functor $\mathbb{A}_{\mathscr{K}} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathrm{Set}]$, and hence is full and faithful. To deduce that $\overline{T}$ itself is full and faithful, consider the composite

$$\hom_{\mathbb{A}_{\mathscr{K}}}(A, A') \xrightarrow{\alpha} \hom_{\mathbb{C}}(\overline{T}(A), \overline{T}(A')) \xrightarrow{\beta} \hom_{[\mathbb{A}^{\mathrm{op}}, \mathrm{Set}]}(R\overline{T}(A), R\overline{T}(A')),$$

where A and A' are arbitrary objects of $A_{\mathscr{K}}$, and α and β are induced by $\overline{T}$ and R, respectively. To prove that α is a bijection, first observe that such is the composite $\beta\alpha$ so that it suffices to prove that β is injective, that is if $f, g : \overline{T}(A) \longrightarrow \overline{T}(A')$ are two arrows in $\mathbb{C}$ such that $fu = gu$ for every arrow with domain an object in the image of T and codomain $\overline{T}(A)$, then $f = g$. But this follows from the fact that $\overline{T}$ preserves $\mathscr{K}$-colimits and that A can be presented as a colimit of a $\mathscr{K}$-diagram in $\mathbb{A}$. □

Example 1.3. Let $\mathbb{C}$ be a category with finite coproducts so that $\mathscr{K}$ is the class of all finite discrete categories and $\mathscr{K}$-colimits are just the finite coproducts. An object C of $\mathbb{C}$ is said to be *connected* if the functor $\hom_{\mathbb{C}}(C,\ \ )$ preserves finite coproducts. Since in this case $\mathbb{A}_{\mathscr{K}}$ is equivalent to the category $\mathrm{Fam}_{\mathrm{fin}}(\mathbb{A})$ of finite families of objects of $\mathbb{A}$, the above proposition ensures that if

$T : \mathbb{A} \longrightarrow \mathbb{C}$ is a full and faithful functor with $T(A)$ connected for every object A in $\mathbb{A}$, then the induced functor

$$\overline{T} : \mathrm{Fam}_{\mathrm{fin}}(\mathbb{A}) \longrightarrow \mathbb{C} \tag{7}$$

is also full and faithful. In particular, if $\mathbb{A}$ is a terminal category, then $\mathrm{Fam}_{\mathrm{fin}}(\mathbb{A})$ is the category $\mathrm{Set}_{\mathrm{fin}}$ of finite sets, and hence if $\mathbb{C}$ has a connected terminal object (and finite coproducts), then there exists a full and faithful functor

$$\mathrm{Set}_{\mathrm{fin}} \longrightarrow \mathbb{C} \tag{8}$$

which preserves terminal objects and finite coproducts.

Example 1.4. Let $\mathbb{C}$ be a category with filtered colimits so that $\mathscr{K}$ is the class of small filtered categories. An object C in $\mathbb{C}$ is said to be *finitely presentable* if the functor $\hom_{\mathbb{C}}(C, \ \ ) : \mathbb{C} \longrightarrow \mathrm{Set}$ preserves filtered colimits. In this case $\mathbb{A}_{\mathscr{K}}$ is equivalent to the category $\mathrm{Ind}(\mathbb{A})$ [6], the inductive completion of $\mathbb{A}$, and the previous proposition ensures that if $T : \mathbb{A} \longrightarrow \mathbb{C}$ is a full and faithful functor with $T(A)$ finitely presentable for every object A in $\mathbb{A}$, then the induced functor

$$\overline{T} : \mathrm{Ind}(\mathbb{A}) \longrightarrow \mathbb{C} \tag{9}$$

is also full and faithful.

Consider now the functor (8) in the case of $\mathbb{C} = (\mathrm{Bool})^{\mathrm{op}}$, where Bool is the category of Boolean algebras. Clearly, it corresponds to the power set functor

$$P : (\mathrm{Set}_{\mathrm{fin}})^{\mathrm{op}} \longrightarrow \mathrm{Bool}\,; \tag{10}$$

moreover, since every finite boolean algebra is finitely presentable, it induces a full and faithful functor

$$\mathrm{Ind}((\mathrm{Set}_{\mathrm{fin}})^{\mathrm{op}}) \longrightarrow \mathrm{Bool}\,, \tag{11}$$

which is well known to be in fact an *equivalence*, since we also know that every finitely presentable (= finitely generated) Boolean algebra is finite.

Define now an object C of a category $\mathbb{C}$ with a terminal object 1 to be *finite* if there exist a natural number n, a coproduct $n \cdot 1$ of the terminal 1 in $\mathbb{C}$ n times and an isomorphism

$$C \simeq n \cdot 1\,. \tag{12}$$

More generally, C is said to be *profinite* if it is a filtered limit of finite objects. We will denote by $\mathbb{C}_{\mathrm{fin}}$ and $\mathbb{C}_{\mathrm{profin}}$ respectively the corresponding full subcategories of $\mathbb{C}$. What we have in fact proved is the following:

Theorem 1.5. *Let $\mathbb{C}$ be a category satisfying the following conditions:*

1. *$\mathbb{C}$ has a terminal object 1 and finite coproducts of it preserved by the functor*

$$\hom_{\mathbb{C}}(1, \ \ ) : \mathbb{C} \longrightarrow \mathrm{Set} \tag{13}$$

   (i. e., 1 is connected in $\mathbb{C}_{\mathrm{fin}}$);

2. *$\mathbb{C}$ has filtered limits of finite objects preserved by the functor*

$$\hom_{\mathbb{C}}(\ \ ,C) : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \tag{14}$$

for every finite object C (i. e., every finite object is finitely presentable in $(\mathbb{C}_{\mathrm{profin}})^{\mathrm{op}}$).

Then there exists a functor

$$H : (\mathrm{Bool})^{\mathrm{op}} \longrightarrow \mathbb{C} \tag{15}$$

which preserves terminal objects, finite coproducts of finite objects and filtered limits; moreover, such a functor is unique up to a unique natural isomorphism and full and faithful, and hence induces an equivalence

$$(\mathrm{Bool})^{\mathrm{op}} \sim \mathbb{C}_{\mathrm{profin}}. \tag{16}$$

Using simple properties of the construction Ind( ) it is also easy to prove the following:

Theorem 1.6. *If $\mathbb{C}$ satisfies all the conditions in the previous theorem, then the following conditions are equivalent:*

1. *the functor H preserves limits;*
2. *the functor H preserves pullbacks of finite objects;*
3. *$\mathbb{C}_{\mathrm{fin}}$ is closed under pullbacks in $\mathbb{C}$.*

Definition 1.7. A category $\mathbb{C}$ is said to be a "*category with profinite objects*" if it satisfies conditions 1 and 2 of Theorem 1.5, and any of the equivalent conditions of Theorem 1.6.

2. The Functor I

Let $\mathbb{C}$ be a category with profinite objects and let

$$1 \xrightarrow{\ e_0\ } 2 \xleftarrow{\ e_1\ } 1 \tag{17}$$

be a coproduct diagram in $\mathbb{C}$, 1 being a terminal object. The object 2 has an internal Boolean algebra structure in which e_0 and e_1 are zero and one respectively, since 2 can be considered as the image under the functor H of a two element set, and the functor H preserves finite products. Note that the internal Boolean algebra structure on 2 is uniquely determined as soon as we decide how to choose zero and one. This fixed structure on 2 gives a canonical boolean algebra structure on each hom-set $\hom_{\mathbb{C}}(C,2)$ so that we have in fact a functor

$$I = \hom_{\mathbb{C}}(\ \ ,2) : \mathbb{C} \longrightarrow (\mathrm{Bool})^{\mathrm{op}}. \tag{18}$$

Our purpose in this section is to show that I is a left adjoint to H.

Lemma 2.1. *The map*

$$\phi : \hom_{\mathbb{C}}(C,2) \longrightarrow \hom_{\mathrm{Bool}}(I(2), I(C)) \tag{19}$$

induced by I is a bijection for each object C in $\mathbb{C}$.

Proof. Just repeat standard arguments: recalling that ϕ is none other than composition, let ψ be the map in the opposite direction defined by $\psi(f) = f(1_2)$, i. e. ψ is the evaluation on the identity map on 2. Then:

$$\psi\phi(\alpha) = \psi(\phi(\alpha)) = \phi(\alpha)(1_2) = 1_2\alpha = \alpha \text{ and}$$

$$(\phi\psi(f))(\beta) = \phi(\psi(f))(\beta) = \beta\psi(f) = \beta f(1_2) = f(\beta 1_2) = f(\beta),$$

because f commutes with β, since f is a homomorphism of Boolean algebras and β, being an endofunction on 2, can be expressed as a Boolean term. □

Lemma 2.2. *The functor I has a right adjoint.*

Proof. Let $\mathbb{X}$ be the full subcategory of $(\mathrm{Bool})^{\mathrm{op}}$ whose objects are those Boolean algebras B for which there exists a universal arrow from I to B. Since $\mathbb{X}$ is closed under limits, it suffices to prove that $\mathbb{X}$ contains $1+1$ (the coproduct of two copies of the terminal object - in fact $1+1$ is the four element Boolean algebra). At this point, the previous lemma ensures that it suffices to show that there exists an object C in $\mathbb{C}$ with $I(C) \simeq 1+1$, which is clear, since I preserves coproducts and terminal objects. □

Theorem 2.3. *The functor I is a left adjoint to H.*

3. Admissible Objects

Let

$$(I, H, \eta, \epsilon) : \mathbb{C} \longrightarrow \mathbb{X} \tag{20}$$

be an arbitrary adjunction between categories $\mathbb{C}$ and $\mathbb{X}$ with pullbacks (later we will consider the case where $\mathbb{C}$ and $\mathbb{X}$ are as in the previous section). For a given object C in $\mathbb{C}$, the *induced adjunction*

$$(I^C, H^C, \eta^C, \epsilon^C) : (\mathbb{C} \downarrow C) \longrightarrow (\mathbb{X} \downarrow I(C)) \tag{21}$$

is constructed as follows (see, e.g., [10, (2.3)]):

- if (C, α) is an object of $(\mathbb{C} \downarrow C)$, i. e. $\alpha : A \longrightarrow C$ is an arrow in $\mathbb{C}$, then
$$I^C(A, \alpha) = (I(A), I(\alpha))\,;$$
- if (X, ϕ) is an object of $(\mathbb{X} \downarrow I(C))$, then
$$H^C(X, \phi) = (C \times_{HI(C)} H(X), \pi_1)\,.$$

  where π_1 is given by the pullback

$$\begin{array}{ccc} C \times_{HI(C)} H(X) & \xrightarrow{\pi_2} & H(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle H(\phi)} \\ C & \xrightarrow[\eta_C]{} & HI(C)\ ; \end{array} \tag{22}$$

- $\eta^C = \langle \alpha, \eta_A \rangle : A \longrightarrow C \times_{HI(C)} HI(A)$;
- $\epsilon^C_{(X,\phi)}$ is the composite

$$I(C \times_{HI(C)} H(X)) \xrightarrow{I(\pi_2)} IH(X) \xrightarrow{\epsilon_X} X\,. \tag{23}$$

An object C in $\mathbb{C}$ is said to be *admissible* if ϵ^C is an isomorphism or, equivalently, if H^C is full and faithful. This notion, first used in Galois Theory in [7], is closely related with *semi-left-exactness* in the sense of [3], as explained in [11]: if H is full and faithful, then clearly I is semi-left-exact if and only if every object C in $\mathbb{C}$ is admissible.

Proposition 3.1. *Let $\mathscr{S}$ be a small category and let $F : \mathscr{S} \longrightarrow \mathbb{C}$ be a functor for which $F(S)$ is admissible for each object S in $\mathscr{S}$. Then the limit of F, when exists, is admissible provided I preserves the limits of F and of all functors $G : \mathscr{S} \longrightarrow \mathbb{C}$ of the form*

$$G(S) = F(S) \times_{HIF(S)} H(X)\,, \tag{24}$$

where (X, ϕ) is an object in $(\mathbb{X} \downarrow I(\mathrm{lim}F))$, and the pullback is constructed with $\eta_{F(S)} : F(S) \longrightarrow HIF(S)$ and the composite

$$H(X) \xrightarrow{H(\phi)} HI(\mathrm{lim}F) \xrightarrow{HI(\pi_S)} HIF(S)\,,$$

where π_S is the projection $\mathrm{lim}F \longrightarrow F(S)$.

Proof. We must show that the composite

$$I((\mathrm{lim}F) \times_{HI(\mathrm{lim}F)} H(X)) \longrightarrow IH(X) \longrightarrow X$$

is an isomorphism. However this follows from the fact that so are the composites

$$I(F(S) \times_{HIF(S)} H(X)) \longrightarrow IH(X) \longrightarrow X$$

(for all S in $\mathscr{S}$) and from the canonical isomorphisms

$$I((\mathrm{lim}F) \times_{HI(\mathrm{lim}F)} H(X)) \simeq I((\mathrm{lim}F) \times_{\mathrm{lim}HIF} H(X)) \simeq I(\mathrm{lim}G) \simeq$$
$$\simeq \mathrm{lim}IG = \mathrm{lim}I(F(\ \ ) \times_{HIF(\ \ )} H(X)).$$ □

Proposition 3.2. *Let C_1 and C_2 be admissible objects in $\mathbb{C}$. Then the coproduct $C_1 + C_2$, when exists, is admissible provided either conditions* 1 *and* 2*, or condition* 3 *below hold:*

1. *$\mathbb{X}$ has binary coproducts preserved by H;*
2. *if*

$$\begin{array}{ccc} A_i & \xrightarrow{\beta_i} & H(X_i) \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle H(\phi_i)} \\ C_i & \xrightarrow[\eta_{C_i}]{} & HI(C_i) \end{array} \tag{25}$$

$(i = 1, 2)$ *are pullbacks in* $\mathbb{C}$*, then the coproduct* $A_1 + A_2$ *exists and*

$$\begin{array}{ccc} A_1 + A_2 & \xrightarrow{\beta_1 + \beta_2} & H(X_1) + H(X_2) \\ {\scriptstyle \alpha_1 + \alpha_2}\downarrow & & \downarrow{\scriptstyle H(\phi_1) + H(\phi_2)} \\ C_1 + C_2 & \xrightarrow[\eta_{C_1} + \eta_{C_2}]{} & HI(C_1) + HI(C_2) \end{array} \tag{26}$$

is a pullback;

3. $\mathbb{C}$ *and* $\mathbb{X}$ *have binary coproducts and the coproduct functors*

$$(\mathbb{C} \downarrow C_1) \times (\mathbb{C} \downarrow C_2) \longrightarrow (\mathbb{C} \downarrow (C_1 + C_2))$$
$$(\mathbb{X} \downarrow I(C_1)) \times (\mathbb{X} \downarrow I(C_2)) \longrightarrow (\mathbb{X} \downarrow (I(C_1) + I(C_2)))$$

are equivalences.

Proof. Obvious since I preserves binary coproducts. □

From now on, let us assume that $\mathbb{X}$ is the category of Stone Spaces (= profinite spaces = compact and totally disconnected spaces), which we identify with the category $(\text{Bool})^{\text{op}}$ via the Stone duality; we also assume that $\mathbb{C}$, I and H are as in the previous section so that $\mathbb{C}$ is a category with profinite objects and has pullbacks.

Theorem 3.3. *Let* C *be an object in* $\mathbb{C}$ *such that the functors*

$$C \times (\ \ ) \qquad \text{and} \qquad \hom_{\mathbb{C}}(C, \ \ ) \tag{27}$$

preserve finite coproducts of finite objects. Then $(\mathbb{C} \downarrow C)$ *is a category with profinite objects and the object* C *is admissible.*

Proof. The terminal object in $(\mathbb{C} \downarrow C)$ being $(C, 1_C)$, where 1_C is the identity morphism of C, we have:

$$\hom_{(\mathbb{C}\downarrow C)}((C, 1_C), (C, 1_C) + \cdots + (C, 1_C))$$
$$\simeq \hom_{(\mathbb{C}\downarrow C)}((C, 1_C), (C \times (1 + \cdots + 1), \pi_1)$$
$$\simeq \hom_{\mathbb{C}}(C, 1 + \cdots + 1) \simeq \hom_{\mathbb{C}}(C, 1) + \cdots + \hom_{\mathbb{C}}(C, 1)$$
$$\simeq \hom_{(\mathbb{C}\downarrow C)}((C, 1), (C, 1)) + \cdots + \hom_{(\mathbb{C}\downarrow C)}((C, 1), (C, 1)),$$

i. e., $(\mathbb{C} \downarrow C)$ satisfies condition 1 of Theorem 1.5.

Let $\mathscr{S}$ be a (small) filtered category and $F : \mathscr{S}^{\text{op}} \longrightarrow (\mathbb{C} \downarrow C)$ be a functor; denoting by U the forgetful functor $(\mathbb{C} \downarrow C) \longrightarrow \mathbb{C}$, we have for any finite object in $(\mathbb{C} \downarrow C)$:

$$\hom_{(\mathbb{C}\downarrow C)}(\lim F, (C, 1_C) + \cdots + (C, 1_C))$$
$$\simeq \hom_{(\mathbb{C}\downarrow C)}(\lim F, (C \times (1 + \cdots + 1), \pi_1)$$
$$\simeq \hom_{\mathbb{C}}(U(\lim F), 1 + \cdots + 1) \simeq \hom_{\mathbb{C}}(\lim UF, 1 + \cdots + 1)$$
$$\simeq \operatorname{colim} \hom_{\mathbb{C}}(UF(\ \ ), 1 + \cdots + 1)$$

$\simeq \operatorname{colim} \hom_{(\mathbb{C}\downarrow C)}(F(\ ), (C \times (1 + \cdots + 1), \pi_1))$

$\simeq \operatorname{colim} \hom_{(\mathbb{C}\downarrow C)}(F(\ ), (C, 1_C) + \cdots + (C, 1_C)),$

i. e., $(\mathbb{C} \downarrow C)$ satisfies condition 2 of Theorem 1.5.

Since $(\mathbb{C} \downarrow C)$ satisfies also condition 1 of Theorem 1.5, the functor

$$\mathbb{C}_{\text{fin}} \longrightarrow (\mathbb{C} \downarrow C) \tag{28}$$

sending $1 + \cdots + 1$ to $(C \times (1 + \cdots + 1), \pi_1) \simeq (C, 1_C) + \cdots + (C, 1_C)$ is full and faithful so that we can conclude that $(\mathbb{C} \downarrow C)_{\text{fin}}$ is closed under pullbacks in $(\mathbb{C} \downarrow C)$ since $\mathbb{C}_{\text{fin}}$ is closed under pullbacks in $\mathbb{C}$ and the functor (28) preserves them. Hence $(\mathbb{C} \downarrow C)$ is a category with profinite objects.

Since $(\mathbb{C} \downarrow C)$ is a category with profinite objects, we can construct the adjunction between $(\mathbb{C} \downarrow C)$ and $\mathbb{X}$ just as we did for $\mathbb{C}$, which we will denote by

$$(I^{(C)}, H^{(C)}, \eta^{(C)}, \epsilon^{(C)}) : (\mathbb{C} \downarrow C) \longrightarrow \mathbb{X}\,. \tag{29}$$

Since

$\hom_{\mathbb{X}}(I(C), 1 + 1) \simeq \hom_{\mathbb{C}}(C, H(1 + 1))$

$\simeq \hom_{\mathbb{C}}(C, 1 + 1) \simeq \hom_{\mathbb{C}}(C, 1) + \hom_{\mathbb{C}}(C, 1)$

is a two element set, $I(C)$ must be a terminal object in $\mathbb{X}$. Hence $(\mathbb{X} \downarrow I(C))$ can be identified with $\mathbb{X}$ and, since (29) obviously agrees with (21) and $H^{(C)}$ is full and faithful by Theorem 1.5, this completes the proof. □

Together with the two previous propositions this theorem shows how to build up admissible objects. The most appropriate level of generality seems to be the one described in the following

Definition 3.4. A category $\mathbb{C}$ is said to be *geometric* when it satisfies the following conditions:

1. $\mathbb{C}$ is *complete*;
2. $\mathbb{C}$ is *extensive*, i. e., it has finite coproducts and the coproduct functor

$$(\mathbb{C} \downarrow A) \times (\mathbb{C} \downarrow B) \longrightarrow (\mathbb{C} \downarrow (A + B))$$

   is an equivalence, for each pair of objects A and B in $\mathbb{C}$;
3. all finite objects in $\mathbb{C}$ are finitely presentable in $(\mathbb{C}_{\text{profin}})^{\text{op}}$;
4. $\mathbb{C}$ is connected, i. e. $1 \simeq A + B$ in $\mathbb{C}$, implies either $A \simeq 1$ or $B \simeq 1$.

Remark 3.5.

1. Under condition 1, condition 2 in Definition 3.4 holds if and only if $\mathbb{C}$ has disjoint and universal finite coproducts; in particular the pullbacks in $\mathbb{C}$ are distributive with respect to finite coproducts, and finite coproducts of pullback squares are again pullback squares. For a discussion of extensivity one may consult [1].
2. The reason for using $(\mathbb{C}_{\text{profin}})^{\text{op}}$ instead of $\mathbb{C}^{\text{op}}$ in 3 is to require conditions which hold also in the category Top of topological spaces.
3. Under conditions 1 and 2, condition 4 holds if and only if 1 is connected in $\mathbb{C}$, and if and only if it is connected in $\mathbb{C}_{\text{fin}}$.

4. Every geometric category is a category with profinite objects as follows from conditions 1 and 3.

The following theorem follows now easily.

Theorem 3.6. *Let $\mathbb{C}$ be a geometric category and $\mathbb{X}$, I, H as above (observe that I and H are well defined since $\mathbb{C}$ is a category with profinite objects). Then:*

1. *any finite coproduct of connected objects is admissible;*
2. *if the functor I preserves filtered limits (i. e. if 2 is finitely presentable in $\mathbb{C}^{\mathrm{op}}$), then any filtered limit of finite coproducts of connected objects is admissible.* □

4. Boolean Galois Theories

4.1. **Connected Normal Galois Theory.** Let $k \subset K$ be a (possibly infinite) Galois field extension, $\mathrm{Sub}(K/k)$ the lattice of its subextensions, $\mathrm{Aut}_k(K)$ the Galois group with the Krull topology, and $\mathrm{Sub}(\mathrm{Aut}_k(K))$ the lattice of closed subgroups in $\mathrm{Aut}_k(K)$. The *fundamental theorem of Galois Theory* asserts that there is a lattice isomorphism

$$\mathrm{Sub}(K/k)^{\mathrm{op}} \simeq \mathrm{Sub}(\mathrm{Aut}_k(K)). \tag{30}$$

This theorem can be deduced from the equivalence of categories

$$\mathrm{Spl}(K/k) \cong \mathbb{X}^{\mathrm{Aut}_k(K)}, \tag{31}$$

where $\mathbb{X}$ is the category of Stone Spaces and $\mathrm{Spl}(K/k)$ is the dual category of k-algebras split over K. Recall that A is in $\mathrm{Spl}(K/k)$ means that $K \otimes_k A$ is freely generated by its idempotents over K – or, equivalently, that A is a directed union of subalgebras each of which is a finite products of subextensions of $k \subset K$.

The equivalence (31) was generalized from fields to arbitrary commutative rings by A.R. Magid in [12] (the finite version, even for schemes, was obtained by A. Grothendieck as mentioned in [12]). Later in [7] (see also [8]–[10]) it was shown that there is a purely categorical "Fundamental Theorem of Galois Theory" which includes all these results. However, in order to deduce, say, (31) from the categorical version, one needs to prove

a) the functor $K \otimes_k (\ \ )$ from k-algebras to K-algebras is monadic;
b) K is admissible in the dual category of commutative rings.

Both of these proofs are easy (one may consult [8]): the first one uses Beck's monadicity theorem, and the second one uses Pierce Theory. This suggests that it should be possible to prove (31) in the more general context studied by Y. Diers [4], where the Pierce Theory is in fact developed. However, Theorem 3.3 tells us that admissibility holds even for what we call "categories with profinite objects" (i. e., in an even more general context than in [4]), and from the results of [7] (or of [8]–[10]), we obtain the following.

Let $\mathbb{C}$ be a category with profinite objects, $p : E \longrightarrow B$ an *effective descent morphism* in $\mathbb{C}$ (i. e. a morphism such that the functor

$$E \times_B (\ \ ) : (\mathbb{C} \downarrow B) \longrightarrow (\mathbb{C} \downarrow E)$$

is monadic), and $\mathrm{Spl}(E,p)$ the full subcategory of $(\mathbb{C} \downarrow B)$ whose objects are the (A,α) for which the diagram

$$\begin{array}{ccc} E\times_B A & \xrightarrow{\eta_{E\times_B A}} & HI(E\times_B A) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle HI(\pi_1)} \\ E & \xrightarrow[\eta_E]{} & HI(E) \end{array} \tag{32}$$

is a pullback (where I, H, η are as in the previous section). If (E,p) is I-*normal* (i. e. (E,p) is in $\mathrm{Spl}(E,p)$) and the functors (27) with $C = E$ preserve finite coproducts of connected objects, then there is an equivalence of categories

$$\mathrm{Spl}(E,p) \cong \mathbb{X}^{\mathrm{Aut}(E,p)}, \tag{33}$$

where $\mathrm{Aut}(E,p)$ is the automorphism group of (E,p) in $(\mathbb{C} \downarrow B)$, equipped with the topology induced by the canonical bijection $\mathrm{Aut}(E,p) \simeq I(E\times_B E)$; moreover, the topology in $\mathrm{Aut}(E,p)$ coincides with the appropriate Krull topology.

This result applies not only to the situation considered in [4], but also to, say, topological spaces and to any topos with connected terminal object and filtered limits. Note that in all these examples second of the functors (27) (with $C = E$) preserve finite coproducts if and only if E is connected.

4.2. **"General" Galois Theory.** When there exists the equivalence (33), the object E is also "connected in the sense of I", i. e. $I(E)$ is a terminal object in $\mathbb{X}$. If this is not the case, then $\mathrm{Aut}(E,p) \simeq I(E\times_B E)$ must be replaced with the internal groupoid $\mathrm{Gal}_I(E,p) =$

$$I(E\times_B E\times_B E) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; I(E\times_B E) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; I(E) \tag{34}$$

in $\mathbb{X}$ as it is done in [12] for commutative rings. Furthermore, if (E,p) is not even I-normal as in [10], then $\mathrm{Gal}_I(E,p)$ becomes an *internal pregroupoid* and the category

$$\mathbb{X}^{\mathrm{Gal}_I(E,p)}$$

of internal $\mathrm{Gal}_I(E,p)$-actions still can be defined (see [10] for details). Note that the fundamental theorem of Galois Theory in this case has the simple form

$$\mathrm{Spl}(E,p) \cong \mathbb{X}^{\mathrm{Gal}_I(E,p)} \tag{35}$$

only under the admissibility of E, $E\times_B E$, $E\times_B E\times_B E$ (and not only of E as discussed in [10]). However, if E is a finite coproduct of connected objects in a geometric category, then so are $E\times_B E$ and $E\times_B E\times_B E$, and the same is true for the filtered limits of finite coproducts of connected objects. Hence when $\mathbb{C}$ is a geometric category and $p : E \longrightarrow B$ is an effective descent morphism in $\mathbb{C}$, Theorem 3.6 gives that if E is a finite coproduct of connected objects, then there is the equivalence (35). The same is true when E is a filtered limit of finite coproducts of connected objects and I preserves filtered limits. Note that:

a) If $\mathbb{C}^{op}$ is *locally indecomposable* in the sense of [4], then $\mathbb{C}$ is a geometric category and I preserves filtered limits; moreover, in this case every object in $\mathbb{C}$ is a filtered limit of finite coproducts of connected objects. Therefore in this case we have the equivalence (35) for every effective descent morphism $p : E \longrightarrow B$.

b) The category Top of topological spaces is not so "good" (see Remark 3.5, 2), but we still have the equivalence (35) for every effective descent morphism $p : E \longrightarrow B$ in which E is either a finite coproduct of connected spaces (in the usual sense), or a filtered limit of finite coproducts of connected spaces, provided that the limit satisfies the condition required in Proposition 3.1 – which is true, for example, if all spaces are compact Hausdorff. However, it is easy to see that, for example, the infinite discrete spaces are not admissible (although they are filtered limits of finite coproducts of codiscrete spaces).

c) If $\mathbb{C}$ is the category of compact Hausdorff spaces, then it is easy to show that $\mathbb{C}$ is a geometric category in which every object is admissible, and since all surjections in $\mathbb{C}$ are effective descent morphisms, we have the equivalence (35) for every surjective continuous map $p : E \longrightarrow B$ (see [2]). In this case $\mathbb{C}^{op}$ is again not a locally indecomposable category in the sense of [4].

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