**Georgian Mathematical Journal**
Volume 9 (2002), Number 4, 607–622

# CHAIN THEORIES AND SIMPLICIAL ABELIAN GROUP SPECTRA

FRIEDRICH W. BAUER

**Abstract.** Two different definitions of a chain theory $\mathbf{K}_*$ lead to the same class of derived homology theories $h_*(\ )$. On the category of CW pairs, these are those homology theories $\mathbf{E}_*$ admitting a classifying simplicial abelian group spectrum $\mathbf{E}$. So one has a functor from the category of chain theories into the category of simplicial abelian group spectra.

**2000 Mathematics Subject Classification:** 55U15, 55U10, 55P43.
**Key words and phrases:** Chain theories, classifying spectrum, simplicial abelian group spectrum.

## 0. Introduction

The subject of the present paper are chain theories (Definitions 1.1 and 1.2) and their derived homology theories. Although these definitions are not equivalent, it turns out that they are associated with the same class of homology theories. If the underlying category $\mathfrak{K}^2$ is the category of CW pairs, these homology theories are precisely direct sums of ordinary homology theories with coefficients in an abelian group (cf. 6.3). This is the classical result of R. O. Burdick, P. E. Conner and E. E. Floyd [4]. Here it appears as a corollary of the existence of a simplicial abelian group spectrum $\mathbf{E} = \mathbf{E}_{\boldsymbol{K}_*}$ such that the derived homology $H_*(\boldsymbol{K}_*(\ ))$ of $\boldsymbol{K}_*$ is isomorphic (as a homology theory) to the homology theory $\mathbf{E}_*(\ )$ (Theorem 6.2). In Section 1 we display the two kinds of chain theories together with a proof that they determine the same class of homology theories (as their derived homology). Since a chain theory of the second kind is a special case of a *chain functor* (cf. Proposition 2.1, Theorem 2.2 and [1], [2] concerning further information about chain functors), we felt obliged to include a definition of a chain functor in Section 2.

Sections 3–6 are devoted to a proof of the main Theorem 6.2 based on considerations about special chain functors (namely chain *theories* of the second kind). This proof is technically much simpler than the corresponding proof of a classifying spectrum (which is not any more an abelian group spectrum!) for an arbitrary chain functor (cf. [3]).

As we have already pointed out, we obtain the main result of [4] as a byproduct.

ISSN 1072-947X / $8.00 / 

## 1. Chain Theories

Let $\mathfrak{K}^2$ be any category of pairs of topological spaces like the category of (based or unbased) CW pairs or any simplicial analogue. We agree to omit almost everywhere basepoints (whenever they occur) from our notation.

**1.1. Definition.** A functor $A_* : \mathfrak{K}^2 \longrightarrow \boldsymbol{ch}$ (= category of chain complexes) is a chain theory of the first kind, whenever to each $(X, U) \in \mathfrak{K}^2$ there exists canonically a short exact sequence

$$0 \longrightarrow A_*(U) \xrightarrow{i_\#} A_*(X) \xrightarrow{j_\#} A_*(X, U) \longrightarrow 0. \tag{1}$$

A chain theory of the first kind determines a homology theory

$$h_*(\cdot) = \{h_n, \partial, n \in \mathbb{Z}\} = H_*(A_*(\cdot))$$

by setting

$$h_*(X, U) = H_*(A_*(X, U)) \tag{2}$$

(the derived homology of $A_*$) and a boundary operator

$$\partial : h_n(X, U) \longrightarrow h_{n-1}(U) \tag{3}$$

which is defined as follows: Suppose $c \in Z_n(A_*(X, U))$, then there exists $\bar{c} \in A_*(X)$ such that $j_\#(\bar{c}) = c$. We define

$$\partial[c] = [i_\#^{-1}\, d\bar{c}]. \tag{4}$$

As a first consequence of (1) we notice $h_*(X, X) = 0$.

**1.2. Definition.** A functor $B_* : \mathfrak{K}^2 \longrightarrow \boldsymbol{ch}$ is a chain theory of the second kind, whenever

B1) to each $(X, U) \in \mathfrak{K}^2$ there are inclusions

$$B_*(U) \underset{\subset}{\xrightarrow{i_\#}} B_*(X) \underset{\subset}{\xrightarrow{j_\#}} B_*(X, U) \tag{5}$$

such that for any cycle $z \in B_*(X, U)$ there exist $z' \in B_*(X)$ as well as $\bar{u} \in B_*(U, U)$ such that

$$z \sim j_\# z' + q_\# \bar{u}, \quad q : (U, U) \subset (X, U), \quad dz' \in im\ i_\#.$$

Observe that the homology class of $j_\# z' + q_\# \bar{u}$ does not depend on the choice of $\bar{u}$.

B2) $j_\# z' + q_\# \bar{u} = dx$ in $B_*(X, U) \implies \exists u \in B_*(U)$: $du = i_\#^{-1} dz'$ in $B_{*-1}(U)$.

B3) $z \in Z_n(B_*(X))$, $j_\#(z) \sim 0$ in $B_*(X, U) \implies \exists x \in B_*(X)$, $u \in B_*(U)$, such that $dx = z + i_\# u$.

B4) $B_*(X, X)$ is acyclic; all inclusions $k : (X, U) \subset (Y, V)$ induce monomorphisms.

*A morphism $\lambda : \boldsymbol{K}_* \longrightarrow \boldsymbol{L}_*$ between chain theories of the first or the second kind is simply a natural transformation of functors.*

Again we obtain a homology theory $h_* = \{h_n, \ \tilde{\partial}, \ n \in \mathbb{Z}\}$ with homology groups

$$h_n(X, U) = H_n(B_*(X, U))$$

and the boundary operator

$$\tilde{\partial} : h_n(X, U) \longrightarrow h_{n-1}(U)$$

which is defined as follows: Let $z \in B_*(X, U)$ be a cycle, then $z \sim z' + \bar{u}$ (inclusions omitted from the notation). We deduce $dz' = -d\bar{u}$, hence $dz' \in im\ i_\#$ and we set

$$\tilde{\partial}[z] = [i_\#^{-1}\, dz']. \tag{6}$$

**1.3. Lemma.** 1) *$\partial[z]$ resp. $\tilde{\partial}[z]$ do not depend on the choices involved and are natural homomorphisms.*

2) *The homology sequence of a pair $(X, U) \in \mathfrak{K}^2$ is exact.*

*Proof.* **Ad 1):** We treat only $\tilde{\partial}$, the case of $\partial$ is standard. $z_1' + \bar{u}_1 \sim z_2' + \bar{u}_2 \implies z_1' - z_2' + (\bar{u}_1 - \bar{u}_2) = dx \overset{B2)}{\implies} dz_1' - dz_2' = du,\ u \in B_*(U) \implies [i_\#^{-1} dz_1'] = [i_\#^{-1} dz_2']$.

**Ad 2):** The exactness of the homology sequences is immediate. □

Every chain theory $B_*$ of the second kind determines a chain theory of the first kind and vice-versa:

Let $B_*$ be a chain theory of the second kind, then we define $\hat{B}_*$ by

$$\hat{B}_*(X) = B_*(X), \quad \hat{B}_*(X, U) = B_*(X)/B_*(U)$$

noticing that

$$0 \longrightarrow \hat{B}_*(U) \longrightarrow \hat{B}_*(X) \longrightarrow \hat{B}_*(X, U) \longrightarrow 0$$

is exact.

Suppose that $A_*$ is a chain theory of the first kind, then we define $\tilde{A}_*$ by

$$\tilde{A}_*(X) = A_*(X), \quad \tilde{A}_*(X, U) = A_*(X) \oplus_i cone\ A_*(U).$$

We have inclusions

$$\tilde{A}_*(U) \subset \tilde{A}_*(X) \subset \tilde{A}_*(X, U).$$

**Ad B1):** A cycle $z \in \tilde{A}_*(X, U)$ is of the form $z = z' + \tilde{u},\ \tilde{u} \in cone\ A_*(U)$. Since $\tilde{A}_*(U, U) = cone\ A_*(U)$, B1) is satisfied.

**Ad B2):** Suppose $z' + \tilde{u} = d(x + \tilde{v}),\ \tilde{u}, \tilde{v} \in cone\ A_*(U),\ x \in A_*(X)$, then $z' - dx = d\tilde{v} - \tilde{u} = u \in A_*(U)$, hence

$$i_\#^{-1} dz' = du,$$

ensuring that B2) holds.

**Ad B3):** Let $z \in \tilde{A}_*(X)$ be a cycle, $j_\# z = d(x + \tilde{v}),\ x \in A_*(X),\ \tilde{v} \in cone\ A_*(U)$, then $d\tilde{v} \in im\ i_\#$, hence $d\tilde{v} = i_\# u,\ u \in A_*(U)$, so that $j_\# z = dx + i_\# u$, implying B3).

**Ad B4):** $\tilde{A}_*(X, X) = cone\ A_*(U)$ is acyclic.

Suppose $A_*$ is a chain theory of the first kind, then we have

$$\hat{\tilde{A}}_*(X, U) = \tilde{A}_*(X)/\tilde{A}_*(U) = A_*(X)/A_*(U) = A_*(X, U),$$

hence

$$\hat{\tilde{A}}_* = A_*.$$

If $B_*$ is a chain theory of the second kind, then we have

$$\tilde{\hat{B}}_*(X) = B_*(X),$$

but

$$\tilde{\hat{B}}_*(X,U) = B_*(X) \oplus_i \mathit{cone}\ B_*(U),$$

which does not agree with $B_*(X,U)$, however

**1.4. Lemma.** *There exists a natural homomorphism*

$$\beta : B_*(X) \oplus_i \ \mathit{cone}\ B_*(U) \longrightarrow B_*(X,U)$$

*inducing a natural isomorphism of homology groups.*

*Proof.* We set $\beta \mid B_*(X) = j_\#$ and notice that

$$B_*(U) \xrightarrow{q_\#} B_*(U,U) \subset B_*(X,U)$$

factors over $B_*(U) \subset \mathit{cone}\ B_*(U)$, furnishing a $\beta$. Let $z = z' + \bar{u}$ be a cycle in $B_*(X,U)$, $z' \in B_*(X)$, $\bar{u} \in B_*(U,U)$, then we find a $\tilde{u} \in \mathit{cone}\ B_*(U)$ such that $-dz' = d\tilde{u} = d\bar{u} \in B_*(U)$. It turns out that $\beta(z' + \tilde{u}) \sim z$. Hence $\beta_*$ is epic.

Let $\beta(z' + \tilde{u}) = z' + \bar{u}$, $\bar{u} \in B_*(U,U)$, $\tilde{u} \in \mathit{cone}\ B_*(U)$ be bounding in $B_*(X,U)$. According to B2) we detect $v \in B_*(U)$ such that $dz' = dv$, hence $z' - i_\# v$ is a cycle. We deduce that $(z' - v) + (v + \bar{u}) = d\bar{x}$, $\bar{x} \in B_*(X,U)$, hence, because $v+\bar{u}$ is a bounding cycle in $B_*(U,U)$, $j_\#(z'-v) \sim 0$ in $B_*(X,U)$. According to B3) we deduce $z' - v = dx + w$, $x \in B_*(X)$, $w \in B_*(U)$, hence

$$z' + \tilde{u} = dx + w + v + \tilde{u} = dx + d\tilde{w}, \quad \tilde{w} \in \mathit{cone}\ B_*(U),$$

(observing that $w + v + \tilde{u}$ is a cycle in $\mathit{cone}\ B_*(U)$, hence bounding) implying that $z' + \tilde{u} \sim 0$ in $B_*(X) \oplus_i \ \mathit{cone}\ B_*(U)$. Therefore $\beta_*$ is monic. □

Now we compare the homology of $A_*$ and $\tilde{A}_*$ resp. the homology of $B_*$ and $\hat{B}_*$:

**1.5. Proposition.** 1) *Let $A_*$ be a chain theory of the first kind, then there exists a natural transformation of functors $\alpha : \tilde{A}_* \longrightarrow A_*$ inducing an isomorphism $\alpha_*$ of homology theories.*

2) *Let $B_*$ be a chain theory of the second kind, then there exist natural transformations*

$$\hat{B}_* \xleftarrow{\alpha} \tilde{\hat{B}}_*, \xrightarrow{\beta} B_*$$

*inducing isomorphisms of homology theories.*

*Proof.* **Ad 1):** We set $\alpha : \tilde{A}_*(X) \xrightarrow{=} A_*(X)$ to be the identity and

$$\alpha : \tilde{A}_*(X,U) = A_*(X) \oplus_i \mathit{cone}\ A_*(U) \longrightarrow (A_*(X) \oplus_i \ \mathit{cone}\ A_*(U))/\mathit{cone}\ A_*(U) \approx$$
$$\approx A_*(X)/A_*(U) = A_*(X,U)$$

the projection, which is easily seen to become an isomorphism on the homology level.

Let $[z] = [z' + \tilde{u}] \in H_n(\tilde{A}_*(X,U))$ be given, then

$$\alpha_* \partial[z] = \alpha_*[i_{\#}^{-1} dz'] = [i_{\#}^{-1} dz'] = \partial \alpha_*[z],$$

hence $\alpha_*$ commutes with boundaries.

**Ad 2):** There exists a transformation of functors $\beta : \tilde{\hat{B}}_*(X) \to \hat{B}_*(X)$ and $\beta$ as in Lemma 1.4, for pairs $(X,U)$, assuring us that $\beta_*$ is an isomorphism of homology groups.

Let $[z] = [z' + \tilde{u}] \in H_n(\tilde{\hat{B}}_*(X,U))$ be given, then we calculate

$$\beta_* \partial[z' + \tilde{u}] = [i_{\#}^{-1} dz'] = \partial[j_{\#} z'] = \partial \beta_*[z' + \tilde{u}],$$

ensuring that $\beta_*$ is an isomorphism of homology theories. $\square$

*Remarks.* 1) Proposition 1.5 asserts that chain theories of the first and of the second kind reach precisely the same kind of homology theories.

2) Definition 1.1 describes the classical chain theory, while Definition 1.2 describes a special case of a chain functor, a concept we will briefly record in the next section.

## 2. Chain Functors and Associated Homology Theories

In contrast to the preceding section we start with a homology theory $h_*(\ ) = \{h_n, \partial_n, n \in \mathbb{Z}\}$ and ask if there exists a chain theory of the first kind $\boldsymbol{K}_*$ and an isomorphism of homology theories

$$h_*(\ ) \approx H_*(K_*(\ ))$$

We assume that all homology and all chain theories have compact carriers. Alternatively we can of course work with the category of finite CW pairs.

Due to a result of R. O. Burdick, P. E. Conner and E. E. Floyd ([4] or [2] for a further reference) this implies, for $\mathfrak{K}^2$ = category of CW pairs, that $h_*(\ )$ is a sum of ordinary homology theories i.e., of those satisfying a dimension axiom although not necessarily in dimension 0.

The non-existence of such a chain theory of the first kind (implying the non-existence of a chain theory of the second kind with the correct homology and boundary operator) gives rise to the theory of *chain functors*, cf. [2] for further references.

A chain functor $\boldsymbol{C}_* = \{C_*,\ C'_*,\ l,\ i',\ \kappa,\ \varphi\}$ is

**CH1)** *a pair of functors* $C_*,\ C'_* : \mathfrak{K} \longrightarrow \boldsymbol{ch}$, *natural inclusions* $i' :\ C_*(A) \subset C'_*(X,A)$, $l : C'_*(X,A) \subset C_*(X,A)$ *non-natural chain mappings*

$$\varphi :\ C'_*(X,A) \longrightarrow C_*(X),$$
$$\kappa :\ C_*(X) \longrightarrow C'_*(X,A),$$

*chain homotopies* $\varphi\kappa \simeq 1$, $\quad j_{\#}\, \varphi \simeq l \quad (j :\ X \subset (X,A))$, *as well as an identity*

$$\kappa\, i_{\#} = i', \qquad i :\ A \subset X.$$

**CH2)** *All inclusions* $k :\ (X,A) \subset (Y,B)$ *are supposed to induce monomorphisms. All* $C_*(X,X)$ *are acyclic.*

We denote by $\mathbf{C}_*(X)$ the chain group $C_*(X,X)$ which according to CH2) can be assumed to contain all $C_*(X,U)$ for all pairs $(X,U)$.

Needless to say, that $C'_*$, as well as $\phi$, $\kappa$ are *not* determined by the functor $C_*(\cdots,\cdots)$ but are additional ingredients of the structure of a chain functor.

Instead of the exact sequence (1) in §1, for *chain theories* (of the first kind) we are now, in the case of a *chain functor*, dealing with the sequence

$$0 \longrightarrow C_*(A) \xrightarrow{i'} C'_*(X,A) \xrightarrow{p} C'_*(X,A)/im\ i' \longrightarrow 0 \tag{1}$$

and there exists a homomorphism

$$\begin{aligned} \psi :\ H_*(C'_*(X,A)/im\ i') &\longrightarrow H_*(C_*(X,A)) \\ [z'] \qquad &\longmapsto \quad [l(z') + q_\#(\bar{a})] \end{aligned} \tag{2}$$

where $z' \in C'_*(X,A)$, $dz' \in im\ i'$, $q:\ (A,A) \subset (X,A)$, $\bar{a} \in C_*(A,A)$, $d\bar{a} = -dz'$.

**CH3)** *It is assumed that $\psi$ is epic.*

Since $C_*(A,A)$ is acyclic, $dz' \in im\ i'$, such that $\bar{a}$ exists and $[l(z') + q_\#(\bar{a})]$ is independent of the choice of $\bar{a}$.

This assumption implies that each cycle $z \in C_*(X,A)$ is homologous to a cycle of the form $l(z') + q_\#(\bar{a})$, with $z'$ being a *relative* cycle, the analogue of a classical relative cycle $z \in C_*(X)$ with $dz \in im\ i_\#$, whenever (1) holds, i.e., whenever we are dealing with a chain theory of the first kind.

**CH4**) *We assume*

$$ker\ \psi \subset ker\ \bar{\partial}, \tag{3}$$

$\bar{\partial}:\ H_n(C'_*(X,A)/im\ i') \longrightarrow H_{n-1}(C_*(A))$ *being the boundary induced by the exact sequence* (2)*. Moreover*

$$ker\ j_* \subset ker\ p_*\ \kappa_*, \tag{4}$$

*with, e.g., $\kappa_*$ denoting the mapping induced by $\kappa$ for the homology groups.*

**CH5)** *Homotopies $H:\ (X,A) \times I \longrightarrow (Y,B)$ induce chain homotopies $D(H):\ C_*(X,A) \longrightarrow C_{*+1}(Y,B)$, natural and compatible with $i'$ and $l$.*

These are almost all ingredients of a chain functor we need. The derived (or associated) homology of a chain functor

$$h_*(X,A) = H_*(C_*(X,A))$$

resp. for the induced mappings, is endowed with a boundary operator $\partial:\ H_n(C_*(X,A)) \longrightarrow H_{n-1}(C_*(A))$ determined by $\bar{\partial}$:

We seek for $\zeta \in H_n(C_*(X,A))$ a representative $l(z') + q_\#(\bar{a})$ and set

$$\partial\ \zeta = \bar{\partial}[z'] = [i'^{-1}\ d\ z'].$$

This turns out to be independent of the choices involved.

This $h_*(\ )$ satisfies all properties of a homology theory eventually with exception for an excision axiom. Therefore it is convenient to add:

**CH6)** *Let $p:\ (X,A) \longrightarrow (X',A')$ be an excision map* (*of some kind, e.g. $p:\ (X,A) \longrightarrow (X/A,\star)$*), then $p_* = H_*(C_*(p))$ is required to be an isomorphism.*

This $H_*(C_*(\ )) = h_*(\ )$ turns out to be a homology theory. Moreover, under very general conditions on $\mathfrak{K}^2$, every homology theory $h_*(\ )$ is isomorphic to the derived homology of some chain functor (cf. [2]).

Let $\lambda : C_* \longrightarrow L_*,\quad \lambda' : C'_* \longrightarrow L'_*$ be natural transformations, where $\boldsymbol{C}_*,\ \boldsymbol{L}_*$ are chain functors, compatible with $i'$, $l$ and the natural homotopies of **CH5)**, then we call $\lambda :\ \boldsymbol{C_*} \longrightarrow \boldsymbol{L_*}$ a *transformation of chain functors*. Such a transformation induces obviously a transformation $\lambda_* :\ H_*(\boldsymbol{C_*}) \longrightarrow H_*(\boldsymbol{L_*})$ of the derived homology. This furnishes a category $\mathfrak{Ch}$ of chain functors. A *weak equivalence* in $\mathfrak{Ch}$ is $\lambda :\ \boldsymbol{C_*} \longrightarrow \boldsymbol{L_*}$ which has the homotopy inverse.

For our present purposes we do not have to require that

**CH7)** *all chain complexes $C_*(X, A)$ be free.*

However this is not a very severe restriction as it can be accomplished if it is not automatically fulfilled. This condition is usually part of the definition of a chain functor.

Our main example of a chain functor $\mathbf{C}_*$ in this article is a chain theory of the second kind which has a compact carrier and satisfy **CH5**–**CH7**.

We define $C'_*(X,U) = C_*(X)$ and take for $\varphi : C'_*(X,U) \longrightarrow C_*(X),\ \kappa : C_*(X) \longrightarrow C'_*(X,U)$ the identity. The inclusion $l : C'_*(X,U) \subset C_*(X,U)$ is simply $j_\#$, $j : X \subset (X,U)$ and $i' : C_*(U) \subset C'_*(X,U)$ equals $i_\#$, $i : U \subset X$.

We have trivially

$$j_\#\varphi = l, \quad \varphi\kappa = 1.$$

We have the following correspondences between the definition of a chain theory of the second kind and that of a chain functor:

B2), B3) $\longleftrightarrow$ **CH4**

B1) $\longleftrightarrow$ **CH3**

B4) $\longleftrightarrow$ **CH2**

Summarizing we obtain

**2.1. Proposition.** *A chain theory $\boldsymbol{C}_*$ of the second kind, satisfying* **CH5**–**CH7** *with compact carriers, determines a chain functor $\boldsymbol{C}_* = \{C_*, C'_*, \varphi, \kappa, i', l\}$ with $C'_*(X,U) = C_*(X),\ \kappa = \varphi = identity,\ l = j_\#,\ i' = i_\#$.*

*This correspondence is functorial, embedding the category of chain functors of the second kind as a full subcategory into the category $\mathfrak{C}h$ of chain functors.*

Combining this with [2] Theorem 3.3, we obtain the following assertion:

**2.2. Theorem.** *Let $h_*(\ ) = \{h_n(\ ),\ \partial_n,\ n \in \mathbb{Z}\}$ be a homology theory defined on a category of pairs of topological spaces $\mathfrak{K}^2$; then the following properties of $h_*$ are equivalent:*

1) *$h_*$ is the derived homology theory of a chain theory of the first kind.*

2) *$h_*$ is the derived homology theory of a chain theory of the second kind.*

3) *$h_*$ is the derived homology theory of a chain functor $\boldsymbol{C}_*$, with natural $\varphi$, $\kappa$ and chain homotopies $\varphi\kappa \simeq 1,\ j_\#\varphi \simeq l$.*

*Proof.* The equivalence of 1) and 2) is the subject of Proposition 1.5, while the equivalence of 1) and 3) follows from [2] Theorem 3.3. □

## 3. Simplicial Chains

In what follows we take for $\mathfrak{K}^2$ the category of based CW pairs. The objective of the present and the following sections is to associate with any chain theory of the second kind $\boldsymbol{C}_*$ canonically an abelian group spectrum $\mathbf{E} = \mathbf{E}_{\boldsymbol{C}_*}$ such that the derived homology of $\boldsymbol{C}_*$ is isomorphic to $\mathbf{E}_*(\ )$. For this purpose we assume that $\boldsymbol{C}_*$ satisfies **CH5**, **CH6** and that $\boldsymbol{C}_*$ has a compact carrier.

We start with some elementary properties of $\boldsymbol{C}_*$:

**3.1. Lemma.** 1) *Let $h_*(\ ) = H_*(\boldsymbol{C}_*)(\ )$ be the derived homology of $\boldsymbol{C}_*$, then there exists a natural isomorphism*

$$\Sigma_* : \ h_*(\ ) \approx h_{*+1}(\Sigma\ \ ). \tag{1}$$

2) *Let $CX = X \times I/\{x_0\} \times I \cup X \times \{1\}$ be the (reduced) cone over a space $X$ with top vertex $\star = \{\{x_0\} \times I \cup X \times \{1\}\}$, then there exists a natural homomorphism*

$$\tau = \tau_\star : C_*(X) \longrightarrow C_{*+1}(CX).$$

*Proof.* Follows from the excision resp. the homotopy axiom. □

Take two different cones $C_+X$, $C_-X$ with top vertices $\star_+$, $\star_-$, and identifying $C_-X \cap C_+X = X$, then

$$\Sigma X = C_+X \cup C_-X$$

is the reduced suspension with common basepoint of $C_+X$and $C_-X$ and

$$\Sigma_\# = \tau_{\star_+} - \tau_{\star_-} : C_*(X) \longrightarrow C_{*+1}(\Sigma X)$$

the suspension inducing (1).

For each $c \in C_n(X)$ we find a finite subcomplex $X' = \bigcup\limits_1^N \triangle^{q_j}$, and a $c' \in C_n(X')$, $i : X' \subset X$, such that $i_\# c' = c$. We assume that $\triangle^{q_j}$ are cells in some cell structure of X, of maximal dimension, i.e., that $X'$ does not contain $\triangle^p \supset \triangle^{q_j}$ $p > q_j$.

This is an immediate consequence of the requirement of compact carriers.

Let $c \in C_n(\triangle^q)$, now $\triangle^q$ denoting a standard $q$-simplex, be a chain satisfying $dc \in C_{n-1}(bd\ \triangle^q)$ (inclusions omitted from the notation). We have the subsimplexes $\partial_i \triangle^q \subset bd \triangle^q \subset \triangle^q$. Suppose

$$dc \sim \sum_{i=0}^{q} (-1)^i \partial_i c \quad \text{in } C_{n-1}(bd\ \triangle^q)$$

with $\partial_i c \in C_{n-1}(\partial_i \triangle^q)$ and observe that in this case $\partial_i c$ are prescribed and *not* determined by $c$.

**3.2. Definition.** 1) A chain $c \in C_n(\triangle^q)$, $\triangle^q$ a standard q-simplex, is a s1-chain whenever all $\partial_{i_k} \cdots \partial_{i_1} c$, $1 \leq k \leq q$ are given.

2) A chain $c \in C_n(X)$ is an s-chain (=simplicial chain) whenever

$$\sum_1^m \sigma^q_{i\#} c_i,$$

$c_i \in C_n(\triangle^{q_i})$ an s1-chain $\sigma^{q_i}_i : \triangle^{q_i} \longrightarrow X$ a singular simplex.

We need

**3.3. Proposition.** *Let $z \in C_n(X)$ be a cycle, then there exists a simplicial cycle $\overline{z} \sim z$.*

This assertion follows from

**3.4. Proposition.** *Let $c \in C_n(X)$ be any chain such that $dc$ is simplicial, having a carrier $\hat{X} \subset X'$ not containing maximal dimensional simplexes of $X'$. Then there exists a simplicial $\overline{c} \in C_n(X)$, $d\overline{c} = dc$, $c - \overline{c} \sim 0$ in $C_n(X)$.*

*Proof.* The proof is divided into a "horizontal" and a "vertical" part. In the first (horizontal) part, we replace $c$ up to homology by a chain $\sum \sigma^{q_j}_{\#}\, c_j$, where each $c_j \in C_n(\triangle^{q_j})$ is concentrated in a single simplex. In the second (vertical) part, each $c = \sigma^{q_j}_{\#}\, c_j$ is adapted so that the resulting chain becomes a s1-chain (i.e., all $\partial_{i_k} \cdots \partial_{i_1} c$ are specified).

1) Let $c \in C_n(X')$ be a chain with $dc \in C_{n-1}(\hat{X})$. Suppose that $\triangle^q$ is any cell of maximal dimension of $X' = \bigcup\limits_1^N \triangle^{q_j}$, assuming that all $\triangle^{q_i}$ are of maximal dimension (i.e., there does not exist a cell of higher dimensions in $X'$, containing a given summand). We consider the subcomplex $Y$ consisting of the closure of $Y' = bd\ \triangle^q \setminus (bd\ \triangle^q \cap \hat{X})$ and $X'' = \bigcup bd\ \triangle^{q_i}$, $\triangle^{q_i}$ being a maximal dimensional cell of $X'$. We observe that $(X' \subset X'/Y)_{\#} c \sim b_1 + \bar{b}_1$ with $b_1 \in C_n(\triangle^q/Y)$, $\bar{b}_1$ a chain with carrier $(\overline{X' \setminus \triangle^q})/Y$. We obtain by excision $b \in C_n(\triangle^q, bd\ \triangle^q)$, $\bar{b} \in C_n(\overline{X' \setminus \triangle^q})$ and can assume without loss of generality that $db - c = -d\bar{b}$ and (according to B1) in Definition 1.2) that $b \in C_n(\triangle^q)$, implying that $b + \bar{b} - c \in C_n(X', \hat{X} \cap Y)$, hence $((X', \hat{X} \cap Y) \subset (X', \hat{X} \cup Y))_{\#}(b + \bar{b} - c) \sim 0$. We deduce from the exactness of the homology sequence of the triple $(X', \hat{X} \cup Y, \hat{X} \cap Y)$ the existence of $\bar{z}' \in C_n(\hat{X} \cup Y)$ with $d\bar{z}' \in C_{n-1}(Y \cap \hat{X})$ such that $b + \bar{b} \sim c - \bar{z}'$ in $C_n(X', Y \cap \hat{X})$. This gives rise to an inductive process on the number $N$ of maximal dimensional cells in $X'$. At the end of this process we reach $z' \in C_n(X'', \bar{Y} \cap \hat{X})$, $\bar{Y} = \overline{\bigcup bd\ \triangle^{q_i} \setminus (bd\ \triangle^{q_i} \cap \hat{X})}$, $c_i \in C_n(\triangle^{q_i})$, $dc_i \in C_{n-1}(bd\ \triangle^{q_i})$, $dz' \in C_{n-1}(\bar{Y} \cap \hat{X})$, such that

$$\sum_1^N c_i - c - z' \sim 0 \text{ in } C_n(X', \bar{Y} \cap \hat{X}).$$

Now we treat $z'$ as before $c$, obtaining $z'', \ldots$ and finally $\hat{c} = \sum\limits_1^{\overline{N}} \sigma^{\overline{q}_j}_{j\#} c_j$ such that

$$\hat{c} - c \sim 0, \quad c_j \in C_n(\triangle^{\overline{q}_j}), \tag{2}$$

$\hat{c} \in C_n(X')$ and $dc = d\hat{c}$, with cells $\triangle^{\overline{q}_j}$ which are not necessarily any more of maximal dimension.

2) Let $\bar{c} = c_j$ be any of those summands, $\triangle^{\overline{q}_j} = \triangle^q$. Because of our assumption on the carrier of the boundary of the original chain $c$ in our Assertion 3.4, we conclude that $d\bar{c} \in C_{n-1}(bd\ \triangle^q)$. Applying the preceding process 1) to $d\bar{c}$ yields elements $\overline{c}_i \in C_{n-1}(\partial_i \triangle^q)$ as well as a homology

$$d\bar{c} \sim \sum_0^q (-1)^i \overline{c}_i, \tag{3}$$

i.e., a chain $w \in C_n(bd\ \triangle^q)$ satisfying

$$dw = d\bar{c} - \sum_0^q (-1)^i \overline{c}_i.$$

Now we proceed further with each $\overline{c}_i$ in (3) separately, to the effect that we get a s1-chain $c_j$ with $\sum \sigma_{j\#}^{q_j} c_j - c \sim 0$.

As a result, $\sum \sigma_{j\#}^{q_j} c_j$ is simplicial and displays the required properties.

This completes the proof of 3.4. □

## 4. The Simplicial Abelian Group Spectrum $\mathbf{E}$

We consider the infinite dimensional simplex $\triangle^\infty = (a_0, a_1, \ldots, \star)$ with final vertex $\star$. A q-simplex in $\triangle^\infty$ is either of the form $\triangle^q = (a_{i_0}, \ldots, a_{i_q})$ or $(a_{i_0}, \ldots a_{i_{q-1}}, \star) = \tau_\star(a_{i_0}, \ldots a_{i_{q-1}})$, $i_0 \leq \cdots \leq i_q$. We define $\partial_j \triangle^q = (a_{i_0}, \ldots, \hat{a}_{i_j}, \ldots, a_{i_q})$ resp. for $\partial_j \tau_\star \triangle^q$ and correspondingly $s_j \triangle^q = (a_{i_0}, \ldots, a_{i_j}, a_{i_j}, \ldots, a_{i_q})$.

**4.1. Definition.** A $p$-simplex $e^p \in E_p$, $p \in \mathbb{Z}$, is an assignment, associating to each such $\triangle^q$ in $\triangle^\infty$ a chain $e^p(\triangle^q) \in C_{p+q}(\triangle^q)$ such that:

E1) $C_{p+q-1}(bd\ \triangle^q) \ni de^p(\triangle^q) \sim \sum (-1)^i e^p(\partial_i \triangle^q)$, if $e^p(\triangle^q) \neq 0$.

E2) $e^p(\tau_\star \triangle^q) = \tau_\star e^p(\triangle^q) \in C_{p+q+1}(\tau_\star \triangle^q)$.

E3) There exists $\overline{q}$ such that $e^p(\triangle^m) = 0$ for all $\triangle^m \supset \triangle^{\bar{q}} = (a_0, \ldots, a_{\bar{q}})$.

We define $\partial_j e^p$ $(s_j e^p)$ in the following way: by assumption we have for $e^p(\triangle^q) \neq 0$

$$de^p(\triangle^q) \sim \sum (-1)^i e^p(\partial_i \triangle^q) \in C_{p+q-1}(bd\ \triangle^q),\ e^p(\partial_i \triangle^q) \in C_{p+q-1}(\partial_i \triangle^q),$$

$\partial_i :\ \partial_i \triangle^q \subset \triangle^q$ and set

$$(\partial_i\ e^p)(\triangle^q) = \begin{cases} \partial_{i\#} e^p(\partial_i \triangle^q) \in C_{p+q-1}(\triangle^q) & i \leq q,\ e^p(\triangle^q) \neq 0 \\ 0 & \text{in all other cases} \end{cases}, \tag{1}$$

$$(s_j\ e^p)(\triangle^q) = s_{j\#}(\triangle^q)(e^p(s_j \triangle^q)) \in C_{p+q+1}(\triangle^q),\ s_j : s_j \triangle^q \longrightarrow \triangle^q.$$

This furnishes a simplicial spectrum in the sense of [7] Definition 2.1. In particular $\partial_i e^p \neq 0$ only for finitely many $i \geq 0$.

In order to specify $\mathbf{E}$ for different chain theories $\boldsymbol{C}_*$ we write sometimes $\mathbf{E}_{\boldsymbol{C}_*}$. This construction is functorial: If $\lambda : \boldsymbol{K}_* \longrightarrow \boldsymbol{L}_*$ is a transformation of chain theories (cf. §1), then there exists canonically an induced transformation $\mathbf{E}_\lambda : \mathbf{E}_{\boldsymbol{K}_*} \longrightarrow \mathbf{E}_{\boldsymbol{L}_*}$ in the category of simplicial spectra.

Moreover, $\mathbf{E}$ carries the structure of an abelian group spectrum ([7], Definition 4.1):

$$(e_1^p + e_2^p)(\triangle^q) = e_1^p(\triangle^p) + e_2^p(\triangle^q) \in C_{p+q}(\triangle^q)$$

so that for $\lambda : \boldsymbol{K}_* \longrightarrow \boldsymbol{L}_*$ the induced $\mathbf{E}_\lambda$ is a morphism of simplicial abelian group spectra.

We summarize the above as:

**4.2. Proposition.** *There exists a functor* $\mathbf{E}$ *from the category of chain theories of the second kind into the category of simplicial abelian group spectra.*

**4.3. Lemma.** *Let* $c \in C_*(\triangle^q)$ *be any s1-chain* (*Definition* 3.1)*, with prescribed* $\partial_{i_k} \cdots \partial_{i_1} c, 1 \leq k \leq q$*, then there exists* $e^p \in E_p$ *such that* $e^p(\triangle^q) = c,\ p+q = n$.

*Proof.* We define $e^p(\triangle^q) = c$. By assumption, we have $e^p(\triangle^l)$ for any $\triangle^l \subset \triangle^q$. We set $e^p(\triangle^s) = 0,\ \triangle^q \underset{\neq}{\subset} \triangle^s$. This furnishes immediately a procedure to establish $e^p$ for any $\triangle^t$ and therefore a $e^p$ with the required property. □

**4.4. Lemma.** *Let* $c \in C_n(X)$ *be a chain,* $\sigma_i^{q_i} : \triangle^{q_i} \longrightarrow X$ *the simplicial singular simplexes, representing the cells of a carrier for* $c$ (*cf.* §3)*. Then* $c$ *is simplicial whenever there exist* $e^{p_i} \in E_{p_i}$ *such that*

$$c = \sum \sigma_{i\#}^{q_i}\ e_i^{p_i}(\triangle^{q_i}). \tag{2}$$

*Proof.* Follows immediately from 4.3 and 2) of Definition 3.1 of a simplicial chain. □

## 5. The Simplicial Spectrum $X \wedge \mathbf{E}$

Let $X$ be a based CW space (basepoint, as always, omitted from the notation), then an $n$-simplex, $n \in \mathbb{Z}$ of $X \wedge \mathbf{E}$, is a pair $\sigma^q \wedge e^p,\ p+q = n$, consisting of a singular simplex $\sigma^q : \triangle^q \longrightarrow X$, $q \geq 0$ and a $p$-simplex $e^p \in E_p$, $p \in \mathbb{Z}$. We define

$$\partial_i(\sigma^q \wedge e^p) = \sigma^q \wedge \partial_i\ e^p$$

resp. for $s_j$, the degeneracies. This endows $X \wedge \mathbf{E}$ with the structure of a simplicial spectrum. Since $X$ is supposed to be a CW space, we can take for singular simplexes the characteristic mappings of the cells.

We define

$$\alpha(\sigma^q \wedge e^p) = \sigma_\#^q\ e^p(\triangle^q) \in C_{p+q}(X) \tag{1}$$

and deduce

$$\alpha(\partial_i\ \sigma^q \wedge e^p) = \sigma_\#^q\ \partial_{i\#}\ e^p(\partial_i\ \triangle^q) = \alpha(\sigma^q \wedge\ \partial_i\ e^p). \tag{2}$$

We define the suspension $(\Sigma E)_{p+1}$ by assignments $\triangle^q \longmapsto \Sigma_\# \ e^p(\triangle^q) \in C_{p+q+1}(\Sigma \triangle^q)$. Hence we have a bijection $(\Sigma E)_{p+1} \longleftrightarrow E_p$. Setting $(\Sigma \ e^p)(\ ) = \Sigma_\#(e^p(\ ))$ and (Definition 4.1 E2) writing $\tau_\pm$ for $\tau_{\star_\pm}$ )

$$\alpha(\sigma^q \ \wedge \ \Sigma e^p) = (\Sigma\sigma^q)_\# e^p(\triangle^q)$$

$$\alpha(\Sigma\sigma^q \wedge e^p) = \alpha(\tau_+ \ \sigma^q \wedge e^p) - \alpha(\tau_- \sigma^q \wedge e^p) = \Sigma_\# \alpha(\sigma^q \wedge e^p).$$

We calculate

$$\Sigma_\# \ \alpha(\sigma^q \wedge e^p) = \alpha(\Sigma\sigma^q \wedge e^p) = \alpha(\sigma^q \wedge \Sigma e^p). \tag{3}$$

In order to perform a stable homotopy theory, we have to convert $X \wedge \mathbf{E}$ into a Kan spectrum $F(X \wedge \mathbf{E})$ by taking the free group spectrum generated by the simplexes of $X \wedge \mathbf{E}$ (cf. [7]). A stable mapping $f : \ \triangle^n \longrightarrow F(X \wedge \mathbf{E}), \ n \in \mathbb{Z}$, is a class $\{f_k\}$ of "geometric" mappings $f_k : \ \triangle^{n+k} \longrightarrow F(\Sigma^k \ X \wedge \mathbf{E})$. To each such $f_k$ there corresponds $\omega = \prod_1^k \sigma_i^{q_i+k} \wedge e^{p_i}, \ q_i + p_i = n$ in $F(\Sigma^k \ X \wedge \mathbf{E})$. This product is abelian up to stable homotopy, i.e. if $\tilde{\omega}$ has the same factors as $\omega$ but eventually in a different order, then the stable $\tilde{f}$ is stably homotopic to $f$.

We define

$$\alpha(f_k) = \sum_i \sigma_{i\#}^{q_i+k} \ e^{p_i}(\triangle^{q_i+k}) \in C_{n+k}(\Sigma^k \ X),$$

observing that this is a stable invariant

$$\Sigma_\# \ \alpha(f_k) = \alpha(f_{k+1}).$$

This allows us to speak about $\alpha(f), \ f = \{f_k\}$, as a class of chains $c_k \in C_{n+k}(\Sigma^k \ X)$, which are connected by suspension. In particular $\alpha(f) \sim 0$ means that $\alpha(f_k) \sim 0$ for sufficiently high $k$. In the same manner $f \sim 0$ means that $f_k \simeq 0$ for sufficiently high $k$. If $z \in Z_n(C_*(X))$ is a cycle, then $\alpha(f) \sim z, \ f : S^n \longrightarrow F(X \wedge \mathbf{E})$ means that there exists $f_k : S^{n+k} \longrightarrow F(\Sigma^k \ X \wedge \mathbf{E})$ such that $\alpha(f_k) \sim \Sigma_\#^k \ z$.

This allows us to formulate

**5.1. Lemma.** *1) $d \ \alpha(f_k) = \alpha(f_k \mid bd \ \triangle^{n+k}), \ f_k : \triangle^{n+k} \longrightarrow F(\Sigma^k \ X \wedge \mathbf{E})$.*
*2) $f : S^n \longrightarrow F(X \wedge \mathbf{E}), \ \alpha(f) \sim 0 \Longleftrightarrow f \sim 0$ (stably).*
*3) $z \in Z_n(C_*(X)) \Longrightarrow \exists \ f : S^n \longrightarrow F(X \wedge \mathbf{E}), \ \alpha(f) \sim z$.*

*Proof.* **Ad 1)** We observe

$$\alpha(f_k \mid bd \ \triangle^{n+k}) = \sum_0^{n+k} (-1)^j \ \alpha(\prod_i \sigma_i^{q_i+k} \ \wedge \partial_j \ e^{p_i}) = d\alpha(f_k)$$

because of (2).

**Ad 2)** $\Rightarrow$: Let $f_k : S^{n+k} \longrightarrow F(\Sigma^k \ X \wedge \mathbf{E}), f = \{f_k\}$ be a representative of the stable homotopy class $f$. To $f_k$ there corresponds $\prod_1^s \sigma^{q_i+k} \wedge e^{p_i} \in F(\Sigma^k \ X \wedge \mathbf{E})$, hence

$$\alpha(f_k) = \sum_1^s \sigma_\#^{q_i+k}(e^{p_i}(\triangle^{q_i+k})) \in C_{n+k}(\Sigma^k \ X),$$

which is simplicial due to 4.4. Since $\alpha(f_k) \sim 0$, there exists $c \in C_{n+k+1}(\Sigma^k\ X)$ such that $dc = \alpha(f_k)$. Proposition 3.4 together with Lemma 4.4 yields a simplicial $\hat{c} = \sum_1^m \eta_{j\#}^{s_j+k}\ \overline{e}_j^{t_j}(\triangle^{s_j+k})$, satisfying $d\overline{c} = \alpha(f_k)$. So $\overline{\omega} = \prod\ \eta_j^{s_j+k}\ \wedge\ \overline{e}_j^{t_j}$ is associated with a $G_k : \triangle^{n+k+1} \longrightarrow F(\Sigma^k\ X \wedge \mathbf{E})$, satisfying $G_k \mid bd\ \triangle^{n+k+1} = f_k$. Hence $f_k \simeq 0$ and $f \sim 0$.

$\Leftarrow$: Suppose $f \sim 0$, then there exists a $G_k : \triangle^{n+k+1} \longrightarrow F(\Sigma^k\ X \wedge \mathbf{E})$, such that $f_k = G_k \mid bd\ \triangle^{n+k+1} = f_k \in f$. Hence (1) implies

$$d\ \alpha(G_k) = \alpha(f_k).$$

Since suspension induces an isomorphism on the homology level, $\alpha(f) \sim 0$.

**Ad 3)** Proposition 3.3 and Lemma 4.4 furnish a simplicial $z \sim \overline{z} = \sum_1^s \sigma_\#^{q_i}(e^{p_i}(\triangle^{q_i}))$, $p_i + q_i = n$. Hence there exists a $\omega = \prod_1^m \sigma^{q_i+k} \wedge e^{p_i} \in F(\Sigma^k\ X \wedge \mathbf{E})$, which is associated with a $f_k : S^{n+k} \longrightarrow F(\Sigma^k\ X \wedge \mathbf{E})$, such that

$$\alpha(f_k) = \alpha(\omega) = \Sigma^k\ \overline{z} \sim \Sigma^k\ z.$$

This implies that $\alpha(f) \sim z$ stably, $f = \{f_k\}$. $\square$

We define

$$\mathbf{E}_n(X) = \pi_n(F(X \wedge \mathbf{E})), \quad n \in \mathbb{Z} \tag{4}$$

in the usual way, resp.

$$\mathbf{E}_n(X,U) = \mathbf{E}_n(X \cup\ CU), \quad (X,U) \in \mathfrak{K}^2. \tag{5}$$

More precisely: $(X \cup\ CU) = (X \cup\ CU, \star)$ and the basepoint $\star$ is the common basepoint of $CU$, $U$ and $X$, since $CU$ denotes the reduced cone over a based space $U$. The group operation in (4) is well known and corresponds to the product in $F(X \wedge \mathbf{E})$ stably.

We have a natural homomorphism

$$\begin{aligned} \alpha_* :\ \mathbf{E}_n(X,U) &\longrightarrow H_n(C_*(X,U)) \\ \{f\} &\longmapsto \quad [\alpha(f)]. \end{aligned} \tag{6}$$

More precisely: Let $f :\ S^n \longrightarrow F((X \cup\ CU) \wedge \mathbf{E})$ be a stable mapping of a (stable) $n$-sphere $S^n$, $n \in \mathbb{Z}$, represented by a $f_k :\ S^{n+k} \longrightarrow F(\Sigma^k(X \cup\ CU) \wedge \mathbf{E})$, then $\alpha_k = [\alpha(f_k)] \in H_*(C_*(\Sigma^k(X,U)))$ and

$$\Sigma_*\ \alpha_k = \alpha_{k+1}.$$

By desuspending we obtain a $\alpha_*(\{f\}) \in H_*(C_*(X,U))$. Lemma 5.1. ensures that this is a natural and well-defined homomorphism.

**5.2. Lemma.** *$\alpha_*$ is an isomorphism.*

*Proof.* Suppose $z \in \zeta \in H_n((C_*(X,U)))$, then lemma 5.1.3) yields a

$$f : S^n \longrightarrow F((X \cup\ CU) \wedge \mathbf{E})$$

such that $\alpha(f) \sim z$. Hence

$$\alpha\{f\} = z$$

ensuring that $\alpha_*$ is epic. □

Suppose $\alpha(\{f\}) = 0$, then $\alpha(f) \sim 0$ and lemma 5.1. 2) implies that $f \sim 0$ (stably). Hence $\alpha_*$ is monic.

## 6. The Classifying Spectrum of a Chain Theory

In order to realize that $\alpha_*$ is an isomorphism of homology theories, we have to deal with the boundary operators.

The boundary operator

$$\partial :\ \mathbf{E}_n(X,U) \longrightarrow \mathbf{E}_{n-1}(U)$$

is well-known to be defined by

$$\mathbf{E}_n(X\cup\ CU) \xrightarrow{k_*} \mathbf{E}_n(C_- X \cup CU) \underset{\approx}{\xrightarrow{\varrho_*}} \mathbf{E}_n(\Sigma U) \overset{-\Sigma_*^{-1}}{\approx} \mathbf{E}_{n-1}(U).$$

Here $C_-\ X$ denotes a second cone over $X$ with top vertex $\star_-$, as in §3, $k$ : $(X\cup CU) \subset (C_-\ X \cup CU)$ is an inclusion, while $\varrho : C_-\ X\cup CU \longrightarrow \Sigma U$ is well-known and $\Sigma_*^{-1}$ denotes desuspension.

We need

**6.1. Lemma.** We have

$$\partial\alpha_*\{f\} = \alpha_*\partial\{f\}, \quad \{f\} \in E_n(X,U). \tag{1}$$

*Proof.* To $f_k : S^{n+k} \longrightarrow F(\Sigma^k X \wedge \mathbf{E})$ determining the class $f : S^n \longrightarrow F(X\wedge\mathbf{E})$ there corresponds $\omega_1 = \prod_1^m \sigma^{q_i+k}\ \wedge\ e^{p_i}$, which we, in a first step, change in its stable homotopy class: We assume at first that all $\sigma^{q_i+k} : \triangle^{q_i+k} \longrightarrow \Sigma^k(X\cup CU)$ are cellular (i.e., that they are characteristic mappings of cells). Moreover, we assume that they are of the form either $\eta^{q_i+k} : \triangle^{q_i+k} \longrightarrow \Sigma^k\ X$ or $\overline{\varepsilon}^{\overline{q}_j+k}$ : $\triangle^{\overline{q}_j+k} \longrightarrow C\ \Sigma^k\ U$. Without loss of generality we can assume that $\overline{\varepsilon}^{\overline{q}_j+k} = \tau_\star\varepsilon^{q_j+k}$, $q_j + 1 = \overline{q}_j$. Changing the order in the product $\omega_1$, we assume that $f_k$ is associated with $\omega$ of the form

$$\omega = \prod_1^s \eta^{q_i+k} \wedge e^{p_i} \circ \prod_1^t \tau_\star\varepsilon^{q_j+k} \wedge e^{p_j} = a_1 \circ a_2. \tag{2}$$

We obtain

$$\alpha(\omega) = v + \tau_+ b,\ v \in C_n(X),\ dv \in C_{n-1}(U),\ b \in C_{n-1}(U), \tag{3}$$

$$(X \subset X\cup CU)_\# v = \alpha(a_1),\ (CU \subset X\cup CU)_\# \tau_+ b = \alpha(a_2), \tag{4}$$

moreover, $dv = -d\tau_+\ b = -b + \tau_+\ db$. Hence $\tau_+\ db \in C_{n-1}(U)$, implying $db = 0$ and $dv = -b$.

We denote by $\tau_+$ (resp. $\tau_-$ ) the cone $CU = C_+\ U$ (resp. $C_-\ X$) with top vertex $\star_+$ (resp. $\star_-$).

Since $v - \tau_-\ dv$ is a cycle in $C_-\ X$ and hence bounding, we have

$$v - \tau_-\ dv = d\tau_-\ v.$$

We calculate

$$\alpha(\varrho_\# \, k_\# a_1) + \alpha(\varrho_\# \, k_\# a_2) = v + \tau_+ b$$
$$= v - \tau_- dv + \tau_- dv - \tau_+ dv = d\tau_- v - \Sigma_\# dv.$$

Therefore

$$\alpha(\varrho_\# \, k_\# a_1) + \alpha(\varrho_\# \, k_\# a_2) \sim \Sigma_\# dv \quad \text{in } C_{n+1}(\Sigma U).$$

Hence

$$\alpha(\partial f) = -\Sigma_\#^{-1}\alpha(\varrho_\# \, k_\# a_1) + -\Sigma_\#^{-1}\alpha(\varrho_\# \, k_\# a_2) \sim (U \subset X)_\#{}^{-1} dv \quad \text{in } C_{n-1}(U).$$

On the other hand, by the definition of the boundary operator in $C_*$ (cf. §1)

$$\partial\alpha_*([f]) = [(U \subset X)_\#{}^{-1} dv].$$

This implies

$$\partial\alpha_*\{f\} = \alpha_*\partial\{f\}.$$

□

*Remark.* The sign in the definition of the boundary operator for $\mathbf{E}_*$ disappears if one decides to define

$$E_n(X, U) = E_n(X \cup C_- \, U),$$

exchanging $C_+$ and $C_-$.

We summarize our results, recalling, that we assume that our basic category $\mathfrak{K}^2$ is the category of based CW pairs:

**6.2. Theorem.** *There exists a functor from the category of chain theories of the first kind (as well as of the second kind) assigning to each $\boldsymbol{K}_*$ a simplicial abelian group spectrum $\mathbf{E} = \mathbf{E}_{\boldsymbol{K}_*}$ and an isomorphism of homology theories*

$$\alpha_* : \; \mathbf{E}_*(\;) \xrightarrow{\approx} H_*(\boldsymbol{K}_*(\;)). \tag{5}$$

*Proof.* This is an immediate consequence of 1.5, 5.2 and 6.1. □

**6.3. Corollary** (O. Burdick, P. E. Conner E. E. Floyd). *A homology theory $h_*(\;)$ is associated with a chain theory (of the first or of the second kind) if and only if*

$$h_*(\;) \approx \bigoplus_{-\infty}^{+\infty} H_{*+k}(\;; G_k), \tag{6}$$

*where $G_k$ are abelian groups.*

*Proof.* If (6) holds, then the summands in (6) are (shifted) singular homology, therefore the existence of a chain theory $\boldsymbol{K}_*$ of the first (and therefore also of the second kind) satisfying $h_*(\;) \approx H_*(\boldsymbol{K}_*(\;))$ is well-known. The other direction is a consequence of 6.2 and the following

**6.4. Lemma.** *Every simplicial abelian group spectrum* $\mathbf{E}$ *is homotopy equivalent* (*in the category of abelian group spectra*) *to a direct sum of suspended Eilenberg- MacLane spectra*

$$\mathbf{E} \simeq \bigoplus_{-\infty}^{+\infty} \Sigma^k \, K(G_k). \tag{7}$$

*Proof.* The proof follows the same pattern as that of [5] Proposition 2.18. for simplicial sets. For the adaptation of this proof to the case of simplicial *spectra* instead of simplicial *sets*, one has to employ the results about chain complexes in [6] instead of those in [5]. □

*Remark.* Every chain functor $\boldsymbol{K}_*$ (§2) admits a classifying spectrum $\mathbf{E}$ (satisfying (5)), however this is not an abelian group spectrum (cf. [2] or [3]). The present proof employs special properties of a chain theory of the second kind, hence of a chain functor, with very special properties of $\kappa$ and $\varphi$ (cf. Proposition 2.1).

## References


1. F. W. Bauer, Chain functors with isomorphic homology. *Homology Homotopy Appl.* **3**(2001), No. 2, 37–53.
2. F. W. Bauer, The Boardman category of spectra, chain complexes and (co-)localizations. *Homology Homotopy Appl.* **1**(1999), No. 3, 95–116.
3. F. W. Bauer, Classifying spectra and generalized homology theories. *Ann. Mat. Pura Appl. (4)* **164**(1993), 365–399.
4. R. O. Burdick, P. E. Conner, and E. E. Floyd, Chain Theories and their derived homology. *Proc. Amer. Math. Soc.* **19**(1968), 1115–1118.
5. P. G. Goerss ans J. F. Jardine, Simplicial homotopy theory. *Progress in Mathematics,* 174. *Birkhäuser Verlag, Basel,* 1999.
6. J. F. Jardine, Generalized étale cohomology theories. *Progress in Mathematics,* 146. *Birkhäuser Verlag, Basel,* 1997.
7. D. Kan and G. W. Whitehead, The reduced join of two spectra. *Topology* **3**(1965), suppl. 2, 239–261.


(Received 17.06.2002)

Author's address:
Fachbereich Mathematik
Johann Wolfgang Goethe Universität
60054 Frankfurt a.M., Robert-Mayer Str. 8-10
Germany
E-mail: f.w.bauer@mathematik.uni-frankfurt.de