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# A PROPERTY OF THE DEGREE FILTRATION OF POLYNOMIAL FUNCTORS

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**Abstract.** This paper is a detailed version of the note with the same title ([10]). It treats a result related to what is commonly referred to as the artinian conjecture (or finiteness conjecture). This conjecture can be stated in the following way. Consider the category $\mathscr{F}$ of functors from the category of finite dimensional vector spaces over the two element field to that of all vector spaces. Consider its full subcategory of functors whose injective envelopes are finite direct sums of indecomposable injectives. The conjecture is that this subcategory is abelian. In our circumstances the only point to prove is that it is stable under quotients (that this formulation is equivalent to the usual one is easy but not formal).

The result proved in this paper shows that the subobject lattices of standard injectives of the category are "as simple as possible" in what concerns the weight filtrations of unstable modules. It is shown that the filtrations by weights and socles are compatible in an appropriate sense. In addition to the recalled notions and facts, the appendix contains a result showing that certain unstable modules are cyclic.

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## 1. Introduction

Let $\mathscr{F}$ be the category of functors from the category of finite dimensional vector spaces over the two element field to that of all vector spaces, and let $\mathscr{F}_\omega$ be the full subcategory of analytic functors.

In this paper, two natural filtrations on standard injectives of the category $\mathscr{F}_\omega$ are compared. The first filtration is given by the Eilenberg-MacLane degree which is specific of the situation we deal with in [2, 12, 4].

Let us recall that the notion of a functor of a degree not exceeding $n$ is defined in the following manner. We denote by $\Delta$ the endofunctor of the category $\mathscr{F}$ determined by

$$\Delta(F)(V) = \ker(F(V \oplus \mathbf{F}_2) \to F(V)) .$$

Then a functor $F$ is said to have a degree not exceeding $n$ if $\Delta^{n+1}(F) = 0$. We also define the notion of the largest subfunctor of a degree not exceeding $n$ of an arbitrary functor $F$ denoted by $t_n(F)$. A functor $F$ is said to be analytic if it is the direct limit of functors $t_n(F)$.

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The second filtration is that of Loewy or the socle filtration. It is defined in any abelian category. Let us recall its definition. The socle of an object $F$ is the largest subobject $\mathrm{Soc}(F) = \mathrm{Soc}_0(F) \subset F$ which is a direct sum of simple objects. $\mathrm{Soc}_n(F)$ is defined iteratively by $\mathrm{Soc}_n(F) = \pi^{-1}\mathrm{Soc}(F/\mathrm{Soc}_{n-1}(F))$, where $\pi : F \to F/\mathrm{Soc}_{n-1}(F)$ is the canonical projection. If $F$ is analytic, this filtration is convergent.

If we consider the functor $V \mapsto \mathbf{F}_2^{V^*}$, then it is easy to show that both filtrations coincide. The functor $V \mapsto \mathbf{F}_2^{V^*}$, which represents $F \mapsto F(\mathbf{F}_2)^*$, is a standard injective object of $\mathscr{F}$. This result shows that both filtrations are compatible in an appropriate sense.

Let $I_E$ be a functor representing the functor $F \mapsto F(E)^*$ from the category $\mathscr{F}$ to the category $\mathscr{E}$, i.e., such that there is a natural equivalence of functors

$$\mathrm{Hom}_{\mathscr{F}}(F, I_E) \cong F(E)^* .$$

The functor $I_E$ is given by the formula

$$V \mapsto \mathbf{F}_2^{\mathrm{Hom}(V,E)},$$

where the term on the right is the set of functions (set maps) from $\mathrm{Hom}(V, E)$ to $\mathbf{F}_2$. These objects are the standard injective cogenerators of the category.

**Theorem 1.1.** *Let $E$ be an elementary abelian 2-group of dimension $d$. If $n \geqslant d2^d$, then all simple subfunctors of the quotient $I_E/t_{n-1}(I_E)$ are of degree $n$. In other words, the socle of $I_E/t_{n-1}(I_E)$ is a finite direct sum of simple functors of degree $n$.*

This problem has been studied by both authors for a few past years, but it is only recently that its proof has been completed.

An independent proof was given by G. Powell for $\dim(E) = 2$ [11]. Since functors $I_E$ are cogenerators for the category $\mathscr{F}$, the result extends to any analytic functor $F$ whose socle is finite, i.e., to a direct sum of a finite number of simple functors. In fact, such a functor $F$ embeds into a finite direct sum of functors $I_E$ [12]. Thus the degree and Loewy filtrations on $F$ are the filtrations induced by those of $I_E$.

**Theorem 1.2.** *Let $F$ be a functor whose injective envelope is a direct summand in a finite direct sum of indecomposable injectives. Let $n$ be a sufficiently large integer. The socle of $F/t_{n-1}(F)$ is a finite direct sum of simple functors of degree $n$.*

This result suggests the following question.

**Question 1.1.** Let $F$ be a functor whose injective envelope is a direct summand in a finite direct sum of indecomposable injectives. Does then the quotient $t_n(F)/t_{n-1}(F)$ admit a filtration whose quotients are Weyl functors when $n$ is sufficiently large?

The above problems can be posed analogously in the category of unstable modules over the Steenrod algebra. It is in fact this category in which we are going to work. Recall that the quotient of the category of unstable modules by the subcategory of nilpotent modules is equivalent to the category of analytic functors, i.e., of direct limits of polynomial functors [2] (Part 1). As the functor $I_E$ is the image, under the equivalence of categories $\mathcal{U}/\mathcal{N}il \cong \mathcal{F}_\omega$, [2], of the mod 2 cohomology $H^*E$ of the elementary abelian 2-group $E$, the problem posed is analogous to one concerning the mod 2 cohomology $H^*E$ of an elementary abelian 2-group $E$ which has been being studied in detail in the last eight years as regards its properties as an unstable module and unstable algebra over the Steenrod algebra. The properties of the latter algebras, as unstable algebras, are the source of fundamental results in homotopy theory [8, 6].

An analog (see Appendix A) of the degree filtration on $I_E$ is obtained as follows. The cohomology $H^*E$ is a Hopf algebra (and likewise $I_E$ is a functor to Hopf $\mathbf{F}_2$-algebras) and the corresponding filtration is none other than the primitive filtration. In that case the primitive filtration is described in the following way. The algebra $H^*E$ is isomorphic to $S^*(E^*)$ which is primitively generated as a Hopf algebra. In particular, $P_n(H^*E) \cong (P_1(H^*E))^n$, the primitive elements lie in $H^1E \cong E^*$ and its images under the iterates of the squaring morphism. To the notion of degree of a polynomial functor there corresponds that of weights for an unstable module. One can thus formulate the corresponding statements. They are the ones through which we are going to show the results.

An advantage is that in this context we are able to use Steenrod operations. There exists a notion of a maximal weight vector for an element in a reduced unstable module defined by the equivalence of categories with polynomial functors. The main results of this paper are mostly about the behavior of such maximal weight vectors under certain Steenrod operations and about their interpretation for an element in a polynomial algebra. The proof rests on the properties of the standard base of Weyl modules, and on the fact that the action of the Steenrod algebra is especially easy to calculate on the elements of a particular base called semi-standard.

## 2. A particular Case

In this section, a particular case is studied. The reduction of Theorem 1.1 to this case is done in the next sections. We begin by recalling the statement. Let $V$ be an $\mathbf{F}_2$-vector space of dimension $d$. We thus have $H^*V \cong \mathbf{F}_2[x_1, \dots, x_d]$, where $\{x_1, \dots, x_d\}$ is a base of $V^*$. This notation will be used in all that follows.

The reader not familiar with the terminology and results on unstable modules, weights and $H^*V$ could start by reading Appendix A which contains reminders necessary for our further discussion and he could consult Appendix B for the operations $P^s_t$.

In what follows, by simple subobjects of $H^*V/P_{n-1}(H^*V)$ we understand simple $\mathcal{N}il$-closed reduced unstable modules as objects in $\mathcal{U}/\mathcal{N}il$. The simplicity is equivalent to the statement that the quotient by any nontrivial submodule is a nilpotent unstable module.

First we reformulate Theorem 1.1 as follows.

**Theorem 2.1.** *The socle of $P_n(H^*V)/P_{n-1}(H^*V)$ is a subobject of the socle of $H^*V/P_{n-1}(H^*V)$. It coincides with it if $n \geqslant d2^d$. Therefore all simple subobjects of $H^*V/P_{n-1}(H^*V)$ are of weight exactly $n$ when $n \geqslant d2^d$.*

This implies Theorem 1.1 since the objects contained in the socle of $P_n(H^*V)/P_{n-1}(H^*V)$ considered as an object of $\mathcal{U}/\mathcal{N}il$ are of weight $n$ as unstable modules (A.5).

To prove Theorem 2.1 it suffices to show that the following proposition is true.

**Proposition 2.2.** *Let $d = \dim(V)$. Suppose that $n > d2^d$. Then for each element $x \in P_n(H^*V)$ such that $x \notin P_{n-1}(H^*V)$ one can find a Steenrod operation $\beta$ such that $\beta(x) \in P_{n-1}(H^*V)$ and $\beta(x) \notin P_{n-2}(H^*V)$.*

The following lemma describes the action of Steenrod operations on certain classes

$$p \in P_n(H^*V) \subset H^*V \cong \mathbf{F}_2[x_1, \ldots, x_d]$$

such that $p \notin P_{n-1}(H^*V)$. It shows that Proposition 2.2 holds true for these classes. It is shown further that one can be reduced to working with this type of classes.

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a 2-regular partition with respect to columns. We thus have $\mu_i - \mu_{i+1} \leqslant 1$ for all $i$. By A.4 we also have $\mu_1 \leqslant d$. Let $\lambda = (\lambda_1, \ldots, \lambda_h)$ be its conjugate (or associated) partition, which means that

$$\lambda_j = \mathrm{Card}\{i \mid \mu_i \geqslant j\} ,$$

and therefore $t = \lambda_1$ and $h = \mu_1$. The partition $\lambda$ is 2-regular, i.e., $\lambda_i > \lambda_{i+1}$ for all $i$. This notation is used in what follows.

Let $S$ be a set of integers $\{h_1, \ldots, h_t\}$ with $h_i < h_j$ for $i < j$. Also, suppose that the sequence $h_i$ increases and for each $i$ we have the inequality

$$2^{h_i} > \sum_{1}^{i-1} \mu_j 2^{h_j} .$$

This condition ensures that the operation $\varrho$ to be introduced below acts on the considered elements as a derivation (*cf.* Appendix B).

**Lemma 2.3.** *Suppose that the polynomial $p \in \mathbf{F}_2[x_1, \ldots, x_d]$ is of weight exactly $n$ and is a sum of monomials of the form*

$$x_1^{k_1} \ldots x_d^{k_d} ,$$

*where:*

- *each exponent $k_i$ can be written as*

$$k_i = \sum_j 2^{b_{i,j}} ,$$

  *where the integers $b_{i,j}$, which are not necessarily pairwise distinct (we thus do not insist on this being the 2-adic decomposition of $k_i$), belong to the set $\{h_1, \ldots, h_t\}$;*
- *the number of integers $b_{i,j}$ equal to $h_\ell$ is $\mu_\ell$,*

*Suppose finally that:*

- $n > d2^d$ *and*
- *if* $i < j$ *and* $\mu_i = \mu_j$, *then the class* $P^{h_i}_{h_j - h_i}(p)$ *is of weight* $n-1$ *at most.*

*Then there exists an operation* $P^s_t = \varrho$ *such that* $\varrho(p) \in P_{n-1}(H^*V)$, *but* $\varrho(p) \notin P_{n-2}(H^*V)$.

Thus the hypothesis is that the monomials occurring in $p$ have exponents that can be expressed as a sum of $2^{h_\ell}$. The power $2^{h_\ell}$ could appear several times in the exponent of a given $x_i$ and altogether appears exactly $\mu_\ell$ times. Such a monomial is of weight strictly less than $n$ if the same power $2^{h_\ell}$ appears several times in the exponent of the same $x_i$. In other words, if the decomposition of the exponents into sum of powers of 2 given above is the 2-adic decomposition, then the monomial is of weight $n$, and it is not otherwise.

*Proof.* Let us choose a monomial

$$m = x_1^{k_1} \dots x_d^{k_d}$$

of weight $n$, whose coefficient in the polynomial $p$ is nonzero, such a monomial existing since $p$ is of weight $n$ (Appendix A). □

**Lemma 2.4.** *There exists a pair* $(a, b)$ *of elements of* $S$ *such that for any integer* $i$, $1 \leqslant i \leqslant d$, *either*

- *both* $2^a$ *and* $2^b$ *both appear in the given decomposition of* $k_i$ *into a sum of powers of* 2 *or*
- *none of them do.*

This is an elementary enumeration argument. To an integer $h_j \in S$ we assign the subset of $\{1, \dots, d\}$ of those indices $i$ for which $2^{h_j}$ appears in the decomposition of $k_i$ into a sum of powers of 2 – by hypothesis, it appears at most once since the monomial is of weight $n$. There are $2^d$ subsets. Therefore when $\lambda_1 = t > 2^d$, the same subset must appear twice.

However $n > d2^d$. Under this hypothesis it can be easily proved that for an integer partition $\mu$, 2-regular with respect to columns, we have $\lambda_1 = t > 2^d$.

Lemma 2.4 is thus proved.

Now suppose $a < b$, and put $\varrho = P^s_t$ with $s = a$ and $a + t = b$. The following proposition completes this part of the proof.

**Proposition 2.5.** *The element* $P^{b-a}_a(p)$ *is nonzero and of weight exactly* $n-1$.

*Proof.* Lemma 2.3 implies that the class $P^a_{b-a}(p)$ is of weight $n-1$ at most.

It is of weight exactly $n-1$ because of the following observations:

- By the inequality

$$2^{h_i} > \sum_{1}^{i-1} \mu_j 2^{h_j}$$

the operation $\varrho$ acts on the considered elements as a derivation (*cf.* Appendix B). More precisely, on a monomial satisfying the two first conditions of Lemma 2.3 we have

$$\varrho(x_1^{k_1}\dots x_d^{k_d}) = \sum_{b_{\ell,k}=a} x_1^{k_1}\dots x_\ell^{k_\ell-2^a+2^b}\dots x_d^{k_d}.$$

- Let $m$ be the monomial considered above and let $T\subset\{1,\dots,d\}$ be the set of indices $i$ such that the decomposition of the exponent $k_i$ of $x_i$ comprises $2^a$ and $2^b$ exactly once, with $\tilde{k}_i = k_i - 2^a - 2^b$. Calculations show that
$$\varrho(m) = \sum_{i\in T} x_i^{\tilde{k}_i+2^{b+1}} \prod_{j\neq i} x_j^{k_j}\ .$$
- This term is nonzero and of weight exactly $n-1$. To show that $\varrho(p)$ is also such, it suffices to show that applying $\varrho$ to another monomial appearing in $p$, say, to
$$m' = x_1^{k'_1}\dots x_d^{k'_d}\ ,$$
we cannot obtain the monomials on the right-hand side of the above equation.
- However applying the operation $\varrho$ to $m'$ means replacing the power $2^b$ by the power $2^a$ in the exponent of a variable $x_i$, and summing up over all occurrences of this situation. Thus we obtain
$$\varrho(\Pi_i\ x_i^{k'_i}) = \sum_{i\in Z} x_i^{k'_i-2^a+2^b}\,\Pi_{j\neq i}x_j^{k'_j}\ ,$$
where $Z$ is the set of those indices $i$ for which $2^a$ appears in the 2-adic decomposition of the exponent $k'_i$.
- If $m'$ is of weight $n$, then the terms from the right-hand side of the above equation are of weight $n$ if $2^b$ does not appear in the 2-adic decomposition of $k'_i$, but, by hypothesis, the terms of weight $n$ are annihilated. Suppose now that $2^b$ does appear in the decomposition for all $i\in Z$. We then have
$$\varrho(m') = \sum_{i\in Z} x_i^{\tilde{k}'_i+2^{b+1}} \prod_{j\neq i} x_j^{k_j}\ .$$

  If a term on the right-hand side of this equation is equal to a term on the right-hand side of an analogous equation for $\varrho(m)$, then, clearly, $m=m'$.

  Therefore the monomials $m'$ of weight $n$ cannot produce the terms annihilating a monomial of $\varrho(m)$.
- It remains to consider the monomials $m'$ of weight $n-1$. When the operation $\varrho$ is applied to $m'$, it acts as a derivation, in particular, $\varrho(m') = \sum_i \varrho(x_i^{k'_i}\prod_{j\neq i} x_j^{k'_j})$. Moreover, $\varrho(x_i^{k'_i})$ is nonzero if and only if $2^a$ appears in the 2-adic decomposition of the exponent $k'_i$.
- If $2^b$ also appears in the decomposition, then the obtained factor is of weight not exceeding $n-2$.

- If it is not so, then the obtained terms are monomials of weight $n-1$. The exponent of a variable $x_i$ contains $2^b$ in its 2-adic decomposition; however the terms of $\varrho(m)$ do not have this property.

This completes the argument. □

Now we have to explain how to reduce an arbitrary class to classes of this type. The first part of the reduction is given below, while the rest is completed in the next section. But, preliminarily, let us recall some facts.

Simple objects of the category $\mathscr{U}/\mathscr{N}il$ are described in [1, 12] in the following manner. One gives a list of representatives of $\mathscr{N}il$-closed reduced unstable modules which are simple as objects of $\mathscr{U}/\mathscr{N}il$. The simplicity in $\mathscr{U}/\mathscr{N}il$ is equivalent to the condition that the quotient by each nontrivial submodule is a nilpotent unstable module. In what follows, by abuse of the language, we will often talk about **simple reduced modules**. We keep the notations $\mu$ and $\lambda$ introduced previously for a pair of conjugate partitions, if $\mu$ is 2-regular with respect to columns, then $\lambda$ is 2-regular. Partitions that are 2-regular with respect to columns classify irreducible representations of the symmetric group $\mathscr{S}_n$ over $\mathbf{F}_2$. To a 2-regular partition $\mu$ with respect to columns we assign the Young symmetrizer $s_\mu \in \mathbf{F}_2[\mathscr{S}_n]$. The associated simple representation is isomorphic to $s_\mu \in \mathbf{F}_2[\mathscr{S}_n]$. The element $s_\mu$ is not determined uniquely. For example, to the partition $\overbrace{(1,\dots,1)}^{n \text{ times}}$ there corresponds the trivial representation of dimension 1, and the associated element $s_\mu$ (unique in this case) is the sum of all the elements of the group $\mathscr{S}_n$. For more details the reader is again referred to [12, 9] and [4].

Simple reduced modules are indexed by 2-regular partitions with respect to columns, with $n$ being any nonnegative integer.

Let $F(1)$ be the free unstable module on a generator of degree 1. It can be identified with the submodule of the polynomial algebra $\mathbf{F}_2[u]$ generated by the class $u$. As a graded $\mathbf{F}_2$-vector space it has a base consisting of elements $u^{2^n}$.

The simple reduced unstable module associated to a partition $\mu$ is isomorphic to

$$s_\mu F(1)^{\otimes n}$$

and is denoted by $S_\mu(F(1))$.

For example, to the partition $\overbrace{(1,\dots,1)}^{n \text{ times}}$, the module $\Lambda^n(F(1))$ is assigned.

Another description of simple reduced modules is given in [9] in the context of functors. Let $\lambda$ be the partition conjugate to $\mu$. Then the functor is a subquotient of the functor

$$V \mapsto \bigotimes_{i=1,\dots,d} \Lambda^{\lambda_i}(V).$$

This is a unique largest composition factor of a subobject of the tensor product determined as the kernel of a certain map. This kernel, denoted by $W_\mu(V)$, is called the Weyl module. In Section 3 we will easily convert this construction within the framework of unstable modules and describe generators of these modules as modules over the Steenrod algebra.

To prove Proposition 2.2, the class $x$ must be replaced by a class $p$ which satisfies the hypotheses of Lemma 2.3. The first stage of this process rests on the general facts from the theory of modules.

Suppose we are given $x \in P_n(H^*V)$ with $x \notin P_{n-1}(H^*V)$ and let $\bar{x} \in P_n(H^*V)/P_{n-1}(H^*V)$ be its reduction. The socle of this module – considered in an evident way as an object from $\mathscr{U}/\mathscr{N}il$ [12, Chapter 5] – is a direct sum of the form $\bigoplus_\mu S_\mu(F(1))^{a_\mu}$, the sum being taken over the partitions of $n$, 2-regular with respect to columns. The summand $S_\mu(F(1))^{a_\mu}$ is called the isotypic component associated to $\mu$. Each nontrivial submodule of $P_n(H^*V)/P_{n-1}(H^*V)$ intersects the socle nontrivially. Therefore we can find a Steenrod operation $\omega$ such that $\omega(\bar{x})$ is nonzero and lies in the socle of $P_n(H^*V)/P_{n-1}(H^*V)$.

Let $\omega(\bar{x}) = \sum_\mu z_\mu$, where $z_\mu$ is from the isotypic component associated to $\mu$. Let $I_\mu$ be the left ideal of $\mathscr{A}$ which is the annihilator of $z_\mu$. If $z_\mu$ is nonzero, then the unstable module $\mathscr{A} z_\mu$ is isomorphic to $S_\mu(F(1))$ in $\mathscr{U}/\mathscr{N}il$ (by the simplicity of $S_\mu(F(1))$). Let $\alpha$ be a partition (2-regular with respect to columns) of $n$. Then by the simplicity of $S_\mu(F(1))$ the classical arguments of the theory of modules show that the ideal $I_\alpha$ is not contained in the intersection of the ideals $I_\mu$, $\mu \neq \alpha$. Hence, once more using the classical arguments, we obtain

**Proposition 2.6.** *There exists a Steenrod operation $\theta$ such that:*

- *the reduction of $y = \theta(x)$ in $P_n(H^*V)/P_{n-1}(H^*V)$ is nonzero and lies in the socle of the module;*
- *moreover, it is in the isotypic component of its socle.*

Now it is necessary to modify the class $y$ so that it have the properties required by Lemma 2.3. This essentially depends on the structure of simple objects of $\mathscr{U}/\mathscr{N}il$.

## 3. Generators of Weyl Modules and Simple Modules

This section begins with a more explicit description of Weyl modules, simple reduced modules and their generators. A part of what follows can be found in [3, 1, 12] and [9].

The difference from [9], where simple reduced modules are described in the context of functors, is that we will need information on the generators of the modules in question. Finally, in [1] not Weyl modules but their duals are used. This being so, to a large extent the results below could be deduced from these two references and from [3].

The following theorem describes the main properties of Weyl modules.

**Theorem 3.1.** *Let $\mu$ be a partition of the integer $n$, 2-regular with respect to columns, and let $\lambda = (\lambda_1, \ldots, \lambda_h)$ be the conjugate partition. The Weyl module $W_\mu(F(1))$ is a submodule of the tensor product*

$$\Lambda^{\lambda_1}(F(1)) \otimes \ldots \otimes \Lambda^{\lambda_h}(F(1)) .$$

*It is defined as the kernel of a map into a direct sum of tensor products of exterior powers. The element*

$$w_\mu = \bigotimes_{i=1}^{i=d} u \wedge u^2 \wedge u^4 \wedge \ldots \wedge u^{2^{\lambda_i - 1}}$$

*is a generator as a module over the Steenrod algebra of the Weyl module $W_\mu(F(1))$. As an object of the category $\mathscr{U}/\mathscr{N}il$ the Weyl module has a unique largest composition factor, i.e., its maximal semisimple quotient is simple. Moreover, this quotient is of degree exactly $n$.*

The map defining $W_\mu(F(1))$ as a kernel is described in detail in [9] in the context of functors. Here it is briefly described in the context of modules. This construction coincides with that of James' theorem on kernels.

Consider the module $\Lambda^a(F(1)) \otimes \Lambda^b(F(1))$, $a \geqslant b$, and for $1 \leqslant t \leqslant b$ consider the map

$$\psi_{a,b,t} : \Lambda^a(F(1)) \otimes \Lambda^b(F(1)) \to \Lambda^{a+t}(F(1)) \otimes \Lambda^{b-t}(F(1))$$

which is the composite of the maps

$$\Lambda^a(F(1)) \otimes \Lambda^b(F(1)) \xrightarrow{1\otimes\Delta_t} \Lambda^a(F(1)) \otimes \Lambda^t(F(1)) \otimes \Lambda^{b-t}(F(1))$$

$$\Lambda^a(F(1)) \otimes \Lambda^t(F(1)) \otimes \Lambda^{b-t}(F(1)) \xrightarrow{\text{mult}\otimes\text{Id}} \Lambda^{a+t}(F(1)) \otimes \Lambda^{b-t}(F(1)).$$

The map $\Delta_t$ is induced by the diagonal $F(1) \to F(1) \oplus F(1)$ from $\Lambda^b(F(1))$ to $\Lambda^b(F(1) \oplus F(1)) \cong \bigoplus_{k+\ell=b} \Lambda^k(F(1)) \otimes \Lambda^\ell(F(1))$, composed with the projection to the corresponding summand, and mult denotes multiplication.

We then denote by $\psi_{a,b}$ the map with domain $\Lambda^a(F(1)) \otimes \Lambda^b(F(1))$, which is the product of the maps $\psi_{a,b,t}$, $a \geqslant t \geqslant 1$. The map $\psi_{\lambda_i,\lambda_{i+1}}$ extends to a map with domain $\Lambda^\lambda$ by tensoring on the left and on the right with the identity. Then by summing over the index $i$ from 1 to $d-1$ we obtain a map of

$$\bigotimes_i \Lambda^{\lambda_i}(F(1))$$

to a direct sum of tensor products of exterior powers. Let $\psi_\lambda$ be this map. James' theorem on kernels essentially states the following.

**Theorem 3.2.** *The module $W_\mu(F(1))$ is contained in and equal to the kernel* $\operatorname{Ker} \psi_\lambda$.

This statement evidently supposes that there is another description of Weyl modules, for example, with the aid of Young symmetrizers (for 2-regular partitions) like that given above. This result makes transparent the fact that the element $w_\mu$ and, more generally, the elements $w_\mu(\mathbf{x})$ to be defined below, belong to the Weyl module.

Now it is necessary to refine the assertion about Young symmetrizers.

The following discussion is valid only under the assumption that $\mu$ is 2-regular. There exists an element $\varepsilon_\mu \in \mathscr{S}_n$ such that the module $W_\mu(F(1))$ is isomorphic to the module $\varepsilon_\mu F(1)^{\otimes n}$. There is an explicit formula for $\varepsilon_\mu$ in [3]. It can be chosen as a product $C_\mu R_\mu$, where $C_\mu \in \mathbf{F}_2[\mathscr{S}_n]$ is the sum of elements of the group $\mathscr{S}_n$, which fix the columns of the Young diagram associated to $\mu$, $R_\mu \in \mathbf{F}_2[\mathscr{S}_n]$ is the sum of elements which fix the rows.

For each partition $\mu$ the element $w_\mu$ is written as

$$w_\mu := C_\mu \bigotimes_{i=1}^{i=h} u \otimes u^2 \otimes u^4 \otimes \ldots \otimes u^{2^{\lambda_i - 1}} ,$$

where the columns of the Young diagram correspond to the partition of the set $\{1, \ldots, n\}$ given by $\{1, \ldots, \lambda_1\}, \{\lambda_1 + 1, \ldots, \lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots$.

Since the partition is 2-regular with respect to columns one has

$$w_\mu = C_\mu R_\mu \widetilde{\omega}_\mu$$

with

$$\widetilde{\omega}_\mu := \bigotimes_{i=1}^{i=h} u^{2^{\lambda_i - 1}} \otimes \ldots \otimes u^4 \otimes u^2 \otimes u .$$

This statement is given as an exercise in [3, Subsections 8 8.2, 8.3].

Thus we have

$$W_\mu(F(1)) \cong C_\mu R_\mu F(1)^{\otimes n} .$$

**Theorem 3.3.**

$$W_\mu(F(1)) \cong \mathscr{A} w_\mu.$$

This result has been proved by the second author. The proof is described in detail by P. Krason in his thesis. We do not need such a precise result here. A weaker result follows, valid for any partition $\mu$ and easier to prove. The proof of Theorem 3.3 is given in Appendix A.

The third part of Theorem 3.1 is thoroughly investigated in [1, 9, 12] and can be deduced from [4]. In fact, the simple module $S_\mu(F(1))$ is also given by the formula

$$S_\mu(F(1)) \cong R_\mu C_\mu R_\mu F(1)^{\otimes n} .$$

**Definition 3.4.** An element $x$ of an unstable module $M$ is said to be an $F$-generator if the quotient $M/\mathscr{A}x$ is nilpotent.

By the definition of simple objects in $\mathscr{U}/\mathscr{N}il$ we have

**Lemma 3.5.** *Any nonzero element of $S_\mu(F(1))$ is an $F$-generator.*

The result stated above can be expressed as follows. Let $\mathbf{x} = \{x(i)\}$, $1 \leqslant i \leqslant \lambda_1$, be a strictly increasing sequence of positive integers or zeros. We have

**Proposition 3.6.** *The element*

$$w_\mu(\mathbf{x}) := \bigotimes_{i=1}^{i=d} \bigwedge_{1 \leqslant j \leqslant \lambda_i} u^{2^{x(j)}}$$

*is an $F$-generator of the Weyl module.* These $F$-generators are called semi-standard.

As above, the element $w_\mu(\mathbf{x})$ is of the form $C_\mu R_\mu \widetilde{w}_\mu(\mathbf{x})$ for a certain tensor $\widetilde{w}_\mu(\mathbf{x})$ from $F(1)^{\otimes n}$.

Let $a_1, \ldots, a_n$ be pairwise distinct integers and let $\mathbf{a} = (a_1, \ldots, a_n)$.

**Proposition 3.7.** *The element*

$$u_\mu(\mathbf{a}) = C_\mu R_\mu \bigotimes_{i=1}^{i=n} u^{2^{a_i}} \ ,$$

*is an $F$-generator of* $W_\mu(F(1))$. These generators are called standard.

The degree $\ell$ of a standard $F$-generator is such that $\alpha(\ell) = n$, where $\alpha$ is the number of powers of 2 in the 2-adic decomposition of $\ell$. Recall that $\mu$ is a partition of the integer $n$. This is not the case for the degree of a semi-standard $F$-generator.

Proposition 3.7 is a direct corollary of the observation that the element $\bigotimes_{i=1}^{i=n} u^{2^{a_i}}$ is an $F$-generator for $F(1)^{\otimes n}$. For this assertion shown with the aid of the operations $P_t^s$ described in Appendix B see [12, Subsection 5.5].

The proof of Proposition 3.6 is given below.

Let $\mu$ be a 2-regular partition with respect to columns. The projections of elements considered in Lemma 3.5 and Proposition 3.6 to the simple module $S_\mu(F(1))$ are $F$-generators of $S_\mu(F(1))$. We call them semi-standard and standard $F$-generators. By abuse, the same notation will be kept. The following trivial observation will be used later: if a Steenrod operation annihilates the generator of $W_\mu(F(1))$, it also annihilates the projection of this class to $S_\mu(F(1))$.

**Proposition 3.8.** *Classes of the socle of $H^*V/P_{n-1}(H^*V)$ which are associated to semi-standard generators of $S_\mu(F(1))$ have the properties required by Lemma* 2.3.

While verifying this, it will be shown how to reduce an arbitrary class to a class associated to a semi-standard $F$-generator.

Take an element $x \in P_n(H^*V)$ projecting onto a class $y$ in the isotypic component associated to $\mu$ of the socle of $P_n(H^*V)/P_{n-1}(H^*V)$. Then the unstable module $\mathscr{A}y$ can be identified with a submodule of $S_\mu(F(1))$. The property of simplicity in $\mathscr{U}/\mathscr{N}il$ of $S_\mu(F(1))$ is equivalent to the property that for each pair of nonzero classes $t$ and $s$ in $S_\mu(F(1))$ there exist an operation $\theta$ and an integer $k$ such that $\theta(s) = (\mathrm{Sq}_0)^k(t)$. Hence it can be supposed (possibly having to apply an appropriate Steenrod operation to $y$) that the class $z$ identifies with a standard $F$-generator $u_\mu(\mathbf{a})$ of $S_\mu(F(1))$.

The operations $P_t^s$ do not annihilate standard $F$-generators; by contrast, when chosen appropriately, they annihilate semi-standard $F$-generators. One will reduce to such a generator, but to ensure the other hypotheses of Lemma 2.3 we must proceed starting from a standard $F$-generator.

Therefore we have to construct a Steenrod operation $\phi$ such that

$$\overline{\phi(x)} \in H^*V/P_{n-1}(H^*V)$$

identifies with a semi-standard $F$-generator of $S_\mu(F(1))$, and satisfies the hypotheses of 2.3.

*Proof of Propositions* 3.6 *and* 3.8.

In considering Proposition 3.6, to simplify the notations let us work with the case where $x(i) = i-1$, the proof extending without trouble to the general case. Consider the operation

$$\eta = \prod_{i=1}^{i=\lambda_1} \prod_{j=1}^{j=\mu_i} P^{i-1}_{c_{(i,j)}} ,$$

where one supposes that the integers $a_{(i,j)} := i-1+c_{(i,j)}$ are pairwise distinct and all greater than $\lambda_1$.

An application of the calculation rules given in Appendix B leads to

**Lemma 3.9.** *The class $\eta(w_\mu)$ is a standard $F$-generator $u_\mu(\mathbf{a})$, with $\mathbf{a} = (a_{(1,1)}, a_{(1,2)}, \ldots, a_{(2,1)}, \ldots)$.*

Hence it follows that there exists an operation sending $w_\mu$ to a standard $F$-generator, and thus therefore $w_\mu$ is an $F$-generator. The proof easily extends to $w_\mu(\mathbf{x})$.

Let $u_\mu(\mathbf{a})$ be a standard $F$-generator such as described in Proposition 3.7. To prove Proposition 3.8, we have to revisit the proof and describe an operation $\phi$ and an integer $h$ such that $\phi(u_\mu(\mathbf{a})) = \mathrm{Sq}_0^h w_\mu(\mathbf{x}) = w_\mu(\mathbf{x})^{2^h}$.

The operation is given by the formula

$$\phi = \prod_{1\leqslant i\leqslant\lambda_1} \prod_{1\leqslant j\leqslant\mu_i} P^{a_{\ell_j-i+1}}_{h_i-a_{\ell_j-i+1}} ,$$

where $\ell_j = \lambda_1 + \ldots + \lambda_j$ and $h_i = x(i) + h$, the integer $h$ being sufficiently large for all the quantities $h_i - a_j$ which are positive and strictly greater than $a_j$. These conditions ensure that each operation $P^s_t$ on the product acts only on one term of the tensor.

**Lemma 3.10.** *We have*

$$\phi(u_\mu(\mathbf{a})) = \mathrm{Sq}_0^h(w_\mu(\mathbf{x})) = w_\mu(\mathbf{x})^{2^h} .$$

Indeed,

$$\phi(\bigotimes_{i=1}^{i=n} u^{2^{a_i}}) = \bigotimes_{i=1}^{i=\mu_1} \bigotimes_{1\leqslant j\leqslant\lambda_i} u^{2^{h+x(\ell_i-j))}} ,$$

where $\ell_{i-1} = \lambda_1 + \ldots \lambda_{i-1}$, if $i>1$, and $\ell_0 = 0$. Let

$$\widetilde{\omega}_\mu(\mathbf{x}) = \bigotimes_{i=1}^{i=\mu_1} \bigotimes_{1\leqslant j\leqslant\lambda_i} u^{2^{x(\ell_i-j)}} .$$

As above, the columns of the Young diagram correspond to the partition of the set $\{1,\ldots,n\}$ given by $\{1,\ldots,\lambda_1\}, \{\lambda_1+1,\ldots,\lambda_1+1,\ldots,\lambda_1+\lambda_2\},\ldots$.

Thus we obtain

$$\phi(u_\mu(\mathbf{a}) = C_\mu R_\mu \widetilde{\omega}_\mu(\mathbf{x})^{2^h} .$$

Finally, since the partition is 2-regular with respect to columns, we have, as above,

$$w_\mu(\mathbf{x}) = C_\mu R_\mu \widetilde{\omega}_\mu(\mathbf{x}) .$$

The result follows.

These calculations which have been done in $W_\mu(F(1))$ are valid for the projection to the simple quotient $S_\mu(F(1))$.

Let us now combine the proof of Proposition 3.8 and the end of the proof of Theorem 1.1. We thus repeat the argument as in the end of Section 2 and keep the assumptions on the class $y$ which are made there. It is also supposed that the class $y$ projects to a standard $F$-generator of a module isomorphic to $S_\mu(F(1))$. This can be ensured by replacing $y$ by $\beta(y)$ for an appropriate operation $\beta$. Thus its degree $\ell$ is such that $\alpha(\ell) = n$. According to A.2, it can be written as a sum of monomials $x_1^{k_1} \dots x_d^{k_d}$ with

$$\alpha(k_1) + \dots + \alpha(k_d) = n$$

and

$$k_1 + \dots + k_d = \ell\ .$$

Therefore for each $i$ we can find a subset $S_i \subset \{1, \dots, n\}$ with

$$k_i = \sum_{j \in S_i} 2^{a_j}$$

such that the sets $S_i$ form a partition of $\{1, \dots, n\}$; they evidently depend on the chosen monomial.

Let us apply a Steenrod operation $\phi$ such as that described earlier in this section. The following lemma shows that $p = \phi(y) \in \mathbf{F}_2[x_1, \dots, x_d]$ satisfies the hypotheses of Lemma 2.3.

**Lemma 3.11.** *The class of $p$ in the quotient by $P_{n-1}(H^*V)$ identifies with a semi-standard $F$-generator of $S_\mu(F(1))$. The polynomial $p$ is a sum of monomials of the form*

$$x_1^{k_1} \dots x_d^{k_d}\ ,$$

*where each exponent $k_i$ can be written as*

$$k_i = \sum_j 2^{b_{i,j}}$$

*with $b_{i,j}$ not necessarily pairwise distinct and belonging to the set $\{h_1, \dots, h_t\}$, $t = \lambda_1$, and the number of occurrences of the indices $b_{i,j}$ equal to $h_\ell$ is $\mu_\ell$.*

The calculation rules concerning the operations $P_t^s$ give this lemma.

In what follows one says that the partition $\mu$ is a **maximal weight vector** of $s \in M$, $M$ being a reduced unstable module, if the isotypic component associated to $S_\mu$ of the maximal semisimple quotient of $\mathscr{A}s$ in $\mathscr{U}/\mathscr{N}il$ is nonzero. An element can have several maximal weight vectors.

Lemmas 3.10 and 3.11 imply

**Corollary 3.12.** *If for a unique vector of maximal weight an element $x \in \mathbf{F}_2[x_1, \dots, x_d]$ admits the partition $\mu$ and has, for its image $S_\mu$, an element which identifies with a semi-standard $F$-generator, then it is a polynomial of the form described in Lemma* 3.11.

*Conversely, consider $x \in P_n(H^*V)$, and let $\mu$ be a 2-regular partition of the integer $n$; suppose that $x$ as a polynomial in the variables $x_i$ is a sum of monomials of the form*

$$x_1^{k_1} \dots x_d^{k_d} \ ,$$

*where each exponent $k_i$ can be written as*

$$k_i = \sum_j 2^{b_{i,j}} \ ,$$

*where $b_{i,j}$ belong to a set of pairwise distinct integers $\{h_1, \dots, h_t\}$, $t = \lambda_1$, and finally the number of the indices $b_{i,j}$ equal to $h_\ell$ is $\mu_\ell$. Then the maximal weight vectors of $x$, which are partitions of the integer $n$, are the partitions $\mu'$ (being 2-regular with respect to columns) with $\mu$ less than $\mu'$ with respect to the natural order on the partitions* [3].

The first part of the corollary follows from the fact that because of Lemma 3.9 each semi-standard $F$-generator can be obtained, up to raising to a power of 2, as an image under the action of an operation of a standard $F$-generator in the way described above. After that we apply Lemma 3.10.

The second part is proved as follows. Consider the image of $x$ in the quotient $P_n(H^*V)/P_{n-1}(H^*V)$. The latter is isomorphic to the direct sum

$$\bigoplus_\nu \Lambda^{\nu_1}(F(1)) \otimes \dots \otimes \Lambda^{\nu_d}(F(1)),$$

where $\nu = (\nu_1, \dots, \nu_d)$ varies over the set of $d$-tuples of positive integers or zeros with sum $n$. Every tensor product of exterior powers embeds into $F(1)^{\otimes n}$ [1]. By hypothesis, through this embedding, $x$ has as its image in each of the summands a sum of tensors of the form $u^{2^{a_1}} \otimes \dots \otimes u^{2^{a_n}}$, where $a_i$ belong to the set $\{h_1, \dots, h_t\}$, the number of $a_j$ equal to $h_i$ being $\mu_i$.

Consider the map which sends the unstable module generated by $x$ to its maximal semisimple quotient of weight $n$ (in the sense of the category $\mathcal{U}/\mathcal{N}il$). Each simple reduced unstable module of weight $n$ embeds into $F(1)^{\otimes n}$ [1]. Thus we obtain, by composition, a map $\phi$ with the domain $\mathcal{A}x \subset P_n(H^*V)/P_{n-1}(H^*V)$ and with image a finite direct sum of $F(1)^{\otimes n}$ such that $\phi(x)$ is in the socle of the image and has the same maximal weight vectors as $x$ (in the weight $n$).

The module $F(1)^{\otimes n}$ is injective among the reduced unstable modules of weight $n$ [1]. The map $\phi$ thus extends to a map with a domain which is a finite direct sum of the $F(1)^{\otimes n}$ and an image of the same type. The endomorphism ring of $F(1)^{\otimes n}$ is the algebra $\mathbf{F}_2[S_n]$ [12], each element of the symmetric group acting by permuting the coordinates. It follows that $\phi(x)$ is also a sum of tensors of the form $u^{2^{a_1}} \otimes \dots \otimes u^{2^{a_n}}$, where $a_i$ belong to the set $\{h_1, \dots, h_t\}$, the number of occurrences of $h_i$ among $a_j$ being $\mu_i$.

The result is then a consequence of the description of the base of the modules $W_\mu$ given in [3]; see also Appendix C. This base gives, by projection, a generating system of $S_\mu$. Then we apply the result obtained in Chapter 8 of [3] with the replacement of the vector space $V$ by the unstable module $F(1)$ and its canonical base (see also [9]). But a simple exercise in combinatorics shows that an element which is a sum of tensors of the type described above can only come from the $\mu'$-tableau, with $\mu'$ less than $\mu$ ([3, Chapter 8]). This extends to any embedding of $S_\mu$ into $F(1)^{\otimes n}$ because of the above remarks: one passes from one embedding to another by the map induced by the sum of elements of the symmetric group.

The result follows. □

## Appendix A. Weights for Unstable Modules

An unstable module $M$ is **reduced** if the mapping

$$x \mapsto \mathrm{Sq}_0(x) = \mathrm{Sq}^{|x|}x$$

is injective or, equivalently, if it does not contain any nontrivial suspensions. If $M$ is an unstable algebra, then the term on the right is equal to $x^2$, which explains the terminology. The unstable modules $H^*E$ are cogenerators for reduced unstable modules [2], *i.e.*, each reduced module embeds into a product of $H^*E$'s.

An unstable module $M$ is **nilpotent** if for each $x \in M$ there exists $k$ such that $\mathrm{Sq}_0^k x = 0$.

Recall that in an unstable algebra $\mathrm{Sq}_0 x = x^2$. Therefore, by abuse of the language, we sometimes write $x^{2^s}$ for $\mathrm{Sq}_0^s(x)$.

An unstable module $M$ is $\mathcal{Nil}$**-closed** if it cannot be embedded into a larger reduced module $N$ such that for any $x \in N$ there exists $k$ with

$$\mathrm{Sq}_0^k(x) \in M .$$

The reduced unstable modules are filtered by weights [1]. This filtration corresponds to that by the degree on functors. As said earlier, the filtration of $H^*E$ by weights identifies with the primitive filtration. Using the fact that $H^*E$ are cogenerators, we can induce the weight filtration on arbitrary reduced modules from the primitive filtrations of $H^*E$. This filtration does not depend on the embedding.

Here is an intrinsic characterization of the weight filtration for a reduced module.

**Proposition A.1.** *A reduced module $M$ is of weight less than or equal to $d$ if and only if it is zero in every degree $k$ such that $\alpha(k) > d$, where $\alpha(k)$ denotes the number of powers of 2 in the 2-adic decomposition (in base 2) of $k$. We say that $M$ is of weight $d$ if it is of weight less than or equal to $d$, but is not of weight less than or equal to $d-1$.*

We denote by $w(M)$ the weight of a module. The weight $w(x)$ of an element $x \in M$ is the weight of the submodule generated by it.

Here is a useful lemma.

**Lemma A.2.** *Let $M$ be a reduced unstable module of weight less than or equal to $n$. An element $x \in M$ is of weight $n$ if and only if there exists an operation $\theta$ such that $\alpha(|\theta(x)|) = n$.*

The algebra $H^*V$ is a Hopf algebra. Let $P_n(H^*V)$ be the $n$-th term of the primitive filtration of $H^*V$.

On $H^*V$ the weight filtration is the same as the primitive filtration. We have the following result which permits us to calculate the weight of a submodule generated by an element $x$ of $H^*V$.

**Proposition A.3** ([1]). *Let $V$ be an $\mathbf{F}_2$-vector space of dimension $d$. Let $x_1, \ldots, x_d$ be a base of $V^*$. Then $H^*V \cong \mathbf{F}_2[x_1, \ldots, x_d]$. Consider*

$$x = \sum_{(a_1,\ldots,a_d)} u_{a_1,\ldots,a_d}\, x_1^{a_1} \ldots x_d^{a_d}\ , \quad x \in H^*V,$$

*the sum being taken over a set of multiindices. The weight of the submodule generated by $x$ is equal to*

$$\sup_{(a_1,\ldots,a_d)}\ \alpha(a_1) + \ldots + \alpha(a_d)\ ,$$

*the supremum being taken over those multiindices $(a_1, \ldots, a_d)$ for which $u_{a_1,\ldots,a_d}$ is nonzero.*

Consequently we have:

- In a degree $\ell$ such that $\alpha(\ell) = n$, we have
$$(P_n(H^*V)/P_{n-1}(H^*V))^\ell \cong (P_n(H^*V))^\ell\ ;$$
- in degrees $\ell$ such that $\alpha(\ell) \geqslant n+1$ the two modules are trivial;
- the quotient map is not injective in degrees $\ell$ such that $\alpha(\ell) < n$.

The following result is classical.

**Lemma A.4.** *The quotient $P_n(H^*V)/P_{n-1}(H^*V)$ is isomorphic to the direct sum*

$$\bigoplus_\nu\ \Lambda^{\nu_1}(F(1)) \otimes \ldots \otimes \Lambda^{\nu_d}(F(1)),$$

*where $\nu = (\nu_1, \ldots, \nu_d)$ varies over the set of $d$-tuples of positive integers or zeros with sum $n$.*

The case $d = 1$ is very classical, and the general case is deduced by means of the tensor product. It is also a particular case of the description of the primitive filtration on a "very nice" unstable algebra, i.e., one of the form $U(M)$, where $U$ denotes the functor of Steenrod-Epstein [13], $M \cong F(1)^{\oplus d}$.

Recall that in an abelian category the socle of an object is the largest semisimple subobject (i.e., a direct sum of simple objects) contained in this object. In what follows the simplicity is obviously understood in the sense of the category $\mathscr{U}/\mathscr{N}il$.

**Lemma A.5.** *The unstable module*

$$\Lambda^{\nu_1}(F(1)) \otimes \ldots \otimes \Lambda^{\nu_d}(F(1))\ ,$$

*with $\sum_i \nu_i = n$, has a finite Jordan–Hölder series in $\mathscr{U}/\mathscr{N}il$. Simple subobjects of this module are*

- *of weight exactly $n$, i.e., not of weight less than or equal to $n-1$,*
- *and their associated partitions $\lambda$ are of length at most $d$.*

The first part is deduced from [12, 5.3.5, 5.3.6], or [4] using the equivalence of categories $\mathscr{U}/\mathscr{N}il \cong \mathscr{F}_\omega$. The second part results from the case $\mu_1 = \mu_2 = \ldots = 1$; i.e., from the fact that in the category $\mathscr{U}/\mathscr{N}il$ the only simple subobjects of $F(1)^{\otimes n}$ are of weight exactly $n$ ([1], [12, 5.6.3]). For the third part see [1, Section 6]. The reader will observe that in this reference simple objects are indexed by dual partitions.

## Appendix B. Operations $P_t^s$

The operation $P_t^s \in \mathscr{A}$ [7] is dual to the element $\xi_t^{2^s}$ of the Milnor dual of $\mathscr{A}$. The following properties follow from the definition of $P_t^s$:

- its action on $F(1)$ is given by $P_t^s(u^{2^s}) = u^{2^{s+t}}$;
- and by $P_t^s(u^{2^v}) = 0$ if $v \neq s$;
- on $F(1)^{\otimes k}$ the action of $P_t^s$ is given by the observation that it acts as a derivation on tensors which are products of classes of the form $u^{2^v}$ with $v \geqslant s$;
- given a tensor of the form $x \otimes y$, where $x$ is a product of classes of the form $u^{2^v}$ with $v \geqslant s$ and $y$ is a product of classes of the form $u^{2^v}$ with $v < s$, we have $P_t^s(x \otimes y) = P_t^s(x) \otimes y + x \otimes P_t^s(y)$;
- from this it follows that on a tensor $u^{2^{a_1}} \otimes \ldots u^{2^{a_k}}$, where the integers $a_i$ belong to a set of integers $\{h_1, \ldots, h_k\}$, where the differences $|h_i - h_j|$ are large compared to $s$, $P_t^s$ acts as a derivation if $h_i \neq h_j$.

## Appendix C. Proof of Theorem 3.3

Here we give a proof of Theorem 3.3, which is different from that given in a broad outline by the second author in a letter to N. Kuhn and which is detalized by P. Krason in his thesis.

Let us recall some notations of Section 3.

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of the integer $n$, and let $\mu = (\mu_1, \ldots, \mu_h)$ be the conjugate partition. Note that $t = \mu_1$ and $h = \lambda_1$. Suppose that $\mu$ is a 2-regular partition of the integer $n$. Recall two partitions of the set $[n] = \{1, \ldots, n\}$.

The first, $C_\lambda$, consists of $\mu_1$ subsets $C_i$, $1 \leqslant i \leqslant \mu_1$,

$$C_i = \left\{ \sum_{j=1}^{j=i-1} \lambda_j + 1, \ldots, \sum_{j=1}^{j=i-1} \lambda_j + \lambda_i \right\}, \text{ thus } \#C_i = \lambda_i.$$

The second, $R_\lambda$, consists of $\lambda_1$ subsets $R_i$, $1 \leqslant i \leqslant \lambda_1$, and

$$R_i = \left\{ i, i + \lambda_1, \ldots, i + \sum_{1,\ldots,\lambda_i - 1} \lambda_j \right\}, \text{ thus } \# R_i = \mu_i.$$

Now we describe the first standard $\lambda$-tableau [3] as follows:

| | | | |
|---|---|---|---|
| 1 | $\lambda_1+1$ | $\cdots\cdots$ | $n$ |
| 2 | $\lambda_1+2$ | $\cdots\cdots$ | |
| $\vdots$ | $\vdots$ | | |
| $\lambda_2$ | $\lambda_1+\lambda_2$ | | |
| $\vdots$ | | | |
| $\lambda_1$ | | | |

The subset $C_i$ (*resp.* $R_i$) is the set of integers occurring in the $i$-th column (*resp.* row) which is composed of $\lambda_i$ (*resp.* $\mu_i$) boxes.

Let us consider the Young subgroups stabilizing $C_\lambda$ and $R_\lambda$. They are also denoted by $C_\lambda$ and $R_\lambda$ by abuse of the notation and are respectively isomorphic to

$$\mathscr{S}_{\lambda_1} \times \cdots \times \mathscr{S}_{\lambda_h} \quad \text{and} \quad \mathscr{S}_{\mu_1} \times \cdots \times \mathscr{S}_{\mu_t}$$

which are elements of the standard base of $W_\lambda(F(1))$, [3, 9]. We denote as usual by $u^{2^n}$ the elements of the standard base of $F(1)$. Let $t$ be a function defined on the set $[n] = \{1, \ldots, n\}$ with values in $\mathbf{N}$. Suppose that $t$ is strictly increasing (resp. increasing) on each subset of the partition $C_\lambda$ (resp. $R_\lambda$).

Denote (by abuse) $C_\lambda = \sum_{\sigma \in C_\lambda} \sigma$, and define an element $w_t \in W_\lambda(F(1))$ by

$$w_t = C_\lambda \sum_{t'} \otimes_1^n u^{2^{t'(i)}},$$

where the sum runs over all the functions $t' : [n] \to \mathbf{N}$, which take the same set of values as $t$ on each subset occurring in the partition $R_\lambda$.

**Theorem C.1** ([3]). *The set of elements $w_t$, where $t : [n] \to \mathbf{N}$ represents the subset of functions which are strictly increasing (resp. increasing) on each subset appearing in the partition $C_\lambda$ (resp. $R_\lambda$), is a base of $W_\lambda(F(1))$.*

On these elements a binary relation is introduced, which, by abuse, is called a lexicographic order; this abuse takes place since the relation is not antisymmetric, but is "strict". We say that $w_k < w_\ell$ if:

- $\sum_{1 \leqslant i \leqslant n} 2^{k(i)} = \sum_{1 \leqslant i \leqslant n} 2^{\ell(i)}$ and, ordering $k(i)$ and $\ell(j)$ by decreasing order, say, $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$, and $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_n$, we have $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2, \ldots, \alpha_u = \beta_u$, et $\alpha_{u+1} > \beta_{u+1}$.

We now show that there exists an operation $\eta$ such that $\eta(w_\mu) = w_t$ modulo smaller elements of the standard base under the lexicographic order. This gives the result. The operation $\eta$ is defined by

$$\eta = \prod_{i=1}^{i=\lambda_1} \prod_{j=1}^{j=\mu_i} P^{i-1}_{t(i+\ell_{j-1})-i+1}$$

with, as above, $\ell_i = \lambda_1 + \ldots + \lambda_i$.

By the definition of $P^s_t$, we have the following relation, where $\Delta$ is the diagonal of $\mathscr{A}$:

$$\Delta^k(P^s_t) = \sum_{b_1 + \ldots + b_k = t} \otimes_u Q_t(b_u)\ ,$$

where $Q_t(b_u)$ is the Milnor dual operation of $\xi_t^{b_u}$. Hence, by applying $P_t^s$ to the tensor $\otimes_{1\leqslant u\leqslant k} u^{2^{a_u}}$ we obtain

$$P_t^s(\otimes_{1\leqslant u\leqslant k} u^{2^{a_u}}) = \sum_{b_1+\ldots+b_k=t} \otimes_u Q_t(b_u)(u^{2^{a_u}}) .$$

Except for the identity, only the dual operation of $\xi_t^{b_u}$ with $b_u = 2^{a_u}$ acts nontrivially on the class $u^{2^{a_u}}$.

Thus the effect of the operation $P^{i-1}_{t(i+\ell_{j-1})-i+1}$ on a tensor product of classes $u^{2^\ell}$ can be written as follows :

- either as replacements, in the tensor, of the power $u^{2^{i-1}}$ by the power $u^{2^{t(i+\ell_{j-1})}}$ summed over all $u^{2^{i-1}}$;
- or as replacements of the subtensor equal to $\otimes_{1\leqslant u\leqslant k} u^{2^{a_u}}$, with $2^{a_1}+\ldots+2^{a_k}=2^{i-1}$ and $k>1$, by the subtensor

$$\otimes_{1\leqslant u\leqslant k} u^{2^{t(i+\ell_{j-1})-i+1+a_u}}$$

summed over all possible occurrences.

The terms obtained in the second case are (strictly) less – in the sense given above – than those obtained in the first. They correspond to smaller elements in the standard base.

Thus if we apply all operations $P_t^s$ appearing in the definition of $\eta$, then according to the first rule we obviously obtain $w_t$. All the remaining terms decompose into (strictly) smaller elements of the base.

We note that this type of operations has also been considered by G. Walker and R. Wood [14].

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