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ON CONSTRUCTION OF SIGNATURE OF QUADRATIC FORMS ON INFINITE-DIMENSIONAL ABSTRACT SPACES

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Abstract. The signature of the Poincaré duality of compact topological manifolds with local system of coefficients can be described as a natural invariant of nondegenerate symmetric quadratic forms defined on a category of infinite dimensional linear spaces.

The objects of this category are linear spaces of the form $W = V \oplus V^*$ where V is abstarct linear space with countable base. The space W is considered with minimal natural topology.

The symmetric quadratic form on the space W is generated by the Poincaré duality homomorphism on the abstract cochain group induced by nerves of the system of atlases of charts on the topological manifold.

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1. Introduction

There are two difficulties in writing the correct Hirzebruch formula (see [2]) for topological manifolds and families of representations of the fundamental group. First, we must define the rational Pontryagin characteristic classes. Such a definition needs the Novikov theorem about topological invariance of the rational Pontryagin classes.

Second, the classical construction of the signature must be modified for topological manifolds. It seems to be impossible to define the Poincaré duality as a homomorphism of finitely generated vector spaces because homology groups themselves are defined using either singular chains or a spectrum of open coverings of manifolds. In both cases one should deal with infinitely generated vector spaces although the homology here turns out to be finitely generated spaces.

We want to develop techniques related to infinitely generated vector spaces obtained as a direct h or an inverse H limit of a spectrum of finite-dimensional vector spaces. These spaces can be endowed with natural (minimal) topologies. In some special cases it is possible to define for a linear and continuous mapping $f : H \oplus h \to h \oplus H$ its signature, an integer number. General results will be applied to the spectrum of chain and cochain groups related to a system of open coverings of the topological manifold. So for a topological manifold X and a given unitary representation σ of the fundamental group $\pi_1(X)$ the signature $\operatorname{Sign}_\sigma X$ can be defined in this way.

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Under some natural condition of continuity it is possible to define the signature of a family of operators f_Y, parameterized by the topological space Y, as an element of the $K-$theory of Y. So our results probably will lead to the construction of the signature for a topological manifold and a family of representations of the fundamental group.

Our work has three parts. The first two sections are devoted to analysis of vector spaces obtained as a direct and an inverse limit of a spectrum of finite-dimensional vector spaces. We begin by studying some properties of the topology of these spaces. The most important property is Proposition 1 saying that any continuous map $H \to h$ is finite-dimensional. The next step is proving a parameterized analog of this property (Theorem 1). The second section is devoted to the definition of the signature for a bijective selfadjoint mapping $f : H \oplus h \to h \oplus H$. We briefly discuss correctness of the definition.

In the third section the obtained results are applied to the definition of the signature for a topological manifold and a representation of the fundamental group. The main technical difficulty here is the procedure of reducing a long self-adjoint complex (the cone of the Poincaré duality homomorphism) to a single operator $f : H \oplus h \to h \oplus H$. To do this, we use some splitting technique.

The proof of Theorem 1 was done by P. S. Popov while the other results were done jointly.

2. Spectrum of Vector Spaces

2.1. **Topology and continuous mappings.** We study vector spaces over the field $\mathbf{K}$ of real or complex numbers. All finite-dimensional vector spaces are endowed with a natural topology, all mappings between these spaces are considered to be linear and (consequently) continuous.

For the sequence h_n consisting of vector spaces and the set of linear mappings

$$\rho_n : h_n \to h_{n+1}$$

we can define the linear space h as a direct limit: $h = \varinjlim(h_n, \rho_n)$. Then for any n there is the natural linear map $i_n : h_n \to h$. The definition of direct and inverse limits in an abstract category can be found in [1].

For the given sequence h_n let us consider the sequence of dual spaces $h_n^* = \mathrm{Hom}(h_n, \mathbf{K})$ and mappings

$$\rho_n^* : h_{n+1}^* \to h_n^*.$$

For this sequence we can construct the inverse limit $H = \varprojlim(h_n^*, \rho_n^*)$ and the family of maps $\pi_n : H \to h_n^*$.

Let us endow the limit spaces with minimal topologies. They are the weakest topologies for which π_n and i_n are continuous.

The base of the topology on h can be described in the following way. Let $U_n \subset h$ be an open set such that

$$\rho_n^{-1}(U_{n+1}) = U_n.$$

Then this sequence defines an element $\varinjlim(U_n, \rho)$ of the base of the topology.

The base of the topology on H consists of sets $\pi^{-1}(U_n)$, where U_n is an open set in h_n^*.

Endowed with the topology described here, the spaces H and h are dual in the following sense:

$$h = \mathrm{Hom}(H, \mathbf{K}),$$
$$H = \mathrm{Hom}(h, \mathbf{K}),$$

where $\mathrm{Hom}(*, K)$ denotes the set of all linear continuous functionals. Because of this duality continuous mappings between these spaces have many nice properties. For example, any linear map $f : H \to H$ is a dual, $f = g^* = (f^*)^*$. Moreover, the weak topology on H or h, generated by all continuous linear functionals (elements of h or H respectively), coincides with the initial one.

Proposition 1. *Let $f : H \to h$ be a linear and continuous mapping with respect to the topologies introduced above. Then, first, the dimension of* $\mathrm{Im}\, f$ *is finite. Second, it is possible to choose an index n such that the diagram*

$$\begin{tikzcd} H \arrow[r] \arrow[d, "f"] & h_n^* \arrow[dl, "\tilde{f}"] \\ h & \end{tikzcd}$$

can be completed by some linear map $\tilde{f}$.

Let us consider an open set $U \subset h$ which is bounded in each h_n. More precisely, $\rho_n^{-1}(U)$ is bounded. Consider the open set $f^{-1}(U)$ in H. This set can be obtained as a union $f^{-1}U = \bigcup \pi_{m_\alpha}^{-1} V_\alpha$ for some index m_α and open sets $V_\alpha \subset h_{m_\alpha}^*$. Hence there is an index m such that $f^{-1}U \subseteq \pi_m^{-1} V$.

Let $v \in H$ be such that $\pi_m v = 0 \in h_m^*$. If we prove that $f(v) = 0 \in h$, then our proposition will follow from this fact immediately. If $f(v)$ is not zero, then there exists a positive constant λ such that

$$\lambda f(v) \notin U. \tag{1}$$

But $\pi_m \lambda f(v) = 0$ and consequently $\lambda v \in f^{-1}U$. This contradiction to (1) proves the proposition. $\square$

In what follows we want to extend this result to the case where f is continuously parameterized by points of a compact space X. It is the main aim of this section. But first we would like to describe the limit spaces h and H in a rather particular situation.

Let h_n be $\mathbf{K}^n$ and $\rho_n = \mathrm{diag}\{1, 1, \dots, 1, 0\}$. Then

$$h = \bigoplus_{n=1}^{\infty} \mathbf{K} = \{\{a_1, \dots, a_n, 0, \dots\},\ a_i \in \mathbf{K}\}$$

and

$$H = \prod_{n=1}^{\infty} \mathbf{K} = \{\{a_1, \dots, a_n, \dots\},\ a_i \in \mathbf{K}\}.$$

We can enumerate the elements of the base of the topology on h in the following way. Let b_i be an arbitrary sequence of positive real numbers. The corresponding open set $U_{\{b_1,\dots,b_n,\dots\}}$ in h is defined as

$$U_{\{b_1,\dots,b_n,\dots\}} = \{\{a_1,\dots,a_n,\dots\} \in h : |a_i|b_i < 1, i = 1,\dots,\infty\}.$$

Let b_i be a sequence of positive real numbers, where only a finite number of elements is nonzero. The corresponding open set $U_{\{b_1,\dots,b_n,\dots\}}$ in H is defined as

$$U_{\{b_1,\dots,b_n,\dots\}} = \{\{a_1,\dots,a_n,\dots\} \in H : |a_i|b_i < 1, i = 1,\dots,\infty\}.$$

Note that only a finite number of inequalities is important.

The pairing between H and h also has a simple description

$$\langle\{a_1,\dots,a_n,\dots\};\{b_1,\dots,b_n,\dots\}\rangle = \sum a_i b_i.$$

Since one of the vectors has only a finite number of nonzero components, this sum has sense. We can see that topologies on H and h have a nice dual description in terms of this pairing.

In a general situation the structure of direct and inverse limits looks very much like this particular case. Let us consider h_n. This space can be decomposed into the direct sum $h'_n \oplus h''_n$, where

$$h''_n = \{v \in h_n : \exists m : (\rho_{n+m} \dots \rho_{n+1}\rho_n)(v) = 0 \in h_{n+m+1}\}.$$

The space h'_n can be decomposed further: $h'_n = h'''_n \oplus \mathrm{Im}\rho_{n-1}$. The space $h = \varinjlim(h_n, \rho_n)$ is naturally isomorphic to $\bigoplus_{n=1}^{\infty} h'''_n$ and

$$H = \varprojlim(h^*_n, \rho^*_n) \equiv \prod_{n=1}^{\infty} h^{*'''}_n.$$

Thus the next assertion is proved.

Proposition 2. *Any infinite-dimensional direct limit of finite-dimensional spaces endowed with the minimal topology is isomorphic to* $\bigoplus_{n=1}^{\infty}\mathbf{K}$. *Any infinite-dimensional inverse limit of finite-dimensional spaces endowed with the minimal topology is isomorphic to* $\prod_{n=1}^{\infty}\mathbf{K}$.

From now on we will use this identification of infinite-dimensional limits with the model cases $\bigoplus_{n=1}^{\infty}\mathbf{K}$ and $\prod_{n=1}^{\infty}\mathbf{K}$.

Let us define "Dimensional Rank" $DR(v) \in N$ for

$$v \in h, v = \{v_1,\dots,v_n,0,\dots\}$$

as a maximal integer index n such that $v_n \neq 0$. DR of a set S is equal to $\sup_{s\in S} DR(s)$.

Lemma 1. *Let g be a continuous map from a compact space Y to h. Then there exists an integer N such that $\forall x \in Y \quad DR(g(x)) < N$.*

Proof. Let us assume the contrary. Let there b a sequence of points $x_i \in Y$ such that $DR(g(x_i)) \to \infty$. Since Y is a compact space, it is possible to choose a point $x \in Y$ and a subsequence x_{i_j} such that $x_{i_j} \to x$. Also, by choosing a subsequence, we can guarantee that

$$DR(g(x_{i_{j+1}})) > DR(g(x_{i_j})).$$

For a given neighborhood U of $g(x)$ all points $g(x) \in h$ except a finite number must be contained in the set U. But, using (1), we can construct an open set V such that $\forall j \quad g(x_{i_j}) \notin V$. It is contradiction to the continuity of g. $\square$

Let us define the natural topology on the space $\mathrm{Hom}(H, h)$ of all continuous linear homomorphisms. An element of a base of the topology can be obtained from $\epsilon \in \mathbf{R}$ and $a, b \in H$ in the following way:

$$U_{\epsilon,a,b} = \{\xi \in \mathrm{Hom}(H, h) : |\langle \xi(a), b\rangle| < \epsilon\}.$$

Now we can formulate a theorem which is the parameterized version of Proposition 1.

Theorem 1. *Let $f : Y \to \mathrm{Hom}(H, h)$ be a continuous mapping from a compact space Y. Then it is possible to choose an index n such that the following diagram*

$$\begin{tikzcd} H \arrow[r] \arrow[d, "f(x)"'] & h_n^* \arrow[dl, "\tilde{f}(x)"] \\ h & \end{tikzcd}$$

can be completed by a map $\tilde{f} : Y \to \mathrm{Hom}(h_n^*, h)$. Also, there exists a number N such that $\forall x \in Y \quad DR(\mathrm{Im}\, f(x)) < N$.

Proof. We will prove this theorem in the infinite-dimensional case as in the finite-dimensional one the assertion is trivial.

First we will prove that there is a number N such that $DR(\mathrm{Im}\, f(x)) < N$. Let us assume the contrary. Suppose that there are sequences $x_i \in X$ and $e_i \in H$ such that

$$DR(f(x_{i+1})(e_{i+1})) > DR(f(x_i)(e_i)).$$

Using Proposition 1, we can choose e_i such that

$$e_i = \{0, \dots 0, 1, 0, \dots\} \in H = \prod_{n=1}^{\infty} \mathbf{K}, \tag{2}$$

where the nonzero value is positioned at the n_i-th place.

We can also find an element $a \in H$ such that

$$\langle f(x_i)(e_i), a\rangle \to \infty. \tag{3}$$

The mapping f is continuous and therefore $\langle f(*), a\rangle$ is continuous as a map from Y to $\mathrm{Hom}(H, \mathbf{K}) = h$. Using Lemma 1, we can see that functions of type $\langle f(x_i)(e_i), a\rangle$ are continuous on Y and uniformly bounded for all e_i having type (2) because every continuous mapping $Y \to h$ is defined by the finite number of continuous scalar "coordinate" functions.

Therefore condition (3) cannot hold – we have got the contradiction. This proves that there exists a number N such that $\forall x \in X \quad DR(\mathrm{Im}\, f(x)) < N$.

The existence of $\tilde{f}$ with the requested properties is now a trivial consequence of Lemma 1. □

2.2. **Signature.** Consider the spaces j, h which are direct limits of spectra of finite-dimensional spaces, and the dual spaces $J = j^*, H = h^*$.

Let a mapping
$$A : H \oplus j \to (H \oplus j)^* \equiv h \oplus J$$
be a linear continuous self-adjoint bijection.

We state that it is possible to define an integer number $\mathrm{Sign}(A)$, the signature of this mapping.

In the matrix form A has the following appearance:
$$A = \left\{ \begin{array}{cc} A_{Hh} & A_{jh} \\ A_{HJ} & A_{jJ} \end{array} \right\},$$
$$A : \left\{ \begin{array}{c} H \\ \oplus \\ j \end{array} \right\} \to \left\{ \begin{array}{c} h \\ \oplus \\ J \end{array} \right\},$$
where the matrix elements satisfy the conditions
$$(A_{Hh})^* = A_{Hh},$$
$$(A_{jJ})^* = A_{jJ},$$
$$(A_{HJ})^* = A_{jh}.$$

By Proposition 1 the operator $A_{Hh} : H \to h$ has finite-dimensional image. It is possible to decompose $h = h_1 \oplus h_2$ so that $h_2 = \mathrm{Im}\, A_{Hh}$. The space H is decomposed by the duality
$$H = h^* = (h_1 \oplus h_2)^* = H_1 \oplus H_2, \quad \dim H_2 = \dim h_2 < \infty.$$
The operator A_{Hh} has the following appearance:
$$A = \left\{ \begin{array}{cc} A_{H_1h_1} & A_{H_2h_1} \\ A_{H_1h_2} & A_{H_2h_2} \end{array} \right\},$$
where $A_{H_1h_1} = 0, A_{H_1h_2} = 0, A_{H_2h_2} = 0$, and the operator $A_{H_2h_2}$ is a self-adjoint bijection acting between two finite-dimensional vector spaces.

Let us represent the spaces $H \oplus j, h \oplus J$ as
$$H \oplus j = (H_1 \oplus H_2) \oplus j = H_1 \oplus (H_2 \oplus j) = H' \oplus j',$$
$$h \oplus J = (h_1 \oplus h_2) \oplus J = h_1 \oplus (h_2 \oplus J) = h' \oplus J'.$$
Accordingly, with this new decomposition the operator A has the matrix form
$$A = \left\{ \begin{array}{cc} A_{H'h'} = 0 & A_{j'h'} \\ A_{H'J'} & A_{j'J'} \end{array} \right\}.$$
So without loss of generality we can consider $A_{Hh} = 0$.

Let us decompose the space j:
$$j = \mathrm{Ker}\, A_{jh} \oplus \overset{\perp}{\mathrm{Ker}}\, A_{jh} \equiv j'_1 \oplus j'_2.$$

One can note that the mapping A_{jh} is epimorphic and the dual map A_{HJ} is monomorphic. These facts follow from the bijectivity of the operator A. The space J can be decomposed by the duality

$$J = (j_1' \oplus j_2')^* = J_1 \oplus J_2 = \overset{\perp}{\mathrm{Im}}\, A_{HJ} \oplus \mathrm{Im}\, A_{HJ}.$$

Under this decomposition $H \oplus j = H \oplus j_1' \oplus j_2'$ and the dual one, the operator A has the matrix form

$$A = \left\{ \begin{array}{ccc} 0 & 0 & B \\ 0 & b & ? \\ B^* & ? & ? \end{array} \right\},$$

where the operator B (and consequently B^*) is a bijection, and $b = b^*$. We want to prove that b is also a bijection. Then b will be finite-dimensional. The signature of A can be defined as

$$\mathrm{Sign}\, A \overset{def}{=} \mathrm{Sign}\, b.$$

Let us prove that b is a bijection. It is a rather simple fact from general linear algebra. Namely, let the mapping C be the bijection of the abstract linear spaces $L_1 \oplus L_2$ and $L_3 \oplus L_4$ having the matrix form:

$$C = \left\{ \begin{array}{cc} 0 & C_1 \\ C_2 & ? \end{array} \right\}$$

and C_2 is a bijection. Then C_1 is also a bijection.

By applying this observation twice we can prove that b is a bijection. But this operator acts between the direct and the inverse limit of sequences of finite-dimensional vector spaces. The algebraic dimension of the first space can be finite or countable, the dimension of the second one can be finite or continuum. Hence b is finite-dimensional and therefore the signature of A is defined.

Thus the following theorem is proved.

Theorem 2. *For a linear continuous self-adjoint bijection*

$$A : H \oplus j \to (H \oplus j)^* \equiv h \oplus J$$

there exists a correctly defined number $\mathrm{Sign}(A)$*, which does not depend on the addition of a direct hyperbolic summand and coincides with the signature for the finite-dimensional case.*

To prove the correctness of our definition of the signature, we could use the following observation. Let there be two different decompositions of the space $H \oplus j$ into the direct sum: $H_i \oplus j_i, i = 1, 2$. Then these decompositions have a difference in the finite part:

$$H_1 \oplus j_1 = H_2 \oplus j_2 = H' \oplus j' \oplus V,$$

where the space V is finite-dimensional, and $H' \subset H_i, j' \subset j_i$, $i = 1, 2$. Then the correctness of our definition follows from finite-dimensional linear algebra.

Let us note that Theorem 1 can probably lead us to the definition of a continuous family of invertible self-adjoint operators f_Y as an element of the $K-$functor of Y.

But there are some difficulties here even in proving the correctness of this definition. It is the direction for our future work.

3. Duality and Topological Manifolds

Let $\mathscr{U} = \{U_\alpha\}$ be a finite covering of a compact topological manifold X. Let $N_{\mathscr{U}}$ be the nerve of the covering $\mathscr{U}$. The nerve $N_{\mathscr{U}}$ determines a finite simplicial polyhedron. Hence chain and cochain complexes with the local system of coefficients defined by a finite-dimensional representation σ of the fundamental group $\pi_1(X)$ can be determined.

Consider a refining sequence of coverings $\mathscr{U}_n = \{U_\alpha^n\}$, $\mathscr{U}_{n+1} \succ \mathscr{U}_n$. This means that $U_\alpha^{n+1} \subset U_\beta^n$ for proper $\beta = \beta(\alpha)$. Hence the function $\beta = \beta(\alpha)$ defines the simplicial mapping

$$\pi_n^{n+1} : N_{\mathscr{U}_{n+1}} \longrightarrow N_{\mathscr{U}_n},$$

the mapping of simplicial chains and cochains complexes

$$\begin{aligned}(\pi_n^{n+1})_* &: C_*\left(N_{\mathscr{U}_{n+1}}\right) \to C_*\left(N_{\mathscr{U}_n}\right),\\ (\pi_n^{n+1})^* &: C^*\left(N_{\mathscr{U}_n}\right) \to C_*\left(N_{\mathscr{U}_{n+1}}\right).\end{aligned}$$

and the homomorphism of the homology and cohomology groups

$$\begin{aligned}(\pi_n^{n+1})_* &: H_*\left(N_{\mathscr{U}_{n+1}}\right) \to H_*\left(N_{\mathscr{U}_n}\right),\\ (\pi_n^{n+1})^* &: H^*\left(N_{\mathscr{U}_n}\right) \to H_*\left(N_{\mathscr{U}_{n+1}}\right).\end{aligned}$$

Using this information, the spectral chains and cochains of X can be defined as

$$\begin{aligned}C^*(X) &= \lim_{\rightarrow} C^*\left(N_{\mathscr{U}_n}\right),\\ C_*(X) &= \lim_{\leftarrow} C_*\left(N_{\mathscr{U}_n}\right).\end{aligned}$$

Spectral homologies and cohomologies of X can be constructed as

$$\begin{aligned}H^*(X) &= \lim_{\rightarrow} H^*\left(N_{\mathscr{U}_n}\right),\\ H_*(X) &= \lim_{\leftarrow} H_*\left(N_{\mathscr{U}_n}\right).\end{aligned}$$

A natural question is about the connection between these spectral homologies and cohomologies and homology groups of $C^*(X), C_*(X)$. The simple argument based on the Mittag–Leffler condition shows that

$$\begin{aligned}H_*(X) &= H\left(C_*(X)\right),\\ H^*(X) &= H\left(C_*(X)\right).\end{aligned}$$

Note that one can choose the refining sequence of coverings in such a way that each covering would have multiplicity equal to $N+1$, $N = \dim(X)$. The Poincaré duality can be defined as the intersection operator with the cycle $D_n \in C_n\left(\tilde{N}_{\mathscr{U}_n}\right)$, where $(\pi_n^{n+1})_*(D_{n+1}) = D_n$. And the Poincaré homomorphism D is induced by the intersection operator with the cycle $D_\infty = \lim\limits_{\leftarrow}(D_n)$.

We have the following picture

$$\begin{tikzcd} C_0 & C_1 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"'] & C_n \arrow[l, "d_n"'] \\ C^n \arrow[u, "D_0"] \arrow[r, "d_n"] & C^{n-1} \arrow[u, "D_1"] \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_1"] & C^0 \arrow[u, "D_n"] \end{tikzcd}$$

where the homorphism of the Poincaré duality induces an isomorphism in homologies and the following conditions hold:

$$\begin{gathered} d_{k-1}d_k = 0, \\ d_kD_k + D_{k-1}d^*_{n-k+1} = 0, \\ D_k = (-1)^{k(n-k)/2}D^*_{n-k}. \end{gathered}$$

Now let us construct the cone of the operator D for the special case $n = 4l$

$$\begin{gathered} 0 \longleftarrow C_0 \xleftarrow{A_1} C_1 \oplus C^n \xleftarrow{A_2} \cdots \\ \cdots \xleftarrow{A_{2l}} C_{2l} \oplus C^{2l+1} \xleftarrow{A_{2l+1}} C_{2l+1} \oplus C^{2l} \xleftarrow{A_{2l+2}} \cdots \\ \cdots \xleftarrow{A_n} C_n \oplus C^1 \xleftarrow{A_{n+1}} C^0 \longleftarrow 0\,. \end{gathered}$$

It is an acyclic and self-adjoint complex with the differential

$$A_k = \left\{ \begin{array}{cc} d_k & D_{k-1} \\ 0 & d^*_{n-k+2} \end{array} \right\}.$$

To reduce to the case of a single operator, we must, in the long exact complex

$$\cdots \xleftarrow{A_{k-1}} H_k \oplus h_k \xleftarrow{A_k} H_{k+1} \oplus h_{k+1} \longleftarrow 0$$

split the linear space $\operatorname{Im} A_k = \operatorname{Ker} A_{k-1}$ in the following way:

$$H_k \oplus h_k = \operatorname{Im} A_k \oplus H_k^{'} \oplus h_k^{'},$$

where the spaces $H_k^{'}$ and $h_k^{'}$ can be represented as direct and inverse limits of spectra of finite-dimensional vector spaces.

The left side of the long complex is split by the duality. After every splitting the complex becomes shorter. So for the interesting case $2l$ this complex is reduced to two dual spaces and a linear self-adjoint bijection A

$$0 \longleftarrow H \oplus j \xleftarrow{A} h \oplus J \longleftarrow 0,$$

Here results of the previous section can be applied to the definition of the signature $\operatorname{Sign}_\sigma(X)$.

The procedure of splitting will be described here. First, let us note that the image of A_k is equal to the kernel of A_{k-1}. It means that this space is a closed subspace of $H_k \oplus h_k$. There are two possibilities for a closed subspace of an inverse limit of finite-dimensional vector spaces. Its dimension can be finite or continuum.

The operator A_k has the following form:

$$A_k = \left\{ \begin{array}{cc} A_{HH} & A_{hH} \\ A_{Hh} & A_{hh} \end{array} \right\}.$$

The bottom-left corner is finite-dimensional, so without loss of generality we can assume that $A_{Hh} = 0$. We can do this by "cutting and pasting" some finite-dimensional space.

Hence we can assume that

$$A = A_k = \left\{ \begin{matrix} A_{HH} & A_{hH} \\ 0 & A_{hh} \end{matrix} \right\}.$$

The sequence is exact and therefore A_k is monomorphic: $\operatorname{Ker} A_k = 0$. Consequently,

$$\operatorname{Ker} A_{HH} = 0,$$
$$\operatorname{Ker} A_{hH}|_{\operatorname{Ker} A_{hh}} = 0,$$
$$\operatorname{Im} A_{hH}|_{\operatorname{Ker} A_{hh}} \cap \operatorname{Im} A_{HH} = 0.$$

Let

$$H_1 = \operatorname{Im} A_{HH} \subset H.$$

Then

$$A : H \oplus h \longrightarrow H_0 \oplus H_1 \oplus h.$$

The operator $A_{hH}| \operatorname{Ker} A_{hh}$ has closed and finite-dimensional image. The possibility of the dimension of $\operatorname{Im} A_{hH}| \operatorname{Ker} A_{hh}$ to be continuum is forbidden by the smaller (only countable) dimension of h. The splitting can be easily constructed now.

A further object for our work is the construction of the signature for families or, rather, for complexes of C^*-modules obtained as a direct or inverse limit of free finitely generated modules over a C^*-algebra. This direction of investigation is very promising.

This work has not been completed for the time being. We have done only a small step in this direction (Theorem 1). One of the main obstacles on this way is the reduction of a long self-adjoint complex to a single operator. We think that these difficulties in our work have a technical character.

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