Georgian Mathematical Journal
Volume 9 (2002), Number 3, 481–494

HOLOMORPHIC FRAMINGS FOR PROJECTIONS IN A BANACH ALGEBRA

MAURICE J. DUPRÉ AND JAMES F. GLAZEBROOK

Abstract. Given a complex Banach algebra, we consider the Stiefel bundle relative to the similarity class of a fixed projection. In the holomorphic category the Stiefel bundle is a holomorphic locally trivial principal bundle over a certain Grassmann manifold. Our main application concerns the holomorphic parametrization of framings for projections. In the spatial case this amounts to a holomorphic parametrization of framings for a corresponding complex Banach space.

2000 Mathematics Subject Classification: 46H99, 46H35, 58B25, 32L05.
Key words and phrases: Banach algebra, projections, Stiefel bundle, Grassmannian, Banach homogeneous space.

1. Introduction

A fundamental aspect of operator theory concerns the geometry of spaces of projections in a Banach algebra as considered by several groups of authors [1], [4], [8], [15], [23], [25], [29]. The techniques involve using certain Grassmann manifolds and the various constructions are applicable to a broad class of operator algebras and associated orbit spaces which need not necessarily be manifolds in the usual sense (see, e.g., [7]). More specifically, for a given Banach(able) algebra A, let $P(A)$ denote the set of idempotents of A which can be considered as abstract projection operators. We then define an equivalence relation "$\sim$" in $P(A)$ by $p \sim q$ if and only if $pq = q$ and $qp = p$. If A is an algebra of linear operators on a vector space, then $p \sim q$ if and only if p and q have the same image. Thus the space $\mathrm{Gr}(A) = P(A)/\sim$ may be regarded as the generalization of the Grassmannian of subspaces of a given vector space. If $V(A)$ denotes the set of proper partial isomorphisms of A, then there is a natural map $\mathrm{Im} : V(A) \to \mathrm{Gr}(A)$. Restriction to the similarity class $\mathrm{Sim}(p, A)$ of a fixed projection p in A, leads to *the Stiefel bundle* $V(p, A) \to \mathrm{Gr}(p, A)$, as introduced in [8]. Thus the space $V(p, A)$ may be regarded as a manifold of framings or bases for A, that is well known to be the case in finite dimensions (see, e.g., [19]). In [8], we showed that both $V(p, A)$ and $\mathrm{Gr}(p, A)$ are analytic Banach manifolds, and $V(p, A) \to \mathrm{Gr}(p, A)$ is an analytic principal bundle.

In operational calculus and systems control theory it is important to consider the parametrization of subspaces of a given Banach space within a suitable mapping class along with the possible extension to another differentiability class.

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

The continuous to smooth case exemplified the main results of [9], [8], the finite dimensional version having been studied in [12]. Here we discuss the role of $V(p, A)$ in the complex analytic (holomorphic) category as the title suggests. Related work pertaining to holomorphic operator–valued functions and families of subspaces is the subject of [13], [14], [15], [17], [30], for instance. Our use of the infinite dimensional Stiefel manifold $V(p, A)$ and the Grassmannian $\mathrm{Gr}(p, A)$, provides a new approach to this subject.

Throughout we denote by $\mathcal{L}(E, F)$, the complex Banach space of bounded linear operators between complex Banach spaces E and F. When $E = F$, we denote by $\mathcal{L}(E)$ the resulting complex Banach algebra.

2. The Stiefel Bundle of a Banach Algebra

Let A be a topological algebra with identity 1 with $G(A)$ denoting the group of units of A. Let $P(A)$ be the set of idempotents in A so that $\hat{p} = 1-p \in P(A)$ and the map sending $x \in A$ to $1-x$, is an affine homeomorphic involution of A which maps $P(A)$ to itself. There is a natural partial order on $P(A)$ where we say that $p \prec q$ if $qp = p$, and therefore "$\prec$ & $\succ$" define an equivalence relation "$\sim$" on $P(A)$. The resulting set of equivalence classes is called *the Grassmannian of* A and is denoted by $\mathrm{Gr}(A)$. $\mathrm{Gr}(A)$ is a space with the quotient topology due to the natural quotient map

$$\mathrm{Im} : P(A) \longrightarrow \mathrm{Gr}(A), \tag{2.1}$$

which by [8] Proposition 4.1 is an open map (see also [25]).

Let $*$ denote the inner automorphic action of $G(A)$, that is, $g * a = gag^{-1}$, for $a \in A$ and $g \in G(A)$. This action induces an action on $\mathrm{Gr}(A)$ where for $x \in \mathrm{Gr}(A)$ and $g \in G(A)$, we take gx to be the result of the inner automorphic action of $G(A)$ on $\mathrm{Gr}(A)$, such that we have

$$\mathrm{Im}(g * c) = \mathrm{Im}(gcg^{-1}) = g\,\mathrm{Im}(c). \tag{2.2}$$

Definition 2.1. We say that $u \in A$ is a *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$ and $vuv = v$, in which case we call v an *pseudoinverse* for u. In general such a pseudoinverse is not unique. In the following we let $W(A)$ denote the set of all partial isomorphisms of A.

If $u \in W(A)$ has a pseudoinverse v, then clearly $v \in W(A)$ with pseudoinverse u, and it is easy to see that both vu and uv belong to $P(A)$. Even though v is not uniquely determined by u alone, it is uniquely determined once u, vu and uv are all specified. If $p \in P(A)$, then we take $W(p, A) \subset W(A)$ to denote the subset of all partial isomorphisms u of A having a pseudoinverse v satisfying $vu = p$. Likewise, $W(A, q)$ denotes the subset of all partial isomorphisms u of A having a pseudoinverse v satisfying $uv = q$. Now for $p, q \in P(A)$, we set

$$\begin{aligned} W(p, A, q) &= W(p, A) \cap W(A, q) \\ &= \Big\{ u \in qAp : \exists v \in pAq, \;\; vu = p \;\text{ and }\; uv = q \Big\}. \end{aligned} \tag{2.3}$$

The map Im of (2.1) extends to a well–defined map $\mathrm{Im}_A : W(A) \to \mathrm{Gr}(A)$ which is constant on $W(A, q)$ with value $\mathrm{Im}(p)$, as shown in [8, Proposition 5.1].

Recall that $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A. For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$, whereby it follows that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 2.2. Let $u \in W(A)$. We call u a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$ where p and q are similar.

Let $G(p) = G(pAp)$ and let $V(A)$ be the set of all proper partial isomorphisms of A. If $p \in P(A)$, then we take $V(p, A)$ to denote the set of all proper partial isomorphisms of A having a pseudoinverse $v \in W(q, A, p)$ with $q \in \mathrm{Sim}(p, A)$. With regards to (2.3) this condition is expressed by

$$V(p, A) := \bigcup_{q \in \mathrm{Sim}(p,A)} W(p, A, q). \tag{2.4}$$

Let $\mathrm{Gr}(p, A)$ denote the image of $\mathrm{Sim}(p, A)$ under the map Im in (2.1), thus defining $\mathrm{Gr}(p, A)$ as the Grassmannian naturally associated to $V(p, A)$.

Henceforth, we specialize A to be a complex Banach algebra with identity 1. Accordingly, $G(A)$ is taken to be the complex Banach Lie group of units of A. For the theory of fiber bundles in the Banach space category, see, e.g., [6], [7], [11]. The particular case of holomorphic Banach bundles is considered in [3], [21], [30]. We proceed to draw upon the main results of [8] where 'analytic' is now replaced by 'holomorphic' for those objects to which the term applies.

The first observation in the holomorphic category concerns analyticity of the map Im in (2.1) and follows directly from [8, Propositions 4.1, 4.2 and 7.1].

Proposition 2.1. $\mathrm{Im} : P(A) \longrightarrow \mathrm{Gr}(A)$ *is a surjective holomorphic equivariant open map which admits local holomorphic sections. For a given* $p \in P(A)$*, the fiber over* $\mathrm{Im}(p)$ *is the linear flat* $p + pA\hat{p}$*.*

As shown in [8], $V(p, A)$ is a complex Banach submanifold of A, and the map $\mathrm{Im}\,|V(p, A)$ is a continuous open map which induces a homeomorphism $V(p, A)/G(p) \cong \mathrm{Gr}(p, A)$. The manifold $\mathrm{Gr}(p, A)$ is open and closed in $\mathrm{Gr}(A)$ and $\mathrm{Gr}(A)$ is a discrete union of these (see below) .

Theorem 2.1 ([8, Theorem 6.1]**).** *The Grassmannian* $\mathrm{Gr}(p, A)$ *is a complex Banach manifold modeled on the space* $\hat{p}Ap$ *and*

$$G(p) \hookrightarrow V(p, A) \xrightarrow{\mathrm{Im}} \mathrm{Gr}(p, A), \tag{2.5}$$

is a locally trivial holomorphic principal $G(p)$*–bundle. Furthermore,* $\mathrm{Gr}(A)$ *is a complex Banach manifold, the action of* $G(A)$ *on* $\mathrm{Gr}(A)$ *induced by the inner automorphic action of* $G(A)$ *on* A*, is a holomorphic action, and* $\mathrm{Im} : V(p, A) \to \mathrm{Gr}(p, A)$*, is a* $G(A)$*–equivariant map where* $G(A)$ *acts on the left of* $V(p, A)$ *by multiplication.*

Next we recall the notion of a complex homogeneous Banach space following, e.g., [27], [15].

Definition 2.3. Let M be a complex Banach manifold and G a complex Banach Lie group acting on M via $\nu : G \times M \to M$. We say that (G, ν, M) is a *complex Banach homogeneous space* if:

(1) The map $\nu : G \times M \to M$ is holomorphic.
(2) Consider the map $\nu^x : G \to M$ which satisfies $\nu^x(g) = \nu(g, x)$, for $x \in M$. Then there exist local holomorphic sections whose domains cover M and map biholomorphically onto a complex submanifold N of G. Additionally, the isotropy subgroup $G_x = (\nu^x)^{-1}(x)$ is a complex Banach Lie subgroup of G.

Let $A[p]$ denote the commutant of p in A. On recalling $\hat{p} = 1 - p$, we have

$$A[p] = pAp + \hat{p}A\hat{p}, \tag{2.6}$$

and it is straightforward to see that $A[p] + pA\hat{p} = pAp + A\hat{p}$, and $A([p])$ is analytically complemented in A by $\hat{p}Ap + pA\hat{p}$. From these relations, we deduce the following Banach Lie subgroups of $G(A)$:

(i) For each $p \in P(A)$, $G(\mathrm{Im}(p)) := G(A[p] + pA\hat{p}) = G(pAp + A\hat{p})$ is the isotropy subgroup of $\mathrm{Im}(p)$ in $\mathrm{Gr}(p, A)$.
(ii) For each $p \in P(A)$, $G(A[p]) = A[p] \cap G(A)$ is the isotropy subgroup for *the inner automorphic action* of $G(A)$ on $P(A)$.
(iii) $H(p) = (p + A\hat{p}) \cap G(A)$ is the isotropy subgroup for *the analytic (left) multiplication* of $G(A)$ on $V(p, A)$ and $H(p)$ as an open subset of $p + A\hat{p}$, is a complex Banach submanifold of $G(A)$.

In particular, $H(p) \subset G(\mathrm{Im}(p)) \subset G(A)$ is an inclusion of Banach Lie subgroups.

Theorem 2.2. *As complex Banach homogeneous spaces, the Grassmannian* $\mathrm{Gr}(p, A) = G(A)/G(\mathrm{Im}(p))$, *and the Stiefel manifold* $V(p, A) = G(A)/H(p)$. *Moreover, there exists a holomorphic locally trivial fibration*

$$G(\mathrm{Im}(p))/H(p) \hookrightarrow G(A)/H(p) \longrightarrow G(A)/G(\mathrm{Im}(p)). \tag{2.7}$$

Proof. The first statement regarding $\mathrm{Gr}(p, A)$ was established in [8, Proposition 7.4], the main point being that the isotropy subgroup $G(\mathrm{Im}(p))$ is a complex Banach analytic Lie subgroup of the complex Banach Lie group $G(A)$. Furthermore, the action of $G(A)$ on $\mathrm{Gr}(p, A)$ is analytic (holomorphic). Consequently, the requirements of Definition 2.3 (1) and the last condition in (2), are satisfied. The remainder of (2) in Definition 2.3, is implied by the existence of local analytic sections and local analytic diffeomorphisms (biholomorphisms) from $\mathrm{Gr}(p, A)$ back to $G(A)$, as shown in [8, Theorem 7.1] (cf. [27, Proposition 1.5]).

We have already noted that $V(p, A)$ is a complex Banach manifold. By [8, Lemma 5.1], the left action of $G(A)$ on $V(p, A)$ is transitive and analytic. On the other hand, the right regular representation of A on itself determines an idempotent continuous linear (analytic) map $R(p) : A \to A$. Now $R(\hat{p})$ is continuous, and $(R(p), R(\hat{p})) : A \to Ap \times A\hat{p}$ is a linear isomorphism. But $(R(p), R(\hat{p}))^{-1}$ is the restriction of the addition map to $Ap \times A\hat{p}$, and so $(R(p), R(\hat{p}))$ is a linear homeomorphism. If $\pi_A : Ap \times A\hat{p} \to Ap$, is the first factor projection, then π_A is open. Hence $R(p) = \pi_A(R(p), R(\hat{p}))$ is open onto its image Ap, and its kernel $(A\hat{p})$ splits A. But since $G(A)$ is open in A, it follows that $\nu^p = R(p)|G(A) : G(A) \to V(p, A) = G(A) \cdot p$ is an open map.

Now for any $g \in G(A)$, we have $T_g\nu^p = R(p)$ and the kernel of $T_e\nu^p$ splits $T_eG(A)$, since we have shown $R(p)$ is onto, the image of $T_g\nu^p$ clearly splits $T_{gp}V(p, A)$. Thus the hypotheses of [7], Corollary 5.6 (3) are satisfied, and therefore imply the equivalent properties: ν^p admits a local holomorphic section, $(G(A), \nu^p, V(p, A))$ is a locally trivial holomorphic principal bundle with structure group $H(p)$ which is a complex submanifold of $G(A)$, and $H(p)$ is a complex Banach Lie subgroup of $G(A)$. So it follows that the requirements of Definition 2.3 are fulfilled and $V(p, A) = G(A)/H(p)$ is a complex Banach homogeneous space. The remaining assertion follows from the fact that $H(p)$ is contained as a closed subgroup in $G(\mathrm{Im}(p))$ and thus induces the holomorphic fibration (2.7). □

Theorem 2.3. *The following properties hold:*

1. $P(A)$ *is a discrete union of the complex Banach homogeneous spaces* $G(A)/G(A[p])$.
2. *The holomorphic bundle* $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ *restricts to define a holomorphic locally trivial fibration*

$$G(\mathrm{Im}(p))/G(A[p]) \hookrightarrow G(A)/G(A[p]) \longrightarrow G(A)/G(\mathrm{Im}(p)). \quad (2.8)$$

3. $\mathrm{Gr}(A)$ *is a discrete union of the complex Banach homogeneous spaces* $G(A)/G(\mathrm{Im}(p))$. *Equivalently,* $\mathrm{Gr}(A)$ *is a discrete union of the* $\mathrm{Gr}(p, A)$.
4. $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ *is a holomorphic locally trivial bundle.*

Proof. Firstly, we recall that $G(A)$ acts transitively and holomorphically on $P(A)$ by its inner automorphic action. We have noted in (ii) above that for each $p \in P(A)$, $G(A[p])$ is the isotropy subgroup of p under the inner automorphic action of $G(A)$. Since $A([p])$ is analytically complemented in A by $\hat{p}Ap + pA\hat{p}$, the same principles used proving Theorem 2.2 together with [8], Proposition 4.1, imply (1)–(4). □

3. Holomorphic Parametrization by Stein Spaces

In this section we will combine Theorems 2.2 and 2.3 with a version of the Oka principle in [3] valid in infinite dimensions (cf. [16]). Concerning the definition and properties of Stein spaces we refer to [18].

Theorem 3.1 ([3, Theorem 8.4]). *Let X be a Stein space and let $P \to X$ be a holomorphic principal bundle. Let Y be a closed subvariety of X and U a holomorphically convex domain in X. If a continuous section $f : X \to P$ is such that $f|Y = g$ is holomorphic, then f is homotopic to a holomorphic section in the space of continuous sections that induce g on Y. If $f : U \to P$, $g : Y \to$*

P, are holomorphic sections such that $f|Y \cap U = g|Y \cap U$, then f can be approximated by holomorphic sections $h : X \to P$ with $h|Y = g$ (uniformly on compacta in U) if and only if it can be approximated by continuous sections h with $h|Y = g$.

Remark 3.1. Observe that Theorem 3.1 requires the bundle to be a principal bundle. There does not seem to be an analogous result for more general holomorphic fiber bundles in the infinite dimensional setting.

Together with Theorems 2.2 and 2.3, Theorem 3.1 will provide a straightforward development of results concerning the holomorphic parametrization of projections in A by Stein spaces. However, we will also need a particular concept which combines the *neighborhood extension property* of [6] with the *section extension property* of [5]:

Definition 3.1. Let $\xi \to X$ be a bundle over a space X. We say that ξ has the *absolute extension property* (AEP), if for every closed set $Y \subset X$, and every section $s \in \Gamma(Y, \xi)$, there exists a section $t \in \Gamma(X, \xi)$ with $t|Y = s$.

Lemma 3.1. *Let $\xi \to X$ be a bundle over a paracompact Hausdorff space X.*

1. *If ξ locally has the AEP, then ξ has the AEP (globally).*
2. *In particular, if ξ is locally trivial with fiber a contractible Banach manifold M topologically embedded as a neighborhood retract in a Banach space E, then ξ has the AEP.*

Proof. Part (1) follows from [6, Corollary 3.2]. To establish (2) we observe that by (1) and the fact ξ is locally trivial, we need only prove it in the case that ξ is trivial. So let $Y \subset X$ be a closed subset and let $f : Y \to \xi$ be a continuous section regarded as a continuous map $f : Y \to M$. By hypothesis, f extends to a continuous section $g : X \to E$. Let T be an open neighborhood of M in E that retracts back onto M. Since $g^{-1}(T)$ is open and $Y \subset g^{-1}(T)$, we obtain a neighborhood extension of f to some open set V containing Y. In other words, there exists a continuous map $h : V \to M$ extending f. Using the normality of X, we have a continuous function $r : X \to [0, 1]$ such that $r|Y = 1$, and $r = 0$ outside a closed neighborhood Z of X which is contained in V.

Next, we choose a contracting homotopy $H : M \times [0, 1] \to M$ such that $H(x, 1) = x$, and $H(x, 0) = c$, a constant. Then we form the map $H(h, r) : V \to M$, which extends f to V. It takes the constant value c outside of the neighborhood Z and hence extends to all of X on specifying the value c at the points of $X \backslash Z$. In this way, f achieves a continuous extension to all of X, and thus represents a global section $f : X \to \xi$. □

Remark 3.2. Regarding (2) of Lemma 3.1, we remark that the main result of [10] provides a wide class of Banach manifolds realizable as embedded open subsets of a Banach space on which they are modeled. Recall that $G(A)$ is open in A as a Banach submanifold.

Henceforth, when we take X to be a Stein space, we will always assume it is paracompact and Hausdorff.

Theorem 3.2. *Let G be a complex Banach Lie group and H, K closed complex Banach Lie subgroups of G with $H \subset K$. Suppose X is a Stein space and $\psi : X \to G/K$ is a holomorphic map. Let Y be a closed subvariety of X and U a holomorphically convex (open) domain in X.*

(1) *Suppose there exists a continuous map $f : X \to G/H$ lifting ψ through $G/H \to G/K$, such that $g = f|Y$ is holomorphic. Then in order that f is homotopic to a holomorphic map in the space of continuous maps lifting ψ through $G/H \to G/K$, that restrict to g on Y, it suffices $(*)$ that f admits a continuous lift $\tilde{f}$ through $G \to G/H$ and $\tilde{g} = \tilde{f}|Y$ is holomorphic.*
(2) *Suppose $\eta : U \to G/H,\ \mu : Y \to G/H$, are holomorphic lifts of ψ such that $\eta|Y \cap U = \mu|Y \cap U$ and they are approximated by continuous lifts h (of ψ) with $h|Y = \mu$. Then in order for η to be approximated by holomorphic lifts $h : X \to G/H$ with $h|Y = \mu$ (uniformly on compacta in U) it suffices $(**)$ that η, μ admit lifts $\tilde{\eta} : U \to G,\ \tilde{\mu} : Y \to G$ with $\tilde{\eta}$ continuous and $\tilde{\mu}$ holomorphic, and such that $\tilde{\eta}|Y \cap U = \tilde{\mu}|Y \cap U$.*
(3) *If H is contractible and is embedded as a neighborhood retract of a Banach space, then $(*)$ and $(**)$ hold.*

Proof. Firstly, in view of [18, (Ch. V, Theorem 1)], the conditions on Y and U imply that each of these inherit the Stein property as subsets of X. With the intention of applying Theorem 3.1, we will need to consider the holomorphic principal bundles $P_K = (G, \pi_K, G/K, K)$ and $P_H = (G, \pi_H, G/H, H)$.

To establish (1), consider the lift $\tilde{f} : X \to G$ of f. We can view $\tilde{f}$ as a continuous section of the holomorphic principal bundle $\psi^* P_K \to X$, and $\tilde{g} = \tilde{f}|Y$ holomorphic. By Theorem 3.1, there exists a homotopy $\tilde{f}_t : X \to G,\ t \in [0, 1]$, satisfying $\tilde{f}_t|Y = \tilde{g},\ \tilde{f}_0 = \tilde{f}$, and $\tilde{f}_1$ is holomorphic . Each of these can then be projected to G/H.

In the case of (2), we start by considering $\tilde{\mu} : Y \to G$ as a holomorphic section of the holomorphic principal bundle $\mu^* P_H \to Y$. At the same time $\tilde{\mu}$ defines a holomorphic section over $Y \cap U$ of $\eta^* P_H \to U$. Next, we view $\tilde{\eta}$ as a continuous section of $\eta^* P_H \to U$, such that $\tilde{\mu}|Y \cap U = \tilde{\eta}|Y \cap U$ is holomorphic. Applying Theorem 3.1 relative to $\eta^* P_H$ and to $\tilde{\mu}|Y \cap U$, we find that $\tilde{\eta}$ is homotopic to a holomorphic section, still denoted by $\tilde{\eta}$, that agrees with $\tilde{\mu}$ on $Y \cap U$. Thus by viewing $\tilde{\mu}$ and $\tilde{\eta}$ as holomorphic sections of $\psi^* P_K$ over Y and U respectively, (2) follows from Theorem 3.1.

We proceed to establish (3) in view of the assumptions on H. In the case of $(*)$, we observe that $g^* P_H \to Y$ is a holomorphic bundle. Thus by Theorem 3.1, $\tilde{g} : Y \to G$ viewed as a section, is homotopic to a holomorphic section still denoted by $\tilde{g}$. Now $f^* P_H \to X$ is a principal bundle whose fiber is contractible. Applying Lemma 3.1 (2) to $f^* P_H$, we can extend $\tilde{g}$ to a continuous lift $\tilde{f}$ of f, such that $\tilde{g} = \tilde{f}|Y$ is holomorphic.

Next we deal with $(**)$. By Theorem 3.1, we obtain a holomorphic section of $\mu^* P_H \to Y$ (as μ is holomorphic by hypothesis) which provides $\tilde{\mu}$. Because

$Y \cap U$ is closed in U and the latter is paracompact since it is open in X, we can apply Lemma 3.1 (2) to extend $\tilde{\mu}$ from $Y \cap U$ to all of U. Then Theorem 3.1 is applied to $\eta^* P_H \to U$ to obtain $\tilde{\mu}$ homotopic to a holomorphic lift $\tilde{\eta}$ relative to $Y \cap U$, and with the subsequent property that $\tilde{\eta}|Y \cap U = \tilde{\mu}|Y \cap U$. □

Remark 3.3. Recall that by [22], the general linear group of an infinite dimensional Hilbert space is contractible. Further examples of Banach spaces having contractible general linear groups, may be found in [24].

Corollary 3.1. *Suppose $G(A[p])$ is contractible. Let $\psi : X \to \mathrm{Gr}(A)$ be a holomorphic map. If there exists a continuous map $f : X \to P(A)$ lifting ψ through $P(A) \to \mathrm{Gr}(A)$ such that $g = f|Y$ is holomorphic, then f is homotopic to a holomorphic map in the space of continuous maps lifting ψ through the map $P(A) \to \mathrm{Gr}(A)$ that induce g on Y.*

Proof. Since we intend to prove this componentwise, it is sufficient to assume X is connected. We have already noted that $G(A[p])$ is open as a subset of the Banach subalgebra $A[p]$. With regards to Theorem 2.3 and in particular (2.8), we view ψ as lifted through

$$G(A)/G(A[p]) = G/H \longrightarrow G(A)/G(\mathrm{Im}(p)) = G/K.$$

Then since $P(A)$ (respectively, $\mathrm{Gr}(A)$) is a discrete union of the G/H (respectively, G/K) and $H = G(A[p])$ is assumed contractible, the result follows directly from Theorem 3.2. □

The next corollary pertains to a parametrization of the holomorphic framing of projections in $P(A)$ implemented via the Stiefel manifold $V(p, A)$.

Corollary 3.2. *Suppose $H(p)$ is contractible. Let $\psi : X \to \mathrm{Gr}(p, A)$ be a holomorphic map. If there exists a continuous map $f : X \to V(p, A)$ lifting ψ through $V(p, A) \to \mathrm{Gr}(p, A)$ such that $g = f|Y$ is holomorphic, then f is homotopic to a holomorphic map in the space of continuous maps lifting ψ through the map $V(p, A) \to \mathrm{Gr}(p, A)$ that induce g on Y.*

Proof. Likewise, we have noted that $H(p)$ is also open as a subset of $p + A\hat{p}$. With regards to Theorem 2.2 and in particular (2.7), we view ψ as lifted through

$$G(A)/H(p) = G/H \longrightarrow G(A)/G(\mathrm{Im}(p)) = G/K,$$

and using the assumed contractibility of $H = H(p)$, the result follows directly from Theorem 3.2. □

4. The Banach Grassmannian $\mathrm{Gr}(F, E)$

To exemplify matters, let us set $A = \mathcal{L}(E)$ where E is a complex Banach space admitting a decomposition

$$E = F \oplus F^c, \qquad F \cap F^c = \{0\}, \tag{4.1}$$

where F, F^c are closed subspaces of E. Here we choose $p \in P(E) = P(\mathcal{L}(E))$, such that $p \in P(E)$ and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces.

Next, consider the subspace

$$\widetilde{W}(p, A) = \widetilde{W}(F, E) \subseteq \mathcal{L}(F, E), \tag{4.2}$$

consisting of injective linear maps T with closed images which split E. So if $T \in \widetilde{W}(F, E)$, then T is injective and there is a (continuous) projection $q \in P(E)$ such that for $T(F) = q(E)$, we have

$$E = \operatorname{Im} T \oplus \operatorname{Ker} q, \qquad \operatorname{Im} T \cap \operatorname{Ker} q = \{0\}. \tag{4.3}$$

Following [8], [9] there is a complex submanifold $\mathcal{V}(p, E) \subset \widetilde{W}(F, E)$ defined by

$$\mathcal{V}(p, E) := \Big\{ T \in \widetilde{W}(F, E) : \exists q \in \operatorname{Sim}(p, \mathcal{L}(E)), q(E) = T(F) \Big\}. \tag{4.4}$$

The assignment $T \mapsto$ Im T, defines a holomorphic locally trivial GL(F)-principal bundle

$$\mathrm{GL}(F) \hookrightarrow \mathcal{V}(p, E) \longrightarrow \mathrm{Gr}(F, E), \tag{4.5}$$

where Gr(F, E) denotes *the Banach Grassmannian* of closed subspaces W which split E and are similar to F (see [9], [28], [30]).

Observe that our $V(p, \mathcal{L}(E))$ is essentially the same as $\mathcal{V}(p, E)$, because a member of $\mathcal{V}(p, E)$ is a linear homomorphism of $F = p(E)$ onto a closed splitting subspace of E similar to F, whereas $u \in V(p, \mathcal{L}(E))$ belongs to $\mathcal{L}(E)$. In other words, for $u \in V(p, \mathcal{L}(E))$, we have $u|F \in \mathcal{V}(p, E)$. It follows that the restriction map which sends u to its restriction to F, defines a biholomorphic map

$$\varphi : V(p, \mathcal{L}(E)) \longrightarrow \mathcal{V}(p, E) = V(p, \mathcal{L}(E))|_F, \tag{4.6}$$

where the inverse is simply the composition with p. For $\mathcal{V}(p, E)$, the map to Gr(F, E) is the assignment of T to its image Im T, as we have noted. Whereas for $V(p, \mathcal{L}(E))$, we have the map Im onto Gr(p, A) where the images are identified. In either event, the base space is the same Banach Grassmannian Gr$(F, E) =$ Gr$(p, \mathcal{L}(E))$, and in this instance the fibration (2.5) with $G(p) =$ GL(F) is identified with (4.5). Thus the assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, may aptly be called a *spatial correspondence* such that the following diagram commutes :

$$\begin{array}{ccc} V(p, \mathcal{L}(E)) & \xrightarrow{\varphi} & \mathcal{V}(p, E) \\ {\scriptstyle \mathrm{Im}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Im}} \\ \mathrm{Gr}(p, \mathcal{L}(E)) & \xrightarrow{=} & \mathrm{Gr}(F, E) \end{array} \tag{4.7}$$

Example 4.1. We may consider Fredholm operators Fred(E, E') for Banach spaces E and E', as based on the notion of "right and left aggregation"; we refer to [30] for details. An operator $T \in$ Fred(E) is stable under compact perturbations and there is a well-defined *index* given by Ind$(T) = \dim \operatorname{Ker}\ T -$ codim Im T. Ind(T) is constant on connected components and is invariant under compact perturbations. For compositions, the index satisfies Ind$(T_1 T_2) =$ Ind$(T_1) +$ Ind(T_2) and there is an induced homomorphism Ind : Fred$(E) \to \mathbf{Z}$.

With regards to (4.1), let

$$\widehat{G} \subset \left\{ \begin{bmatrix} T_1 & * \\ & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(F),\ T_2 \in \mathrm{Fred}(F^c) \right\},$$

be a Banach Lie group that generates a Banach subalgebra $B \subset A = \mathcal{L}(E)$, but with possibly a different norm. Suppose that $\widehat{G}$ acts analytically on $\mathrm{Gr}(B)$, whereby $\widehat{\mathrm{Gr}}(B)$ denotes a typical orbit. Following [7, Corollary 5.6], the map $\nu^x : \widehat{G} \to \widehat{\mathrm{Gr}}(B)$ admits a local analytic section if and only if $\widehat{\mathrm{Gr}}(B)$ is a complex Banach submanifold of $\mathrm{Gr}(B)$, and ν^x is a submersion. Fixing $q \in P(B)$, let $\widehat{\mathrm{Gr}}(q,B) = \widehat{\mathrm{Gr}}(B) \cap \mathrm{Gr}(q,B)$, and let $\widehat{\mathrm{Gr}}_a(B)$ denote a connected component of $\widehat{\mathrm{Gr}}(B)$ for which Ind $T_1 = a$. Accordingly, we define $\widehat{\mathrm{Gr}}_a(q,B) = \widehat{\mathrm{Gr}}_a(B) \cap \mathrm{Gr}(q,B)$. The restriction $\mathcal{V}a(q,B) = \mathcal{V}(q,B)|\, \widehat{\mathrm{Gr}}_a(q,B)$, thus provides a framing for elements of $\widehat{\mathrm{Gr}}_a(q,B)$.

Example 4.2. When $E = H$ is a complex separable Hilbert space, the $\widehat{\mathrm{Gr}}(q,B)$ can be identified with the restricted Grassmannians $\mathrm{G}_{\mathrm{res}}(H)$ in [26]. Here H admits an orthogonal direct sum decomposition $H = H_+ \oplus H_-$, where $H_\pm$ are closed subspaces, and this decomposition is specified by a unitary operator $J : H \to H$ such that $J|H_\pm = \pm 1$. We consider closed splitting subspaces W that are commensurable with H_+ (that is, for which $W \cap H_+$ has finite codimension in both W and H_+). Then the relevant algebra is $B = \mathcal{L}_J(H)$, the Banach algebra of bounded linear operators $T : H \to H$ such that $[J,T]$ is a Hilbert–Schmidt operator. There is a norm $\|\ \|_J$ defined by $\|T\|_J = \|T\| + \|[J,T]\|_2$ and with the topology induced by $\|\ \|_J$, the group of units $G(B)$ is a complex Banach Lie group (see [26]). In particular, the $\widehat{\mathrm{Gr}}_a(q,B)$ are identified with the disconnected pieces of $\mathrm{G}_{\mathrm{res}}(H)$ and $V_a(q,B)$ is a manifold of 'admissible bases' for the latter.

Remark 4.4. Consider a unital C*-algebra A and the standard (free countable dimensional) Hilbert module H_A over A. Now let B denote the A-linear bounded operators with A-linear bounded adjoints. Then B is a Banach algebra for which $G(B)$ retracts onto the subgroup of unitaries $U(B) = U(H_A)$. Following [20], there is a restricted Grassmannian $\mathrm{G}_{\mathrm{res}}(H_A)$ which is a Banach manifold modeled on the Banach space $\mathcal{K}(H_A)$, and whose unitary and topological structures are describable in a way similar to [26]. The structural properties of $\mathrm{G}_{\mathrm{res}}(H_A)$ have a significant bearing on the Riemann–Hilbert problem for analytic vector functions [2], [20], elliptic transmission problems [20], and the K-theory of C*-algebras [31] (see also references therein). It is possible that the Stiefel manifolds $V(q,B)$ may have potential applications to these questions.

In the case of the Banach algebra $A = \mathcal{L}(E)$, we have the following interpretation of Corollary 3.2 in terms of a holomorphic parametrization of the closed splitting subspaces W and their bases where the latter are regarded as elements of $\mathcal{V}(p,E)$:

Corollary 4.3. *Suppose* $H(p)$ *is contractible. Let* $\psi : X \to \mathrm{Gr}(F, E)$ *be a holomorphic map parametrizing closed splitting subspaces* W *of* E *by* X*. If there exists a continuous parametrization of bases for* W *under* ψ*, as given by a continuous map* $f : X \to \mathcal{V}(p, E)$ *lifting* ψ *through* Im *in* (4.7) *such that* $g = f|Y$ *is holomorphic, then such a parametrization is homotopic to a holomorphic parametrization within the space of continuous maps lifting* ψ *through* Im *that induce* g *on* Y*.*

Observe that this latter result is independent of the actual choice of bases ('unconditional', whatever), but does depend on the lift to $\mathcal{V}(p, E)$ which consequently determines the type of basis furnished to each $W \in \mathrm{Gr}(F, E)$, via the basis assigned to F.

5. The Universal Bundle and the Gauss Map

Continuing with $A = \mathcal{L}(E)$, let ρ denote the natural left action of $G(p)$ on F via evaluation which is analytic. For $x \in \mathcal{V}(p, E),\ a \in F$ and $g \in G(p)$, consider the action given by

$$(x, a)g = (x \cdot g, \rho(g^{-1})a). \tag{5.1}$$

Next we will give a specific realization of the quotient of $\mathcal{V}(p, E) \times F$ under the action in (5.1).

Proposition 5.1. *The space*

$$\begin{aligned}\mathcal{E}(F, E) &:= \Big\{(W, v) \in \mathrm{Gr}(F, E) \times E : v \in W\Big\} \subset \mathrm{Gr}(F, E) \times E \\ &= \mathcal{V}(p, E) \times_\rho F, \end{aligned} \tag{5.2}$$

defines a holomorphic Banach bundle $\mathcal{E}(F, E) \to \mathrm{Gr}(F, E)$*, namely the universal (or tautological) bundle whose fiber over* $W \in \mathrm{Gr}(F, E)$ *is simply the subspace* W *itself. Furthermore, there exists a holomorphic Banach bundle* $\mathcal{E}(\mathcal{L}(E)) \to \mathrm{Gr}(\mathcal{L}(E))$*, satisfying* $\mathcal{E}(\mathcal{L}(E))|\,\mathrm{Gr}(F, E) = \mathcal{E}(F, E)$*.*

Proof. Firstly, we can always find a local holomorphic section s of $\mathrm{Im} : P(E) \to \mathrm{Gr}(\mathcal{L}(E))$ and hence a projection $p_W = s(W)$, that provides a local holomorphic splitting

$$\begin{aligned}\mathrm{Gr}(F, E) \times E &\leftrightarrows \mathcal{E}(F, E) \\ (W, v) &\mapsto p_W(v).\end{aligned} \tag{5.3}$$

The existence of such a local holomorphic splitting suffices to show that $\mathcal{E}(F, E)$ is a complex submanifold and a holomorphic subbundle of $\mathrm{Gr}(F, E) \times E$ where a typical fiber is simply the subspace W itself. We also have the assignment

$$\begin{aligned}\mathcal{V}(p, E) \times F &\longrightarrow \mathcal{E}(F, E) \\ (T, v) &\mapsto (T(F), T(v)),\end{aligned} \tag{5.4}$$

as induced by (5.1). Now using [8, Proposition 5.2] together with the fact that the inner automorphic action on $P(\mathcal{L}(E))$ and the map Im admit local holomorphic sections, (5.4) also admits local holomorphic sections. Thus the second equality in (5.2) follows. Since $\mathrm{Gr}(A) = \mathrm{Gr}(\mathcal{L}(E))$ consists of closed splitting subspaces which split E and $\mathrm{Gr}(A)$ is a discrete union of spaces of the type $\mathrm{Gr}(F, E)$, then essentially the same definition as in (5.2) can be used to define $\mathcal{E}(\mathcal{L}(E))$ tautologically. □

We describe how the Grassmannian $\mathrm{Gr}(p, A)$ figures in an infinite dimensional Gauss map construction in a style that may be applicable to the theory of operator-valued functions.

Theorem 5.1. *Let X be a complex submanifold of a complex Banach space E. Then the following hold.*

(1) *There exists a well-defined holomorphic Gauss map $\psi : X \to \mathrm{Gr}(\mathcal{L}(E))$, which when X is connected, maps X holomorphically into $\mathrm{Gr}(F, E)$.*
(2) *For each $x \in X$, the tangent space $T_x X$ is a closed splitting complex subspace of E.*
(3) *There exists a holomorphic Banach bundle isomorphism $TX \cong \psi^*\mathcal{E}(\mathcal{L}(E))$, and likewise when X is connected, we have $TX \cong \psi^*\mathcal{E}(F, E)$.*

Remark 5.1. If X above is finite dimensional, then X is a Stein manifold by [30, Theorem 3.5].

Proof. Let U be an open neighborhood of $x \in X$ together with a biholomorphic map $f : U \to V$, where V is an open subset of some complex Banach space. Given that X is a complex submanifold, we can extend f locally to a holomorphic map of an open subset $W \subset E$, by suitably shrinking U, V and W should this be necessary, and such that $W \cap X = U$. In this way, we produce a holomorphic map $h : W \to V$, such that $h|U = f|U$.

Now we set $r = f^{-1} \circ h$. Then $r : W \to U$ is a holomorphic retraction of W onto U and its derivative $r' : W \to A = \mathcal{L}(E)$, is a holomorphic map. For $x \in W \cap X$, we have by the chain rule that $r'(x)$ is an idempotent whose image is $T_x X$. Thus $p = r'(x)$ and $1 - p$ are linear idempotents in A, and consequently there exists a well–defined holomorphic map $r_U : U \to P(A)$. On recalling the map $\mathrm{Im} : P(A) \to \mathrm{Gr}(A)$ in Proposition 2.1, we now compose to produce a holomorphic map $\psi : X \to \mathrm{Gr}(A)$, such that $\psi|U = \mathrm{Im}(r_U)$ and is independent of the local retraction r. We recall that $\mathrm{Gr}(p, A)$ is both open and closed in $\mathrm{Gr}(A)$ and the latter is a discrete union of the $\mathrm{Gr}(p, A)$. So when X is connected, ψ maps X holomorphically into $\mathrm{Gr}(p, A) = \mathrm{Gr}(F, E)$. Thus the tangent space $T_x X = \psi(x)$ is realized as an element of $\mathrm{Gr}(\mathcal{L}(E))$ where we recall that the latter consists of closed subspaces which split E. Likewise when X is connected, $T_x X = \psi(x)$ is realized as an element of $\mathrm{Gr}(F, E)$ by the above reasoning.

Finally, consider the tangent bundle $\Pi : TX \to X$ and let $v \in TX$. The assignment $v \mapsto (\psi(\Pi(v)), v)$ is holomorphic and defines a holomorphic Banach bundle map $TX \to \mathcal{E}(\mathcal{L}(E))$ that induces a linear homeomorphism fiberwise. Then there is an obvious unique factorization defining a bundle isomorphism with the holomorphic Banach bundle $\psi^*\mathcal{E}(\mathcal{L}(E))$. Consequently, since the fiber over x of $\psi^*\mathcal{E}(\mathcal{L}(E))$ is the (complex) subspace determined by $\psi(x)$, we see that it is simply the subspace T_xX. $\square$

Acknowledgement

We wish to thank the referee for comments and for pointing out several references. We are also grateful to M. Putinar for bringing our attention to reference [30]. J. Glazebrook wishes to acknowledge the support of the Coles Ballenger Grant (EIU) during the preparation of this paper.

References

1. E. Andruchow, G. Corach and D. Stojanoff, Projective spaces of a C^*-algebra. *Integral Equations Operator Theory* **37**(2000), No. 2, 143–168.
2. B. Bojarski and G. Khimshiashvili, Global geometric aspects of Riemann–Hilbert problems. *Georgian Math. J.* **8**(2001), No. 4, 713–726.
3. L. Bungart, On analytic fiber bundles I. Holomorphic fiber bundles with infinite dimensional fibers. *Topology* **7**(1968), 55–68.
4. G. Corach, H. Porta, and L. Recht, Differential geometry of systems of projections in Banach algebras. *Pacific J. Math.* **143**(1990), No. 2, 209–228.
5. A. Dold, Partitions of unity in the theory of fibrations. *Ann. of Math.*(2) **78**(1963), 233–255.
6. M. J. Dupré, The classification and structure of C^*-Algebra bundles. *Mem. Amer. Math. Soc.* **21**(1979), No. 222, 1–77.
7. M. J. Dupré and J. F. Glazebrook, Infinite dimensional manifold structures on principal bundles. *J. Lie Theory* **10**(2000), No. 2, 359–373.
8. M. J. Dupré and J. F. Glazebrook, The Stiefel bundle of a Banach algebra. *Integral Equations Operator Theory* **41**(2001), No. 3, 264–287.
9. M. J. Dupré, J.–Cl. Evard and J. F. Glazebrook, Smooth parametrization of subspaces of a Banach space. *Rev. Un. Mat. Argentina* **41**(1998), No. 2, 1–13.
10. J. Eells and K. D. Elworthy, Open embeddings of certain Banach manifolds. *Ann. of Math.* (2) **91**(1970), 465–485.
11. J. M. G. Fell and R. S. Doran, Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. I. Basic representation theory of groups and algebras. *Pure and Applied Mathematics,* 125. *Academic Press, Inc., Boston, MA,* 1988.
12. J. Ferrer, I. GARCÍA and F. Puerta, Differentiable families of subspaces. *Linear Algebra Appl.* **199**(1994), 229–252.
13. I. Gohberg and J. Leiterer, Cocycles, operator-valued functions and families of subspaces. (Russian) *Mat. Issled.* **8**(1973), No. 2(28), 23–56.
14. I. Gohberg and J. Leiterer, Über Algebren stetiger Operatorfunktionen. *Studia Math.* **57**(1976), No. 1, 1–26.
15. B. Gramsch, Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren. *Math. Ann.* **269**(1984), No. 1, 27–71.
16. B. Gramsch, Oka's principle for special Fréchet Lie groups and homogeneous manifolds in topological algebras of the microlocal analysis. *Banach Algebras* 97 (*Blaubeuren*), 189–204, *de Gruyter, Berlin,* 1998.

17. B. Gramsch and W. Kaballo, Multiplicative decompositions of holomorphic Fredholm functions and Ψ^*-algebras. *Math. Nachr.* **204**(1999), 83–100.
18. H. Grauert and R. Remmert, Theory of Stein spaces. (Translated from the German) *Grundlehren der Mathematischen Wissenschaften,* 236. *Springer-Verlag, Berlin-New York,* 1979.
19. D. Husemoller, Fibre bundles. 3rd ed. *Graduate Texts in Mathematics,* 20. *Springer-Verlag, New York,* 1994.
20. G. Khimshiashvili, Homotopy classes of elliptic transmission problems over C^*-algebras. *Georgian Math. J.* **5**(1998), No. 5, 453–468.
21. P. A. Kuchment, Floquet theory for partial differential equations. *Operator Theory Advances and Applications,* 60. *Birkhaüser Verlag, Basel*, 1993.
22. N. H. Kuiper, The homotopy type of the unitary group of Hilbert space. *Topology* **3**(1965), 19–30.
23. M. Martin and N. Salinas, Differential geometry of generalized Grassmann manifolds in C^*-algebras. *Operator theory and boundary eigenvalue problems* (*Vienna,* 1993), 206–243, *Oper. Theory Adv. Appl.,* 80, *Birkhaüser, Basel,* 1995.
24. B. S. Mityagin, The homotopy structure of the linear group of a Banach space. *Russian Math. Surveys* **25**(1970), No. 5, 59–103.
25. H. Porta and L. Recht, Spaces of projections in a Banach algebra. *Acta Cient. Venezolana* **38**(1987), No. 4, 408–426.
26. A. Pressley and G. Segal, Loop groups and their representations. *The Clarendon Press, Oxford University Press, New York,* 1986.
27. I. Raeburn, The relationship between a commutative Banach algebra and its maximal ideal space. *J. Functional Analysis* **25**(1977), No. 4, 366–390.
28. H. Upmeier, Symmetric Banach manifolds and Jordan C^*-Algebras. *North-Holland Mathematics Studies,* 104. *Mathematical Notes,* 96. *North-Holland Publishing Co., Amsterdam,* 1985.
29. D. R. Wilkins, The Grassmann manifold of a C^*-algebra. *Proc. Roy. Irish Acad. Sect. A* **90**(1990), No. 1, 99–116.
30. M. G. Zaidenberg, S. G. Krein, P. A. Kushment and A. A. Pankov, Banach bundles and linear operators. (Russian) *Uspekhi Mat. Nauk* **30**(1975), No. 5(185), 101–157.
31. S. Zhang, Factorizations of invertible operators and K-theory of C^*-algebras. *Bull. Amer. Math. Soc.* **28**(1993), 75–83.

(Received 17.09.2001)

Authors' addresses:

M. J. Dupré
Department of Mathematics, Tulane University
New Orleans, LA 70118, U.S.A.
E-mail: mdupre@tulane.edu

J. F. Glazebrook
Department of Mathematics, Eastern Illinois University
Charleston IL 61920, U.S.A.

Department of Mathematics
University of Illinois at Urbana–Champaign
Urbana IL 61801, U.S.A.
E-mails: cfjfg@eiu.edu, glazebro@math.uiuc.edu