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PSEUDO-BESSEL FUNCTIONS AND APPLICATIONS

G. DATTOLI, C. CESARANO, AND D. SACCHETTI

Abstract. We employ the decomposition theorem with respect to the cyclic group of order k to introduce families of pseudo-Bessel functions and to study their properties. We also discuss in brief their applications in probability theory and in electromagnetism.

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1. Introduction

The importance of the decomposition theorem and pseudo-hyperbolic and pseudo-trigonometric functions [1] in applications has been recognized only recently, within the context of problems involving arbitrary order coherent states [2]–[5] and the emission of electromagnetic radiation by accelerated charges [4].

The concepts and of [1] have opened a wider scenario on the possibility of employing larger classes of pseudo-type functions and the initial effort has been made in [6], where families of pseudo-Laguerre and pseudo-Hermite polynomials have been introduced.

According to the decomposition theorem (see [1] for details), given an analytic function of the complex variable x, defined in a circular neighborhood of the origin, by using the series expansion

$$g(x) = \sum_n a_n x^n$$

and considering the series

$$g_j(x;k) = \sum_n a_{kn+j} x^{kn+j}$$

where $k, j \in N, \quad 0 \le j < k$, we obtain the representation formula

$$g_j(x;k) = \frac{1}{k} \sum_{l=1}^{k} \frac{g(x\rho_{l,k})}{\rho_{lj,k}}, \tag{1}$$

where

$$\rho_{l,k} = \exp\left(\frac{2\pi i l}{k}\right).$$

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It is evident that $\rho_{l,k}$ are the roots of unity. For $k=2$, $j=0$ and $k=2$, $j=1$ function (1) reduces to the even and odd components, respectively, of the function $g(x)$. In a more general case the function $g_j(x;k)$ reflects the symmetry

$$g_j(x\rho_{1,k};k)=\rho_{j,k}g_j(x;k)$$

with respect to the roots of unity.

A fairly immediate consequence of the previous theorem is relevant to the generating functions of special functions. An example is provided by the case of ordinary cylindrical Bessel functions for which we find [4]

$$\sum_{n=-\infty}^{\infty} t^{kn+j}J_{kn+j}(x)=\frac{1}{k}\sum_{l=1}^{k}\frac{\exp\left[\frac{x}{2}(t\rho_{l,k}-(t\rho_{l,k})^{-1})\right]}{\rho_{lj,k}},\quad |t|\neq 0. \tag{2}$$

By setting $t=\exp(i\theta)$ (2) yields the following modified Jacobi–Anger expansion:

$$\sum_{n=-\infty}^{\infty}\exp(i(kn+j)\theta)J_{kn+j}(x)=\frac{1}{k}\sum_{l=1}^{k}\frac{\exp\left[ix\sin(\theta+\frac{2\pi il}{k})\right]}{\rho_{lj,k}}.$$

The importance of the above results for the derivation of spectroscopic characteristics of radiation emitted by a "crystalline" electron beam moving in magnetic undulators has been stressed in [4]. The extension of relation (2) to generalized Bessel functions [7] is straightforward but interesting too. Indeed, it has been shown that *even for the N-variable Bessel functions* the following decomposition holds:

$$\sum_{n=-\infty}^{\infty} t^{kn+j}J_{jn+k}(\{x\})=\frac{1}{k}\sum_{l=1}^{k}\frac{\exp\left[\sum_{s=1}^{N}\frac{x_s}{2}(t^s\rho_{sl,k}-(t^s\rho_{ls,k})^{-1})\right]}{\rho_{lj,k}},$$
$$|t|\neq 0,\qquad \{x\}=x_1,x_2,\dots,x_N.$$

We have already remarked that along with pseudo-hyperbolic and pseudo-trigonometric functions, further families of pseudo-type functions have been investigated. Pseudo-Hermite polynomials [6], as well as pseudo-hyperbolic functions, have been used in the theory of higher order coherent states [2]–[4], while functions, which can be recognized as belonging to the family of pseudo-Bessel, have been used in [8] within the context of problems relevant to stochastic processes in random walk problems.

First systematic investigations of the properties of this family of Bessel functions has been carried out in [10]. In this paper we will consider the problem showing connections to other important special functions such as Tricomi and Wright functions, and discuss further generalizations. It is worth noting that pseudo-Bessel functions naturally appear in different research fields (see, e.g., [4]–[8]).

2. Pseudo-Bessel Functions

The Bessel–Tricomi functions [7] $C_n(x)$, linked to ordinary cylindrical functions by the relation

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}), \tag{3}$$

are defined by the generating function

$$\sum_{n=-\infty}^{\infty} t^n C_n(x) = \exp(t - \frac{x}{t}), \quad |t| \neq 0, \tag{4}$$

and by the series

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(n+r)!}\,.$$

According to the decomposition theorem we can also conclude that the series

$$C_{n,j}(x;k) = \sum_{r=0}^{\infty} \frac{(-x)^{kr+j}}{(kr+j)!(n+kr+j)!} = \frac{1}{r}\sum_{l=1}^{k} \frac{C_n(x\rho_{l,k})}{\rho_{lj,k}}$$

defines a pseudo-Bessel–Tricomi function. According to (4), an application of the above-mentioned decomposition theorem yields the following result.

Theorem 1. *The generating functions of* $C_{n,j}(x;k)$ $(k, j \in N,\ 0 \leq j < k)$ *are given by*

$$\sum_{n=-\infty}^{\infty} t^n C_{n,j}(x,k) = \exp(t) E_j\left(-\frac{x}{t};k\right), \tag{5}$$

where

$$E_j(x;k) = \sum_{n=0}^{\infty} \frac{x^{kn+j}}{(kn+j)!} = \frac{1}{k}\sum_{l=1}^{k} \frac{\exp(x\rho_{l,k})}{\rho_{lj,k}}$$

is the pseudo-hyperbolic function introduced in [1] *satisfying the properties*

$$\begin{aligned} \frac{d}{dx}E_j(x;k) &= E_{j-1}(x;k), \\ \frac{d}{dx}E_0(x;k) &= E_{k-1}(x;k), \\ \left(\frac{d}{dx}\right)^k E_j(x;k) &= E_j(x;k). \end{aligned} \tag{6}$$

Using (6) and (5), we can derive recurrence relations of the type

$$\begin{aligned} \frac{d}{dx}C_{n,j}(x,k) &= -C_{n+1,j-1}(x,k), \\ nC_{n,j}(x;k) &= C_{n-1,j}(x;k) + xC_{n+1,j-1}(x;k). \end{aligned}$$

Furthermore, on account of (3) it easily follows that the series

$$J_{n,j}(x;k) = \sum_{r=0}^{\infty} \frac{(-1)^{rk+j}(\frac{x}{2})^{2rk+n+j}}{(rk+j)!(rk+j+n)!}$$

can be written in terms of pseudo-Bessel–Tricomi functions, namely

$$J_{n,j}(x;k) = \left(\frac{x}{2}\right)^{n-j} C_{n,j}\left(\frac{x^2}{2};k\right). \tag{7}$$

Since $E_j(x;k)$ is a generalization of the exponential function ($E_0(x;1) = \exp(x)$), we can introduce pseudo-Bessel functions by considering obvious extension of the generating function of ordinary Bessel function. We can use, e.g., the generating function

$$\sum_{n=-\infty}^{\infty} t^{kn+j-w} J_{n;j,w}(x;k) = E_j\left(\frac{xt}{2};k\right) E_w\left(-\frac{x}{2t};k\right), 0 \leq w \leq j < k,$$

to introduce the pseudo-Bessel function $J_{n;j,w}(x;k)$ defined by the series

$$J_{n;j,w}(x;k) = \left(\frac{x}{2}\right)^{kn} \sum_{r=0}^{\infty} \frac{(-1)^{kr+w}\left(\frac{x}{2}\right)^{2ks+w+j}}{(k^2n+ks+j)!(ks+w)!}$$

or, equivalently, by

$$J_{n;j,w}(x;k) = \sum_{l=1}^{k}\sum_{m=1}^{k} \frac{C_n(\frac{x\rho_{m,k}}{2}, -\frac{x\rho_{l,k}}{2})}{\rho_{mw,k}\rho_{lj,k}},$$
$$C_n(x,y) = y^n C_n(xy).$$

Before going further let us recall that the use of the operator $\mathcal{D}_x^{-1}$, which is essentially the inverse of a derivative operator and whose action on a given function $f(x)$ is provided by

$$\mathcal{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-\xi)^{\nu-1} f(\xi)d\xi,$$

can be used to state the following result.

Theorem 2. *Consider the function defined by the series expansion*

$$J_n(x;k,\lambda) = \sum_{r=0}^{\infty} \frac{x^{kr+n}}{(kr+n)!(\lambda r)!}$$

and the Bessel–Wright function $W_n(x|\nu)$ (see [10]). Then the following representation of the function $J_n(x;k,\lambda)$ in terms of the Bessel–Wright function holds true:

$$J_n(x;k,\lambda) = \frac{x^n}{\lambda} \sum_{m=1}^{\lambda} W_n(x\rho_{m\frac{\lambda}{k},\lambda}|\frac{k}{\lambda}). \tag{8}$$

Proof. By considering the operator

$$\mathcal{D}_x^{-\nu} 1 = \frac{x^\nu}{\Gamma(\nu+1)}$$

the use of the decomposition theorem allows us to write

$$J_n(x;k,\lambda) = \mathcal{D}_x^{-n} \sum_{r=0}^{\infty} \frac{\mathcal{D}_x^{-\left(\frac{k}{\lambda}\right)\lambda r}}{(\lambda r)!} = \frac{\mathcal{D}_x^{-n}}{\lambda} \sum_{m=1}^{\lambda} \exp(\mathcal{D}_x^{-\frac{k}{\lambda}} \rho_{m,\lambda}),$$

and noting that

$$\mathcal{D}_x^{-n} \exp(\mathcal{D}_x^{-\nu}) = \sum_{r=0}^{\infty} \frac{x^{\nu r+n}}{r!\Gamma(\nu r+n+1)} = x^n W_n(x|\nu)$$

(8) immediately follows. □

The points we have touched upon in this section show that the use of the concepts and formalism originally suggested in [1] may provide interesting elements of speculation on new classes of functions belonging to the Bessel family. Further considerations on their properties will be presented in the next section.

Before concluding this section, it is worth mentioning the differential equation satisfied by the functions $C_{n,j}(x,k)$. To this aim we recall that ordinary Tricomi functions satisfy the second order differential equation

$$\widehat{P}_n C_n(x) = -C_n(x),$$
$$\widehat{P}_n = x\frac{d^2}{dx^2} + (n+1)\frac{d}{dx}.$$

It is also fairly straightforward to prove that

$$\left(\widehat{P}_n\right)^r C_n(ax) = (-a)^r C_n(ax)$$

and thus it is readily understood that

$$\left(\widehat{P}_n\right)^r C_{n.j}(x,k) = -C_{n,j}(x,k).$$

For further comments the reader is referred to [9], where the problem has been discussed in a more general framework.

3. Concluding Remarks

In the preceding section we have shown that families of pseudo-Bessel functions can be obtained in many different ways. Before discussing possible fields of application let us consider a further example. We recall that the function [7]

$$J_n^{(m)}(x,y) = \sum_{l=-\infty}^{\infty} J_{n-ml}(x) J_l(y) \tag{9}$$

defines a *two-variable Bessel function* generated by

$$\sum_{n=-\infty}^{\infty} t^n J_n^{(m)}(x,y) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right) + \frac{y}{2}\left(t^m - \frac{1}{t^m}\right)\right].$$

Then we can introduce a new generalization.

Definition 1. k-generalized Bessel functions are defined as follows:

$$J_n^{(m)}(x,y|k) = \sum_{l=-\infty}^{\infty} J_{n-mkl}(x)J_{kl}(y).$$

On the other hand, the one parameter generalization of the generalized Bessel functions (9) is given by

$$J_n^{(m)}(x,y;\tau) = \sum_{l=-\infty}^{\infty} \tau^l J_{n-ml}(x)J_l(y)$$

and is generated by

$$\sum_{n=-\infty}^{\infty} t^n J_n^{(m)}(x,y;\tau) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(\frac{t^m}{\tau}-\frac{\tau}{t^m}\right)\right].$$

By the latter relations and the results of Section 2 the following theorem follows easily:

Theorem 3. *The k-generalized Bessel function can be decomposed in the form*

$$J_n^{(m)}(x,y|k) = \frac{1}{k}\sum_{l=1}^{k} J_n^{(m)}(x,y;\rho_{k,l}).$$

We have already remarked that one of the reasons for interest in pseudo-Bessel functions is their use in electromagnetic problems concerning the emission by an relativistic "chrystalline" electron beam propagating in magnetic undulators. By chrystalline beams we mean bunches of electrons with non-randomly distributed positions and having a phase distribution with respect to the field of an undulator, symmetric with respect to the roots of unity (see [4]).

Another important application comes from the theory of random motion with alternating velocities [8]. Within such a context it has been shown that the associated probability law can be expressed by the function

$$S_{i,j}^{(n)}(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2nk+i+j}}{(nk+i)!(nk+j)!}.$$

By the results of the previous sections and (7) we find

$$S_{i,j}^{(n)}(z) = \left(\frac{z}{2}\right)^{-j} I_{i-j,j}(z;n),$$

where $I_{n,j}(x;k)$ denotes the modified version of $J_{n,j}(x;k)$.

The results of this paper hint to interesting applications of these families of Bessel functions. We believe that a further insight into the relevant theory and properties may provide a more thorough understanding of the problems in which they currently appear.

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Authors' addresses:

Giuseppe Dattoli
Unità Tecnico Scientifica Tecnologie Fisiche Applicate
ENEA – Centro Ricerche Frascati
Via E. Fermi, 45, 00044 - Frascati - Roma
Italy
E-mail: dattoli@frascati.enea.it

Clemente Cesarano
Abteilung Angewandte Analysis
Universität Ulm
Helmholtzstr. 18, 89081 Ulm
Germany

Current address:
Unità Tecnico Scientifica Tecnologie Fisiche Applicate
ENEA – Centro Ricerche Frascati
Via E. Fermi, 45, 00044 - Frascati - Roma
Italy
E-mail: cesarano@frascati.enea.it

Dario Sacchetti
Dipartimento di Statistica
Università degli Studi di Roma “La Sapienza”
P.le A. Moro, 2, 00185 – Roma
Italy
E-mail: dario.sacchetti@uniroma1.it