Georgian Mathematical Journal
Volume 9 (2002), Number 3, 461–472

ON IDEALS WITH PROJECTIVE BASES

J. CICHOŃ AND A. KHARAZISHVILI

Abstract. A theorem concerning some descriptive properties of σ-ideals and generalizing the main result of [1] is proved. Various applications of this theorem are also presented.

2000 Mathematics Subject Classification: 03E15, 28A05, 28B20.
Key words and phrases: Ideal of sets, projective base, the perfect subset property, nonmeasurable set.

The article is concerned with those descriptive properties of a σ-ideal of sets, which are implied by the existence of a projective base for this ideal and are closely connected with the existence of nonmeasurable sets. The main result of the article (Theorem 1) generalizes the result of [1] and can successfully be applied to some questions of measure theory and set-theoretic topology (in this connection, cf. the references below and, especially, [2]).

The notation used throughout the paper is fairly standard.

As usual, we denote by ω the set of all natural numbers.

If X is any set, then $[X]^{\leq\omega}$ is the family of all countable subsets of X.

The cardinality of a set X is denoted by $\mathrm{card}(X)$.

The set of all real numbers is denoted by R.

Recall that if X is an arbitrary set, then $P(X)$ is the family of all subsets of X.

If X is an arbitrary topological space, then $B(X)$ is the family of all Borel subsets of X. Respectively, the well-known classes of projective subsets of X are denoted by

$$\Sigma_1(X),\ \Pi_1(X),\ \Sigma_2(X),\ \Pi_2(X),\ldots.$$

It is also convenient to put $\Sigma_0(X) = \Pi_0(X) = B(X)$. As usual, we define

$$\Delta_n(X) = \Sigma_n(X) \cap \Pi_n(X).$$

A Σ_n-space is any Σ_n-subset of a Polish space X, equipped with the topology induced by the topology of X. In particular, according to this definition, a Suslin space is any Σ_1-space.

If $\mathcal{S}$ and $\mathcal{T}$ are any two families of subsets of a given set X, then we denote by $\mathcal{S}\triangle\mathcal{T}$ the family of sets

$$\{Y\triangle Z : Y \in \mathcal{S}\ \&\ Z \in \mathcal{T}\},$$

ISSN 1072-947X / $8.00 /

where, as usual,

$$Y \triangle Z = (Y \setminus Z) \cup (Z \setminus Y).$$

It is clear that if $\mathcal{S}$ is a σ-algebra of sets and $\mathcal{I}$ is a σ-ideal of sets, then $\mathcal{S} \triangle \mathcal{I}$ is a σ-algebra of sets, too.

If X is a metric space and $x \in X$, then $B(x, \varepsilon)$ denotes the open ball with center x and radius ε.

Let X be a topological space and let $\mathcal{I}$ be an ideal of subsets of X. We recall that $\mathcal{I}$ has a Borel base if, for every set $Y \in \mathcal{I}$, there exists a set $Z \in \mathcal{I} \cap B(X)$ such that $Y \subset Z$. In an analogous manner, we say that $\mathcal{I}$ has a Π_n-base if, for every set $Y \in \mathcal{I}$, there exists a set $Z \in \mathcal{I} \cap \Pi_n(X)$ such that $Y \subset Z$.

In this article we investigate those σ-ideals of subsets of Σ_n-spaces, which have Π_n-bases. We are especially interested in situations where such σ-ideals produce nonmeasurable (e.g., in the Lebesgue sense) subsets of an original space.

Notice first that if A is a subset of a Polish space X and

$$A \in \Pi_n(X) \setminus \Sigma_n(X),$$

then the σ-ideal generated by the family of sets $P(A) \cup [X \setminus A]^{\leq \omega}$ is a σ-ideal with some Π_n-base and, obviously, this ideal has no Σ_n-base. We now give a slightly more elaborated example.

Example 1. Let P be a nonempty perfect subset of the real line $\mathbf{R}$, consisting of linearly independent elements (over the field $\mathbf{Q}$ of all rational numbers). It is well known that such a set P exists (in this connection, see, e.g., [3]). Let us fix a natural number $n > 0$ and consider an arbitrary subset A of P belonging to the class $\Pi_n(R) \setminus \Sigma_n(R)$. It is also well known (see, for instance, [4]) that, for any two different real numbers t and q, we have

$$\text{card}\Big((P+t) \cap (P+q)\Big) \leq 1.$$

Now, let us consider the σ-ideal $\mathcal{J}$ of subsets of the real line, generated by the family of all translates of the set A. Then $\mathcal{J}$ is a σ-ideal invariant under the group of all translations of the real line. Evidently, $\mathcal{J}$ has a Π_n-base. On the other hand, $\mathcal{J}$ does not possess a Σ_n-base. To see this, suppose to the contrary that $\mathcal{J}$ has a Σ_n-base. Then there exist a set $B \in \Sigma_n(R)$ and a sequence $\{t_n : n \in \omega\}$ of reals, such that

$$A \subset B \subset \cup\{A + t_n : n \in \omega\}.$$

From these inclusions we get

$$\text{card}\Big((B \cap P) \setminus A\Big) \leq \omega$$

and therefore we obtain $A \in \Sigma_n(R)$, which is impossible.

Moreover, let us remark that if a σ-ideal $\mathcal{I}$ has a projective base, then, for some $n \in \omega$, it has also a Π_n-base. This fact is an immediate consequence of the following simple set-theoretical proposition:

Lemma 1. *Suppose that $(X, \leq)$ is an upward σ-centered partially ordered set, B is a cofinal subset of X and suppose that $B = \cup\{B_n : n \in \omega\}$. Then, for some $n \in \omega$, the set B_n is also cofinal in X.*

We omit an easy proof of this proposition.

We shall say that a class of subsets of a Polish space has the perfect subset property if every uncountable set from this class contains a nonempty perfect subset. Let us recall that, in the theory **ZFC**, the classes Σ_0 and Σ_1 have the perfect subset property. Recall also that the statement

$$\text{the class } \Pi_1 \text{ has the perfect subset property}$$

is independent of the theory **ZFC**. Moreover, it is known that, for each natural number $n > 0$, the theory

$$\mathbf{ZFC} \ \& \ \Sigma_n \text{ has the perfect subset property } \&$$
$$\Pi_n \text{ has not the perfect subset property}$$

is relatively consistent.

Let $\mathcal{A}$ and $\mathcal{S}$ be any two families of sets. We say that $\mathcal{A}$ is $\mathcal{S}$-summable if, for every $\mathcal{A}' \subset \mathcal{A}$, we have $\cup\mathcal{A}' \in \mathcal{S}$ (cf. [2]).

A family $\mathcal{A}$ of sets is called point-finite if $\{A \in \mathcal{A} : a \in A\}$ is finite for each point $a \in \cup\mathcal{A}$.

The following result can be regarded as a stronger version of the main theorem from [1].

Theorem 1. *Suppose that the class Σ_n has the perfect subset property. Let X be an arbitrary Σ_n-space and let $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base. Suppose also that:*

1) $\mathcal{A}$ is a point-finite family of sets;

2) $\mathcal{A}$ is a $(\Sigma_n(X)\triangle\mathcal{I})$-summable family.

Then there exists a family $\mathcal{B} \in [\mathcal{A}]^{\leq\omega}$ such that $(\cup\mathcal{A} \setminus \cup\mathcal{B}) \in \mathcal{I}$.

Proof. Without loss of generality we may assume that X is an uncountable Σ_n-space and the cardinality of the given family $\mathcal{A}$ is less than or equal to the cardinality continuum. Let T be a subset of X which contains no perfect subset and satisfies the equality $\mathrm{card}(T) = \mathrm{card}(\mathcal{A})$ (note that the set T can be realized as a subset of some Bernstein set in the space X). Obviously, we may identify the set T with the set of indices of the given family $\mathcal{A}$. In other words, we can write $\mathcal{A} = \{A_t : t \in T\}$. Furthermore, we put

$$\Gamma = \Big\{(x, t) \in X \times T : x \in A_t\Big\}.$$

Let D be a countable dense subset of X. It is easy to check that

$$\Gamma = \bigcap_{k\in\omega} \bigcup_{d\in D} \left(\Gamma^{-1}\Big(B\Big(d, \frac{1}{k+1}\Big)\Big) \times B\Big(d, \frac{1}{k+1}\Big)\right).$$

For any $k \in \omega$ and for any $d \in D$, let $S_{k,d} \in \Sigma_n(X)$ and $A_{k,d} \in \mathcal{I}$ be subsets of X such that

$$\Gamma^{-1}\Big(B\Big(d, \frac{1}{k+1}\Big)\Big) = S_{k,d} \triangle A_{k,d}.$$

Define

$$A = \bigcup_{k\in\omega} \bigcup_{d\in D} A_{k,d}$$

and, taking into account that $A \in \mathcal{I}$, fix some Π_n-set $A' \in \mathcal{I}$ such that $A \subset A'$. Then the set

$$\Gamma' = \Gamma \cap \Big((X \setminus A') \times X\Big)$$

is a Σ_n-subset of the product space $X \times X$. Hence, the set

$$T_1 = \Big\{t \in T : (\exists x)((x,t) \in \Gamma')\Big\}$$

is a Σ_n-subset of the set T, too. Consequently, T_1 must be countable. Thus we obtain

$$\bigcup_{t\in T_1} A_t \supset \bigcup_{t\in T} A_t \setminus A'.$$

In virtue of the relation $A' \in \mathcal{I}$, this completes the proof. $\square$

Remark 1. Suppose that $\Gamma \subset A \times B$ is a relation with finite vertical sections, i.e.,

$$\mathrm{card}\Big(\{y : (x,y) \in \Gamma\}\Big) < \omega$$

for each $x \in A$. Let $\{V_n : n \in \omega\}$ be a countable family of subsets of B, which separates the points in B, i.e., for any two distinct points $a, b \in B$, there exists $n \in \omega$ such that $\mathrm{card}(V_n \cap \{a,b\}) = 1$. Then we have

$$\Gamma = \bigcup_{f\in 2^\omega} \bigcap_n \Gamma^{-1}\Big(V_0^{f(0)} \cap \ldots \cap V_n^{f(n)}\Big) \times \Big(V_0^{f(0)} \cap \ldots \cap V_n^{f(n)}\Big),$$

where $V^0 = V$ and $V^1 = B \setminus V$ for each set $V \subset B$. This fact enables us to develop some analogues of the above theorem for a wider class of topological spaces. In this connection, recall that a typical example of a non-separable Banach space with a countable family of Borel sets, separating the points, is the classical space l^∞ consisting of all bounded real-valued sequences.

It is possible to apply directly Theorem 1 to the family of all analytic subsets of a Polish space X (put $\mathcal{I} = \{\varnothing\}$), to the σ-algebra of subsets of X with the Baire property, to the σ-algebra of Lebesgue measurable subsets of the real line, and so on. We shall give now some other applications of this theorem.

Theorem 2. *Suppose that the class Σ_n has the perfect subset property. Let $\mathcal{A}$ be an uncountable family of nonempty pairwise disjoint Σ_n-sets. Then there exists a subfamily $\mathcal{C}$ of $\mathcal{A}$ such that $\cup\mathcal{C}$ is not a Σ_n-set.*

Proof. Indeed, let us put $X = \cup\mathcal{A}$. If X is not a Σ_n-space, then there is nothing to prove. Assume now that X is a Σ_n-space. Then we can consider the given family $\mathcal{A}$ with the σ-ideal

$$\mathcal{I} = [X]^{\leq\omega}.$$

Applying Theorem 1 to $\mathcal{A}$ and $\mathcal{I}$, we easily get the required result. □

Theorem 3. *Suppose that X is a Polish space and $\mathcal{I}$ is a σ-ideal of subsets of X with a Borel base. Suppose also that $\mathcal{A} \subset \mathcal{I}$ is a point-finite family of sets, such that $\cup\mathcal{A} = X$. Then there exists a subfamily $\mathcal{C}$ of $\mathcal{A}$ for which*

$$\bigcup\mathcal{C} \notin B(X)\triangle\mathcal{I}.$$

Proof. Indeed, since the class $B(X) = \Sigma_0(X) = \Pi_0(X)$ has the perfect subset property, we may directly apply Theorem 1 to the family $\mathcal{A}$ and to the ideal $\mathcal{I}$. □

Let us consider two immediate consequences of Theorem 3.

Example 2. Let X be a Polish space and let $\mathcal{I}$ denote the σ-ideal of all first category subsets of X. Suppose also that $\mathcal{A} \subset \mathcal{I}$ is a point-finite covering of X. Then, according to Theorem 3, there exists a family $\mathcal{C} \subset \mathcal{A}$ such that the set $\bigcup\mathcal{C}$ does not have the Baire property.

Example 3. Let X be a Polish space and let μ be a nonzero σ-finite Borel measure on X. Denote by μ' the completion of μ and let $\mathcal{I}$ be the σ-ideal of all μ'-measure zero subsets of X. Suppose also that $\mathcal{A} \subset \mathcal{I}$ is a point-finite covering of X. Then, in view of Theorem 3, there exists a family $\mathcal{C} \subset \mathcal{A}$ such that the set $\bigcup\mathcal{C}$ is not measurable with respect to μ'.

Now, for a given family $\mathcal{S}$ of sets, we define

$$\mathcal{S}^- = \Big\{Z : (\forall Z' \subset Z)(Z' \in \mathcal{S})\Big\}.$$

Using this notation, we can formulate the following result.

Theorem 4. *If X is a Polish space and $\mathcal{I}$ is a σ-ideal of subsets of X covering X and possessing a Borel base, then*

$$\Big(B(X)\triangle\mathcal{I}\Big)^- = \mathcal{I}.$$

This result easily follows from Theorem 3 and seems to be natural.

In order to present some other consequences of Theorem 1, we need to recall the concept of a discrete family of sets (in a topological space). Since we deal here only with metrizable topological spaces, this concept will be introduced for metric spaces (see, e.g., [5]).

A family $\mathcal{F}$ of subsets of a metric space E is called discrete if there exists a nonzero $m \in \omega$ such that

$$(\forall F_1, F_2 \in \mathcal{F})\Big(F_1 \neq F_2 \Rightarrow \mathrm{dist}(F_1, F_2) > \frac{1}{m}\Big).$$

Notice that if $\mathcal{F}$ is a discrete family of closed sets and $\mathcal{S} \subset \mathcal{F}$, then $\cup\mathcal{S}$ is a closed set, too. Moreover, if $Z \subset E$ is compact, then the family

$$\Big\{Y \in \mathcal{F} : Y \cap Z \neq \varnothing\Big\}$$

is finite.

In our further considerations we need the following corollary from the well-known Montgomery Lemma (see [6]).

Lemma 2. *Let E be an arbitrary metric space and let $\mathcal{S}$ be any family of open sets in E, such that $\cup\mathcal{S} = E$. Then there exists a sequence $\{\mathcal{F}_n : n \in \omega\}$ of discrete families of closed subsets of E, satisfying the relations:*
1) $(\forall n \in \omega)(\forall F \in \mathcal{F}_n)(\exists U \in \mathcal{S})(F \subset U)$;
2) $\bigcup_{n\in\omega}(\bigcup \mathcal{F}_n) = E$.

The proof of this lemma is presented, e.g., in [6] and [7].

Remark 2. For any infinite cardinal number τ, let us consider the topological sum of the family of spaces $\{[0,1] \times \{\xi\} : \xi < \tau\}$. Let us identify in this sum all points $(0,\xi)$, where $\xi < \tau$, and denote the obtained space by E_τ. It is well known that any metric space E with weight τ can be embedded into the countable product of copies of E_τ (see, e.g., [5]). From this fact, the previous auxiliary proposition (i.e., Lemma 2) can be deduced without using the Montgomery Lemma.

Let E be a metric space. As usual, we denote by $\mathrm{Comp}(E)$ the family of all nonempty compact subsets of E and we equip this family with the Vietoris topology (see, e.g., [5]). Further, let $\mathcal{S}$ be a family of subsets of a given set X, closed under countable unions and countable intersections.

We shall say that a mapping

$$\Phi : X \to \mathrm{Comp}(E)$$

is upper $\mathcal{S}$-measurable if, for every open set $Y \subset E$, the relation

$$\Big\{x \in X : \Phi(x) \cap Y \neq \varnothing\Big\} \in \mathcal{S}$$

is fulfilled. It can easily be checked that, in our case, a mapping Φ is upper $\mathcal{S}$-measurable if and only if it is lower $\mathcal{S}$-measurable, i.e., for every closed set $Z \subset E$, we have

$$\Big\{x \in X : \Phi(x) \cap Z \neq \varnothing\Big\} \in \mathcal{S}.$$

Of course, it is essential here that, for each point $x \in X$, the set $\Phi(x)$ is compact and nonempty.

Lemma 3. *Let the class Σ_n have the perfect subset property, let X be an arbitrary Σ_n-space and let $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base. Suppose also that E is a metric space and*

$$\Phi \ : \ X \to \mathrm{Comp}(E)$$

is a lower $(\Sigma_n(X)\triangle\mathcal{I})$*-measurable mapping. Finally, let* $\mathcal{F}$ *be a discrete family of closed subsets of* E*. Then there exist a countable subfamily* $\mathcal{F}'$ *of* $\mathcal{F}$ *and a set* $A \in \mathcal{I}$*, such that*

$$(\forall x \in X \setminus A)(\Phi(x) \cap (\cup\mathcal{F}) \neq \varnothing \Rightarrow \Phi(x) \cap (\cup\mathcal{F}') \neq \varnothing).$$

Proof. For each set $Z \in \mathcal{F}$, we put

$$A_Z = \Big\{x \in X : \Phi(x) \cap Z \neq \varnothing\Big\}.$$

The compactness of all values of the mapping Φ implies that $\{A_Z : Z \in \mathcal{F}\}$ is a point-finite family. Furthermore, the discreteness of the family $\mathcal{F}$ implies that $\{A_Z : Z \in \mathcal{F}\}$ is a $(\Sigma_n(X)\triangle\mathcal{I})$-summable family. Hence, by Theorem 1, there exist a countable subfamily $\mathcal{F}'$ of $\mathcal{F}$ and a set $A \in \mathcal{I}$, such that

$$\cup\{A_Z : Z \in \mathcal{F}'\} \cup A = \cup\{A_Z : Z \in \mathcal{F}\}.$$

Suppose now that $x \in X \setminus A$ and that $\Phi(x) \cap (\cup\mathcal{F}) \neq \varnothing$. Then we have

$$x \in \cup\{A_Z : Z \in \mathcal{F}\}.$$

Consequently, $x \in \cup\{A_Z : Z \in \mathcal{F}'\}$ and, therefore,

$$\Phi(x) \cap (\cup\mathcal{F}') \neq \varnothing.$$

This completes the proof. $\square$

Lemma 4. *Let the class* Σ_n *have the perfect subset property, let* X *be an arbitrary* Σ_n*-space and let* $\mathcal{I}$ *be a* σ*-ideal of subsets of* X *with a* Π_n*-base. Suppose also that* E *is a metric space and*

$$\Phi \ : \ X \to \mathrm{Comp}(E)$$

is a lower $(\Sigma_n(X)\triangle\mathcal{I})$*-measurable mapping. Finally, let* $\mathcal{F}$ *be a discrete family of closed subsets of* E*. Then there exist a countable subfamily* $\mathcal{F}'$ *of* $\mathcal{F}$ *and a set* $A \in \mathcal{I}$*, such that*

$$(\forall x \in X \setminus A)(\forall Z \in \mathcal{F})\Big(\Phi(x) \cap Z \neq \varnothing \Rightarrow Z \in \mathcal{F}'\Big).$$

Proof. We use Lemma 3 and define three sequences

$$(A_n)_{n\in\omega}, (B_n)_{n\in\omega}, (\mathcal{F}_n)_{n\in\omega}$$

satisfying the following properties:

(1) $A_0 = \{x \in X : \Phi(x) \cap (\cup\mathcal{F}) \neq \varnothing\}$;
(2) $A_0 \supset B_0 \supset A_1 \supset B_1 \supset \ldots$;
(3) $\mathcal{F}_n \cap \mathcal{F}_m = \varnothing$ for all distinct $n, m \in \omega$;
(4) $\mathcal{F}_n \in [\mathcal{F}]^{\leq\omega}$ for each $n \in \omega$;
(5) $A_n \setminus B_n \in \mathcal{I}$ for each $n \in \omega$;
(6) if $x \in B_n$, then $\Phi(x) \cap (\cup\mathcal{F}_n) \neq \varnothing$;
(7) for any $n \in \omega$, we have

$$A_{n+1} = \Big\{x \in B_n : \Phi(x) \cap (\cup(\mathcal{F} \setminus (\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_n))) \neq \varnothing\Big\}.$$

Observe that

$$\cap_{n\in\omega}A_n = \cap_{n\in\omega}B_n = \varnothing.$$

Indeed, if $x \in \cap_{n\in\omega}A_n = \cap_{n\in\omega}B_n$, then the set

$$\Big\{Z \in \mathcal{F} : \Phi(x) \cap Z \neq \varnothing\Big\}$$

must be infinite. But this is impossible, since $\Phi(x)$ is a compact set. Hence, $\cap_{n\in\omega}A_n = \varnothing$. Let us put

$$A = \cup_{n\in\omega}(A_n \setminus B_n),$$
$$\mathcal{F}' = \cup_{n\in\omega}\mathcal{F}_n.$$

Notice that $A \in \mathcal{I}$. If $x \in X \setminus A$, $Z \in \mathcal{F}$ and $\Phi(x) \cap Z \neq \varnothing$, then we have

$$x \in (B_0 \setminus A_1) \cup (B_1 \setminus A_2) \cup \dots .$$

This yields $x \in B_n \setminus A_{n+1}$ for some $n \in \omega$. Therefore, $x \in B_n$ and $x \notin A_{n+1}$. The last two relations imply at once that if $Z \in \mathcal{F}$ and $\Phi(x) \cap Z \neq \varnothing$, then $Z \in \mathcal{F}'$. Thus, the lemma is proved. $\square$

Lemma 5. *Let the class Σ_n have the perfect subset property, let X be an arbitrary Σ_n-space and let $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base. Suppose also that E is a metric space and*

$$\Phi : X \to \mathrm{Comp}(E)$$

is a lower $(\Sigma_n(X)\triangle\mathcal{I})$-measurable mapping. Then there exist a set $A \in \mathcal{I}$ and a closed separable subset F of E, such that

$$(\forall x \in X \setminus A)(\Phi(x) \subset F).$$

Proof. Applying Lemma 2, we can find a double sequence

$$(\mathcal{F}_{n,m})_{n\in\omega,m\in\omega}$$

of families of closed subsets of the space E, such that:

1) $(\forall n \in \omega)(\forall m \in \omega)(\forall Z \in \mathcal{F}_{n,m})(\mathrm{diam}(Z) < \frac{1}{1+m})$;
2) $(\forall m \in \omega)(\cup_{n\in\omega}(\cup\mathcal{F}_{n,m}) = E)$;
3) $(\forall n \in \omega)(\forall m \in \omega)(\mathcal{F}_{n,m}$ is discrete).

Next, we use Lemma 4 and, for any $n, m \in \omega$, we can find a set $A_{n,m}$ and a family $\mathcal{H}_{n,m}$, such that:

a) $\mathcal{H}_{n,m} \in [\mathcal{F}_{n,m}]^{\leq\omega}$;
b) $A_{n,m} \in \mathcal{I}$;
c) $(\forall x \in X \setminus A_{n,m})(\forall Z \in \mathcal{F}_{n,m})(\Phi(x) \cap Z \neq \varnothing \Rightarrow Z \in \mathcal{H}_{n,m})$.

Let us consider the subspace

$$F = \mathrm{cl}\Big(\cap_{m\in\omega}\left(\cup_{n\in\omega}(\cup\mathcal{H}_{n,m})\right)\Big)$$

of the space E and let us put

$$A = \cup_{n,m\in\omega}A_{n,m}.$$

Then it is easy to check that F and A are the required sets, which completes the proof. $\square$

Now, we are able to formulate and prove several consequences of the preceding results, concerning some kinds of measurable functions and selectors.

Theorem 5. *Suppose again that the class Σ_n has the perfect subset property. Let X be an arbitrary Σ_n-space and let $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base. If E is a metric space and*

$$f : X \to E$$

is a $(\Sigma_n(X)\triangle\mathcal{I})$-measurable mapping, then there exists a set $A \in \mathcal{I}$ such that $f(X \setminus A)$ is a separable subspace of E.

Proof. Let us consider the mapping

$$\Phi(x) = \{f(x)\} \quad (x \in X).$$

Then we obviously have

$$\Phi : X \to \mathrm{Comp}(E)$$

and it is easy to see that Φ is lower $(\Sigma_n(X)\triangle\mathcal{I})$-measurable. Hence, we may utilize Lemma 5 to the mapping Φ. In this way we get the desired result. $\square$

Evidently, we can also apply Theorem 5 to the class Σ_1 and to the ideal $\mathcal{I} = \{\varnothing\}$. Then we obtain the following theorem due to Frolik (see [8]).

Theorem 6. *Suppose that X is a Suslin space, E is a metric space and*

$$f : X \to E$$

is a Borel mapping. Then the range of f is a separable subspace of E.

Let us notice that this result can also be easily deduced from Theorem 2.

Remark 3. Let us consider the topology $\mathcal{T}$ on the real line R generated by the family

$$\Big\{U \setminus T : U \text{ is open } \& \ T \text{ is countable}\Big\}.$$

Then the function $f : R \to R$ given by the formula $f(x) = x$ and treated as a mapping from the real line equipped with the standard topology into the real line equipped with the topology $\mathcal{T}$, is a Borel isomorphism, but the range of f is nonseparable. This simple example shows us that, in Theorem 6, the assumption of metrizability of E is rather essential.

Theorem 7. *Suppose that X is a Polish space, E is a metric space and*

$$f : E \to X$$

is a mapping satisfying the following conditions:

a) $(\forall x \in X)(f^{-1}(\{x\}) \in \mathrm{Comp}(E) \cup \{\varnothing\})$;

b) for each closed set $Z \subset E$, we have $f(Z) \in \Sigma_1(X)$.

Then E is a separable space.

Proof. Indeed, let us define

$$\Phi(x) = f^{-1}(x) \quad (x \in f(E))$$

and put $\mathcal{I} = \{\varnothing\}$. Then a straightforward application of Lemma 5 to Φ and $\mathcal{I}$ yields the desired result. $\square$

Let $\mathcal{S}$ be a family of subsets of a given set X, let E be another set and let $f : X \to E$. We shall say that f is an $\mathcal{S}$-step function (with the values in the set E) if there exist a partition $\{A_n : n \in \omega\}$ of X into sets from $\mathcal{S}$ and a sequence $\{e_n : n \in \omega\}$ of elements from E, such that

$$f = \cup_{n\in\omega}(A_n \times \{e_n\}).$$

Of course, here the function f is identified with its graph.

Suppose that the class Σ_n has the perfect subset property. Let X be a Σ_n-space, $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base, E be an arbitrary metric space and let $f : X \to E$. Then the following two conditions are equivalent:

a) f is a $(\Delta_n(X)\triangle\mathcal{I})$-measurable function;

b) f is $\mathcal{I}$-almost equal to a pointwise limit of a sequence of $(\Delta_n(X)\triangle\mathcal{I})$-step functions.

This equivalence directly follows from Theorem 5. The next statement can be obtained analogously.

Theorem 8. *Suppose again that the class Σ_n has the perfect subset property. Let X be a Σ_n-space, $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base, E be any metrizable topological group and let*

$$f : X \to E, \ g : X \to E$$

be any two $(\Delta_n(X)\triangle\mathcal{I})$-measurable functions. Then the sum $f + g$ is a $(\Delta_n(X)\triangle\mathcal{I})$-measurable function, too.

Proof. It suffices to reduce Theorem 8 to the case where E is a separable metrizable topological group. But this can easily be done with the aid of Theorem 5 (we can also directly apply the equivalence of the conditions a) and b) above). $\square$

Theorem 9. *Suppose again that the class Σ_n has the perfect subset property. Let X be a Σ_n-space and let $\mathcal{I}$ be a σ-ideal of subsets of X with a Π_n-base. Suppose also that E is a metric space and let*

$$\Phi : X \to \mathrm{Comp}(E)$$

be a lower $(\Delta_n(X)\triangle\mathcal{I})$-measurable mapping. Then there exists a $(\Delta_n(X)\triangle\mathcal{I})$-measurable selector of Φ.

Proof. According to Lemma 5, there are a Π_n-set $A \in \mathcal{I}$ and a closed separable subspace F of E, such that

$$(\forall x \in X \setminus A)(\Phi(x) \subset F).$$

Let F' denote the completion of the metric space F. Then F' is a Polish space and $\Phi(x)$ is a nonempty compact subset of F' for each $x \in X \setminus A$. Hence, we may apply the classical theorem on measurable selectors (due to Kuratowski and Ryll–Nardzewski [9]) to the restriction $\Phi|(X \setminus A)$ and to the σ-algebra $(\Delta_n(X)\triangle\mathcal{I}) \cap P(X \setminus A)$. In view of this theorem, there exists a $((\Delta_n(X)\triangle\mathcal{I}) \cap P(X \setminus A))$-measurable selector of $\Phi|(X \setminus A)$. Then it is not hard to see that a suitable extension of this selector gives us a $(\Delta_n(X)\triangle\mathcal{I})$-measurable selector of the original mapping Φ. $\square$

Remark 4. A connection between the notion of summability and the existence of measurable selectors is thoroughly investigated in the well-known monograph [2]. In this monograph the paracompactness of any metric space is utilized instead of the Montgomery Lemma, and the further argument is essentially based on the assumption that a given σ-ideal satisfies the countable chain condition. In this respect, it is reasonable to recall here that there are many natural examples of σ-ideals with a Borel base which do not satisfy the countable chain condition.

Note also that some related questions concerning the existence of measurable (or nonmeasurable) selectors for a given partition of the real line are discussed in [10].

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(Received 17.09.2001)

Author's addresses:

J. Cichoń
Institute of Mathematics
University of Wroclaw
pl. Grunwaldzki, 2/4, 50-384 Wroclaw
Poland
E-mail: jci@promat.com.pl

A. Kharazishvili
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Georgia
E-mail: kharaz@saba.edu.ge