Georgian Mathematical Journal
Volume 9 (2002), Number 3, 449–459

ON A GENERALIZATION OF THE DIRICHLET INTEGRAL

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Abstract. Using the theory of spline functions, we investigate the problem of minimization of a generalized Dirichlet integral

$$F_\lambda(u) = \int\limits_{\Omega} \Big(\lambda^2 + \sum_{i=1}^{n} u_{x_i}^2\Big)^{p/2}, \quad 1 < p < \infty,$$

where Ω is a bounded domain of an n-dimensional Euclidean space R_n, $\lambda \geq 0$ is a fixed number, and u_{x_i} is a generalized according to Sobolev with respect to x_i derivative of the function u defined on Ω. Minimization is realized with respect to the functions u whose boundary values on Γ form a preassigned function, and for them $F_\lambda(u)$ is finite.

2000 Mathematics Subject Classification: 35J40.
Key words and phrases: Dirichlet integral, spline functions, Sobolev space, generalized derivative, l-Laplacian.

Let X and Z be linear normed spaces and Y be a Banach space possessing properties E and U. Property E means that in any closed subspace there exists an element g, least deviating from the given one $y \in Y$. Property U shows that such an element g is unique. It is known that the linear normed space possesses property E if and only if it is reflexive, and property U if and only if it is strictly convex (i.e., when the norm of a midpoint of the segment connecting two different points on the unit sphere, is less than 1). Suppose that $\mathbf{t}$ and $\mathbf{a}$ are continuous linear mappings from X into Y and from X into Z, respectively. For a fixed element $z \in \operatorname{Im}\mathbf{a}$ let us consider its pre-image in X, i.e., a set $I_z = \{x \in X : \mathbf{a}(x) = z\}$ and then in Y, i.e., a set $\Delta_z = \mathbf{t}(I_z)$. This is the set of those elements in Y which have the form $y = \mathbf{t}(x)$, where $x \in I_z$. Find $\|t(x)\|_Y$ when $x \in I_z$. If this inf is attained on some element σ from the set I_z, then it is called a spline corresponding to $\mathbf{t}$, $\mathbf{a}$ and z. By the definition,

$$\|\mathbf{t}(\sigma)\|_Y = \inf_{X \in I_z} \|\mathbf{t}(X)\|_Y. \tag{1}$$

Such a definition of a spline was introduced first for Hilbert spaces X, Y, Z ([1], Ch. 4, §4.4). The above-given definition of the spline is given in [2].

In the sequel, instead of $\operatorname{Ker}(\cdot)$ the operator kernel will be denoted by $N(\cdot)$.

Theorem 1. *The spline corresponding to* $\mathbf{t}$, $\mathbf{a}$ *and* $\mathbf{z}$ *exists if the set* $\mathbf{t}(N(\mathbf{a}))$ *is closed in* Y. *It is unique if* $N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$.

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

Proof. Clearly, the set $\Delta_z = \mathbf{t}(I_z)$ is a shift of the subspace $\Delta_0 = \mathbf{t}(I_0) = \mathbf{t}(N(\mathbf{a}))$ by the element $\mathbf{t}(x_0)$, where x_0 is an arbitrary fixed element from I_z. By the condition, any subspace of the space Y possesses properties E and U. Then the shifts of the subspace possess the same properties. (Indeed, let $x \in Y$, $x \notin \Delta_z$ and let x_0 be an arbitrary element from Δ_z. For $x - x_0$ in Δ_0 there exists a best approximation element of g_0, i.e., $\|x - x_0 - g_0\|_Y = \inf_{g\in\Delta_0} \|x - x_0 - g\|_Y = \inf_{g\in\Delta_0} \|x - (x_0 + g)\|_Y = \inf_{\widetilde{g}\in\Delta_z} \|x - \widetilde{g}\|_Y$, i.e., $\widetilde{g_0} = x_0 + g_0$ is the element of the best approximation in Δ_z. Analogously, we can see that $\widetilde{g}_0$ is unique.) Since, by the condition, the set $\Delta_0 = \mathbf{t}(I_0)$ is closed, it is a subspace, and hence its shift Δ_z likewise possesses properties E and U. Let $\overline{y}$ be a best approximation element of zero in Δ_z, i.e., $\|\overline{y}\|_Y = \inf_{y\in\Delta_z} \|y\|_Y$. Thus we have proved the existence of the spline.

Let us prove its uniqueness under the condition $N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$. Suppose that there exist two splines σ and $\widetilde{\sigma} \in I_z$, $\mathbf{a}(\sigma) = \mathbf{a}(\widetilde{\sigma}) = z$, $\mathbf{a}(\sigma - \widetilde{\sigma}) = 0$, i.e., $\sigma - \widetilde{\sigma} \in I_0 = N(\mathbf{a})$. Since $\overline{y}$ is a best approximation unique element, $\mathbf{t}(\sigma) = \mathbf{t}(\widetilde{\sigma}) = y$, $\mathbf{t}(\sigma - \widetilde{\sigma}) = 0$, i.e., $\sigma - \widetilde{\sigma} \in N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$ and $\sigma = \widetilde{\sigma}$. □

Remark. Let y_0 be an arbitrary fixed element, and instead of problem (1) let us consider a somewhat modified problem. We are to find that $\inf_{x\in I_z} \|y_0 + \mathbf{t}(x)\|_Y$. If the conditions of Theorem 1 are fulfilled, then there exists an element $\widetilde{\sigma} \in I_z$ for which

$$\|y_0 + \mathbf{t}(\widetilde{\sigma})\|_Y = \inf_{x\in I_z} \|y_0 + \mathbf{t}(x)\|_Y.$$

Now we need establish the sufficient condition for the set $\mathbf{t}(N(\mathbf{a}))$ from Y to be closed.

Lemma 1. *If X is a Banach space,* $\operatorname{Im}\mathbf{t}$ *is closed in Y, and $N(\mathbf{t})$ is finite-dimensional, then $\mathbf{t}(N(\mathbf{a}))$ is a closed subspace in Y.*

Proof. Since $N(\mathbf{t})$ is finite-dimensional, it has a complement, i.e., there exists a subspace $N(\mathbf{t})^\perp$ in X such that $N(\mathbf{t}) \cap N(\mathbf{t})^\perp = \{0\}$, $X = N(\mathbf{t}) \oplus N(\mathbf{t})^\perp$. Every element $x \in X$ is represented uniquely as $x = u + v$, $u \in N(\mathbf{t})$, $v \in N(\mathbf{t})^\perp$. Denote by $\widetilde{\mathbf{t}}$ the restriction of the mapping $\mathbf{t}$ on $N(\mathbf{t})$. It is clear that $\operatorname{Im}\mathbf{t} = \operatorname{Im}\widetilde{\mathbf{t}}$, and $\widetilde{\mathbf{t}}$ is a continuous one-to-one mapping acting from $N(\mathbf{t})^\perp$ to a closed subspace of the space Y. According to the Banach theorem on the continuity of an inverse mapping, $\mathbf{s} = \mathbf{t}^{-1}$ is a continuous mapping. Denote by $N_a(\mathbf{t})$ the projection of $N(\mathbf{a})$ onto $N(\mathbf{t})^\perp$. We have

$$\begin{aligned} \mathbf{t}(N(\mathbf{a})) &= \mathbf{t}(N_a(\mathbf{t})) = \widetilde{\mathbf{t}}(N_a(\mathbf{t})) \\ &= \mathbf{s}^{-1}(N_a(t)) = \mathbf{s}^{-1}((N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t})^\perp). \end{aligned} \tag{2}$$

Here we have taken into account the fact that $N_a(\mathbf{t}) = ((N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t}))^\perp$. Indeed, if $x \in N_a(\mathbf{t})$, then $x \in N(\mathbf{t})^\perp$, i.e., x is the projection of some element $x_1 \in N(\mathbf{a})$ onto $N(\mathbf{t})$ and $x = x_1 - x_2$, where $x_2 \in N(\mathbf{t})$. This implies that $x \in N(\mathbf{a}) + N(\mathbf{t})$ and hence $x \in (N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t})^\perp$. Conversely, if $x \in (N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t})^\perp$, then $x = x_1 + x_2$, where $x_1 \in N(\mathbf{a})$, $x_2 \in N(\mathbf{t})$. Next, $x_1 = x - x_2$, but x belongs to $N(\mathbf{t})^\perp$ and hence it is the projection of x_1 onto

$N(\mathbf{t})^{\perp}$, i.e., $x \in N_a(\mathbf{t})$. $N(\mathbf{a})$ is closed, and since $N(\mathbf{t})$ is finite-dimensional, $N(\mathbf{a}) + N(\mathbf{t})$ is likewise closed. Indeed, let the sequence $x_n \in N(\mathbf{a}) + N(\mathbf{t})$, $n = 1, 2, \ldots$, converge in X. This sequence can be written as follows:

$$x_n = \overline{x}_n + \sum_{k=1}^{r} \alpha_n^{(k)} t_k, \tag{3}$$

where $\overline{x}_n \in N(\mathbf{a})$, and $t_1, \ldots, t_r$ are linear independent elements from $N(\mathbf{t})$ such that $\mathbf{a}(t_1), \ldots, a(t_r)$ are also linear independent in Z. (Indeed, let $\{t_k\}_{1\le k\le m}$ be a basis in $N(\mathbf{t})$ such that $\mathbf{a}(t_1), \ldots, \mathbf{a}(t_r)$, $r \le m$, is a maximal linear independent subset of the set $\{\mathbf{a}(t_k)\}_{1\le k\le m}$. Then for t_j, $j = r+1, \ldots, m$, we have $\mathbf{a}(t_j) = \sum_{k=1}^{r} \beta_k^{(j)} \mathbf{a}(t_k)$; and $t_j - \sum_{k=1}^{r} \beta_k^{(j)} t_k = \xi_j \in N(\mathbf{a})$, or $t_j = \sum_{k=1}^{r} \beta_k^{(j)} t_k + \xi_j$, $\beta_k^{(j)} \in R$, $\xi_j \in N(\mathbf{a})$, $r+1 \le j \le m$. But, by the definition, $x_n = \widetilde{x}_n + \sum_{k=1}^{m} \gamma_n^{(k)} t_k$, $\widetilde{x}_n \in N(\mathbf{a})$, $\gamma_n^{(k)} \in R$. Taking into account all the above-said, we can see that (3) is valid). By virtue of the convergence of (3) and the continuity of the mapping $\mathbf{a}$, the sequence $\mathbf{a}(x_n) = \sum_{k=1}^{r} \alpha_n^{(k)} \mathbf{a}(t_k)$ converges as well.

Thus we conclude that for all $1 \le k \le r$ the number of sequences $\alpha_n^{(k)}$ converge. Further, according to (3), there exists $\lim \overline{x}_n = \overline{x} \in N(\mathbf{a})$, and if $\lim x_n = x$, then $x \in N(\mathbf{a}) + N(\mathbf{t})$. Hence $N(\mathbf{a}) + N(\mathbf{t})$ is a closed set. $N(\mathbf{t})$ is closed, and since $\mathbf{s}^{-1}$ is a continuous mapping, from (2) it follows that the set $\mathbf{t}(N(\mathbf{a}))$ is closed in Y. $\square$

Let us realize the above-described construction for the following case. Let Ω be a simply-connected bounded domain of an n-dimensional Euclidean space R_n for which Sobolev embedding theorems (see, e.g., [3], pp. 279–280) are valid. For instance, they are valid if Ω is representable in the form of a union of a finite number of star-shaped domains.

Let $X = W_p^1(\Omega)$ be a set of integrable on Ω in the p-th degree, $1 < p < \infty$, functions $u(x) = u(x_1, \ldots, x_n)$ having the integrable in the p-th degree generalized, according to Sobolev, partial derivatives of first order u_{x_i}, $1 \le i \le n$. The set of such functions with the norm

$$\|u\|_{W_p^1(\Omega)} = \left\{ \int_{\Omega} (u_{x_1}^2 + \cdots + u_{x_n}^2)^{p/2} d\sigma \right\}^{1/p} + \left\{ \int_{\Omega} |u|^p d\sigma \right\}^{1/p}, \ d\sigma = dx_1, \ldots, dx_n,$$

is the Banach space, or more precisely, the Sobolev space (its different equivalent norms are well-known (see, e.g., [3]). In the capacity of the space Y we take the Cartesian product of n samples of the space $L_p(\Omega)$ and of one-dimensional space R, i.e., $Y = R \times L_p(\Omega) \times \cdots \times L_p(\Omega) = R \times \prod_n L_p(\Omega)$. Introduce in Y the norm of the element $(\lambda, u_1, \ldots, u_n)$ as follows:

$$\begin{aligned} \|(\lambda, u_1, \ldots, u_n)\|_{Y} &= \|(\lambda^2 + u_1^2 + \cdots + u_n^2)^{1/2}\|_{L_p(\Omega)} \\ &= \left\{ \int_{\Omega} (\lambda^2 + \sum_{i=1}^{n} u_i)^{p/2} dv \right\}^{1/p}. \end{aligned} \tag{4}$$

It is known that $L_p(\Omega)$ is a reflexive and strictly normed space for $1 < p < \infty$. The same will be Y with norm (4). Let us prove that the space Y is strictly convex. To this end, we take in Y different points $u = (\lambda_1, u_1, \dots, u_n)$ and $v = (\lambda_2, v_1, \dots, v_n)$ with norms, equal to unity, and prove that $\|u+v\|_Y < 2$.

By means of the inequality

$$\sqrt{(\lambda_1+\lambda_2)^2 + (u_1+v_1)^2 + \cdots + (u_n+v_n)^2} \\ \leq \sqrt{\lambda_1^2 + u_1^2 + \cdots + u_n^2} + \sqrt{\lambda_2^2 + v_1^2 + \cdots + v_n^2} \tag{5}$$

we get

$$\|u+v\|_Y = \left\{ \int\limits_\Omega ((\lambda_1+\lambda_2)^2 + (u_1+v_1)^2 + \cdots + (u_n+v_n)^2)^{p/2} d\sigma \right\}^{1/p}$$
$$\leq \left\{ \int\limits_\Omega \left(\sqrt{\lambda_1^2 + u_1^2 + \cdots + u_n^2} + \sqrt{\lambda_2 + v_1^2 + \cdots + v_n^2} \right)^p d\sigma \right\}^{1/p}$$
$$\stackrel{\text{def}}{=} \left\{ \int\limits_\Omega (f_1+f_2)^p d\sigma \right\}^{1/p}, \tag{6}$$

where

$$f_1 = \sqrt{\lambda_1^2 + u_1^2 + \cdots + u_n^2}, \quad f_2 = \sqrt{\lambda_2^2 + v_1^2 + \cdots + v_n^2},$$
$$\|f_1\|_{L_p(\Omega)} = \|f_2\|_{L_p(\Omega)} = \|u\|_Y = \|v\|_Y = 1.$$

Consider the following cases:

(1) $f_1 \neq f_2$. By virtue of the strict convexity of the space $L_p(\Omega)$ for $1 < p < \infty$, from (6) we find that

$$\|u+v\|_Y < \|f_1\|_{L_p(\Omega)} + \|f_2\|_{L_p(\Omega)} = \|u\|_Y + \|v\|_Y = 2;$$

(2) $f_1 = f_2$, i.e., $\lambda_1^2 + u_1^2 + \cdots + u_n^2 = \lambda_2^2 + v_1^2 + \cdots + v_n^2$ a.e. on Ω. Suppose first that at least one of the numbers λ_1 and λ_2 is not equal to zero. The equality in inequality (5) is obtained, when the vectors u and v are proportional a.e., and since the first coordinates are the numbers, the coefficient of proportionality is a positive number C, i.e., $v = Cu$. But $\|u\| = \|v\| = 1$ and hence $C = 1$, $v = u$ which is impossible by our supposition.

If $\lambda_1 = \lambda_2 = 0$, then the equality a.e. in (5) is obtained if a.e. $v = C(x)u$, where $C(x) > 0$ for almost all $x \in \Omega$. In this case $u = 0$ a.e. in Ω if and only if $v = 0$ a.e. in Ω. But $|v| = |u|$ a.e., and hence $C(x) = 1$ a.e. in Ω. We obtain $u = v$ and this contradicts our supposition. Thus the strict convexity of the space Y is proved.

In the sequel, we shall need some of the Sobolev embedding theorems on a structure of traces for elements from Sobolev spaces on manifolds of lesser dimensions. As such a manifold we take the boundary Γ of the domain Ω which is required to consist of a finite number of surfaces of the class C^1. By the

Sobolev embedding theorems, for $p > n$ the function f from the space $W^1p(\Omega)$ is continuous everywhere in Ω, including the boundary Γ. Moreover,

$$\max_{x\in\Gamma} |f(x)| \le c\|f\|_{W^1_p(\Omega)}, \tag{7}$$

where c is the constant depending on Γ. If $1 < p \le n$, the function $f \in W^1_p(\Omega)$ has a trace on Γ, which belongs to $L_q(\Gamma)$, where $q < (n-1)p/(n-p)$ and

$$\|f\|_{L_q(\Gamma)} \le c\|f\|_{W^1_p(\Omega)}. \tag{8}$$

Since $(n-1)p/(n-p) > p$, for $1 < p \le n$ the trace of elements of the space $W^1_p(\Omega)$ on Γ belongs to $L_p(\Gamma)$ (the fact that $f|_\Gamma$ is the trace of the element $f \in W^1_p(\Omega)$ on Γ means that in the class of equivalent functions f we can find a function which possesses limit values in the ordinary, or in the L_p sense. The above-mentioned theorems are proved, for example, in [3] or in [4].

As the Banach space Z we introduce the space of $L_p(\Gamma)$-functions, integrable on Γ in the p-th degree. Denote by $\mathbf{a}$ a linear operator from X to Z which to the element $u \in X$ puts into correspondence its trace on the boundary Γ of the domain Ω. Inequalities (7) and (8) imply that $\mathbf{a}$ is a linear continuous operator. The operator $\mathbf{t}$, acting from X to Y, can be defined by the formula

$$\mathbf{t}(u) = (0, u_{x_1}, \dots, u_{x_n}).$$

Clearly,

$$\|\mathbf{t}(u)\|_Y = \left\{ \int_\Omega \left(\sum_{i=1}^n u^2_{x_i} \right)^{p/2} dv \right\}^{1/p}.$$

The right-hand side of the latter equality is the summand of the norm $\|u\|_X$, and hence $\mathbf{t}$ is a linear continuous operator.

In the space Z we fix an element $f \in \operatorname{Im}\mathbf{a}$ and denote by I_f a set of elements u of the space X whose trace on Γ is f, i.e., $\mathbf{a}(u) = f$, $\Lambda = (\lambda, 0, \dots, 0)$ belongs to Y, and the set $\Lambda + t(u) = (\lambda, u_{x_1}, \dots, u_{x_n})$ for all possible $u \in I_f$ is a shift of the set $\mathbf{t}(N(\mathbf{a}))$ onto the element $(\lambda, v_{x_1}, \dots, v_{x_n})$, where v is some fixed element from I_f.

Consider the functional

$$F_\lambda(u) = \left\{ \int_\Omega \left(\lambda^2 + \sum_{i=1}^n u^2_{x_i} \right)^{p/2} d\sigma \right\}^{1/p} = \|\Lambda + t(u)\|_Y \tag{9}$$

defined on I_f and find its inf with respect to all $u \in I_f$. By Theorem 1, the closure of the set $\mathbf{t}(N(\mathbf{a}))$ in Y and the equality $N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$ guarantee the existence and uniqueness of the minimizing element $\widetilde{u} \in I_f$. To prove the closure of the set $\mathbf{t}(N(\mathbf{a}))$, it is sufficient, following Lemma 1, to prove finite dimensionality of $N(\mathbf{t})$ and closure of $\operatorname{Im}\mathbf{t}$ in Y. $N(\mathbf{t})$ is the set of functions from X whose all generalized partial derivatives of first order are zeros. This means that $N(\mathbf{t})$ is a one-dimensional space of constants. It remains to prove the closure $\operatorname{Im}\mathbf{t}$. Let $u^{(m)}$ be a sequence in X such that all the sequences $u^{(m)}_{x_i}$, $1 \le i \le n$, composed of generalized partial derivatives of $u^{(m)}$ converge in $L_p(\Omega)$ to the functions α_i. We have to prove that the vector $(\alpha_1,\dots,\alpha_n)$ is the generalized gradient of a function $u\in X$, i.e., $u_{x_i}=\alpha_i$, $1\le i\le n$. For $1\le i,j\le n$ denote by $(u^{(m)})_{x_ix_j}$ the derivative with respect to x_j of the function $(u^{(m)})_{x_i}$ in a sense of the theory of generalized functions. Then for any finite in Ω function f we have

$$(u^{(m)})_{x_ix_j}(f)=\frac{\partial u^{(m)}}{\partial x_i}\Big(\frac{\partial f}{\partial x_j}\Big)\to\int\limits_\Omega \alpha_i\frac{\partial f}{\partial x_j}d\sigma,$$
$$(u^{(m)})_{x_jx_i}(f)=\frac{\partial u^{(m)}}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big)\to\int\limits_\Omega \alpha_j\frac{\partial f}{\partial x_i}d\sigma.$$

The left-hand sides of these relations are equal to $u^{(m)}(f_{x_ix_j})$ (elements $u^{(m)}\in X$ are considered as regular generalized functions). Hence their right-hand sides are likewise equal:

$$\int\limits_\Omega \alpha_i\,\partial f/\partial x_j d\sigma=\int\limits_\Omega \alpha_j\partial f/\partial x_i d\sigma,$$

i.e., $(\alpha_j)_{x_i}=(\alpha_i)_{x_j}$, where the derivatives are considered in a sense of the theory of generalized functions. It is known ([3], p. 160; [5], Ch. 5, 5.9.3, 5.11.1) that in the case, where Ω is simply connected, there exists a locally integrable in Ω function u whose generalized partial derivatives u_{x_i}, $1\le i\le n$, belong to $L_p(\Omega)$, and $\operatorname{grad}u=(\alpha_1,\dots,\alpha_n)$. But by the Sobolev embedding theorem ([3], pp. 280–282), such a function u belongs to $L_p(\Omega)$, and hence $u\in X$, i.e., $\operatorname{Im}\mathbf{t}$ is closed in Y. The condition $N(\mathbf{a})\cap N(\mathbf{t})=\{0\}$ can be easily verified. Thus we have proved the following

Theorem 2. *Let Ω be a bounded simply connected domain from R_n, whose boundary Γ consists of a finite number of surfaces of the class C^1, f be the defined on Γ function for which the set $I_f=\{u\in W_p^1(\Omega): u|_\Gamma=f\}$ is non-empty, and let λ be an arbitrary fixed real number. Then for the functional F_λ, defined by formula* (9), $\inf_{u\in I_f}F_\lambda(u)$ *is obtained on some element $u\in I_f$.*

Remark 1. Theorem 2 is also valid for the domain which is divided by means of surfaces of the class C^1 into a finite number of simply connected domains. Indeed, the functions $u\in I_f$ will have, according to the above-mentioned Sobolev theorems, on such surfaces a trace belonging to the corresponding space L_p, and the problem of minimization of the functional F_λ can be considered in the obtained simply connected domains. Using minimizing elements, we can construct a minimizing function in the initial domain.

Remark 2. Using the method described above, we can investigate a functional of more general type such as

$$F_{\lambda,s}(u)=\Big\{\int\limits_\Omega(\lambda^2+Du)^{p/2}d\sigma\Big\}^{1/p}, \tag{10}$$

where

$$Du = \sum_{s_1+\dots+s_n=s} c(s_1,\dots,s_n)\Big(\frac{\partial^s u}{\partial x_1^{s_1}\dots\partial x_n^{s_n}}\Big)^2, \quad c(s_1,\dots,s_n)>0,$$

and λ is a fixed function belonging to $L_p(\Omega)$, while the derivatives are understood in the Sobolev sense. Not going into details, we note that for functional (10) we obtain a theorem analogous to Theorem 2. Here we restrict ourselves to case (9).

For the element u minimizing functional (9) we derive the so-called Euler's generalized equation. To this end, we consider some function η belonging to $D(\Omega)$, a space of finite, infinitely differentiable in Ω function, and for the real numerical parameter α we compose the set $\{u+\alpha\eta\}$. Let $p=l+1$, $l>0$. The function u minimizes the functional

$$I_\lambda(u) = \int\limits_\Omega \left(\lambda^2 + \sum_{i=1}^n u_{x_i}^2\right)^{(l+1)/2} d\sigma,$$

and since the trace of the function $u+\alpha\eta$ belonging to $W_{l+1}(\Omega)$ on Γ is f, for any $\alpha\in R$ we have

$$I_\lambda(u+\alpha\eta) \ge I_\lambda(u).$$

Let us prove that

$$I_\lambda(u+\alpha\eta) = \int\limits_\Omega \left(\lambda^2 + \sum_{i=1}^n (u_{x_i}+\alpha\eta_{x_i})^2\right)^{(l+1)/2} d\sigma \tag{11}$$

has a derivative with respect to α for any $\alpha\in R$, and I'_λ can be calculated by differentiation under the integral sign. For this we write the expression

$$\frac{\Delta I_\lambda}{\Delta\alpha} = \frac{I_\lambda(u+(\alpha+\Delta\alpha))\eta - I_\lambda(u+\alpha\eta)}{\Delta\alpha}$$

($\Delta\alpha$ is an increment of α) and transform it by means of the Lagrange mean value theorem. We obtain

$$\begin{aligned}\frac{\Delta I_\lambda}{\Delta\alpha} = (l+1)\int\limits_\Omega &\left(\lambda^2 + \sum_{i=1}^n (u_{x_i}+\alpha_\theta\eta_{x_i})^2\right)^{(l-1)/2}\\ &\times\left[\sum_{i=1}^n (u_{x_i}+\alpha_\theta\eta_{x_i})\eta_{x_i}\right] d\sigma,\end{aligned} \tag{12}$$

where $\alpha_\theta = \alpha + \theta(u_{x_1},\dots,u_{x_n},\alpha)$, $0<|\theta(u_{x_1},\dots,u_{x_n},\alpha)|<|\Delta\alpha|$.

By the Cauchy–Bunjakovsky inequality we find that the integrand is estimated from the above by the expressions

$$\left[\lambda^2 + \sum_{i=1}^n (u_{x_i}+\alpha_\theta\eta_{x_i})^2\right]^{l/2} \left(\sum_{i=1}^n \eta_{x_i}^2\right)^{1/2}$$

and

$$\max\left\{1, \left[\lambda^2 + \sum_{i=1}^{n}(u_{x_i} + \alpha_\theta \eta_{x_i})^2\right]^{l+1/2}\right\}\left(\sum_{i=1}^{n}\eta_{x_i}^2\right)^{1/2}.$$

But

$$\lambda^2 + \sum_{i=1}^{n}(u_{x_i} + \alpha_\theta \eta_{x_i})^2 \leq \left(\lambda^2 + \sum_{i=1}^{n} u_{x_i}^2\right)(1 + |\alpha_\theta|) + (|\alpha_\theta| + \alpha_\theta^2)\sum_{i=1}^{n}\eta_{x_i}^2,$$

and $u \in W_{l+1}$. Therefore we can easily see that the integrand in (12) has a majorant belonging to $L_1(\Omega)$. By the Lebesgue theorem, we can pass in (12) to the limit under the integral sign. Thus from (11) we find that

$$I'_\lambda(u + \alpha\eta) = (l+1)\int\limits_\Omega \left(\lambda^2 + \sum_{i=1}^{n}(u_{x_i} + \alpha\eta_{x_i})^2\right)^{(l-1)/2}\left[\sum_{i=1}^{n}(u_{x_i} + \alpha\eta_{x_i})\eta_{x_i}\right]d\sigma.$$

Since $\alpha = 0$ is the minimum point of the function $I_\lambda(u + \alpha\eta)$, we have that $I'_\lambda(u) = 0$, and hence the function u, minimizing $I_\lambda(u)$, satisfies the generalized Euler equation

$$\int\limits_\Omega \left(\lambda^2 + \sum_{i=1}^{n} u_{x_i}^2\right)^{(l-1)/2}\left(\sum_{i=1}^{n} u_{x_i}\eta_{x_i}\right)d\sigma = 0 \tag{13}$$

for any function $\eta \in D(\Omega)$. For $\lambda = 0$ we get the equation

$$\int\limits_\Omega |\operatorname{grad} u|^{l-1}\left(\sum_{i=1}^{n} u_{x_i}\eta_{x_i}\right)d\sigma = 0. \tag{14}$$

It is seen from (14) that u is a solution of the equation

$$\Delta_l(u) \overset{\text{def}}{=} \operatorname{div}\left(|\operatorname{grad} u|^{l-1}\operatorname{grad} u\right) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|\operatorname{grad} u|^{l-1}u_{x_i}\right) = 0, \tag{15}$$

where partial derivatives $\partial/\partial x_i$ are understood in a sense of the theory of generalized functions. The solution of equation (13) satisfies in the same sense equation (14), where $\operatorname{grad} u$ is replaced by an expression of the type $\left(\lambda^2 + \sum_{i=1}^{n} u_{x_i}^2\right)^{1/2}$. Equation (14) is known in the literature as the l-Laplace equation, and the corresponding operator Δ_l is called the l-Laplacian ([8]). For $\lambda = 0$, $l = 1$ the functional F_λ, defined by formula (9), coincides with the known Dirichlet integral, and Δ_1 is the known Laplace operator.

Let now the function u belonging to the space $W^1_{l+1}(\Omega)$, $l > 0$, satisfy in Ω the equation

$$\{\Delta_l\}_\lambda u = \operatorname{div}\left(\left(\lambda^2 + |\operatorname{grad} u|^2\right)^{(l-1)/2}\operatorname{grad} u\right) = 0 \tag{16}$$

in a sense of the theory of generalized functions. Suppose that the trace of the function u on the boundary Γ of the domain Ω is the function $f \in L_{l+1}(\Gamma)$. Let us prove that in such a case a solution $u \in W^1_{l+1}(\Omega)$ of equation (16) minimizes functional (9) in the class I_f. As above, I_f denotes the set of functions from $W^1_{l+1}(\Omega)$ whose trace on Γ is f. It is clear that the solution u of equation (16) satisfies equation (13) for any function $\eta \in D(\Omega)$. Note first that (13) is also satisfied for any function $\eta \in W^1_{l+1}(\Omega)$ whose trace on Γ is zero. We denote the set of such functions by $\overset{\circ}{W}{}^1_{l+1}(\Omega)$. Indeed, let η be an arbitrary function from $\overset{\circ}{W}{}^1_{l+1}(\Omega)$. Then there exists in Ω a sequence $\eta_n \in D(\Omega)$ such that $\|\eta - \eta_n\|^1_{W_{l+1}}(\Omega) \to 0$ as $n \to \infty$. For $l = 1$ this proposition is known (see, e.g., [6], p. 119). The proof is analogous for any $l > 0$. The integral in the left-hand side of (10) exists if $u,\ \eta \in W^1_{l+1}(\Omega)$. Indeed, from the Hölder inequality we obtain

$$
\begin{aligned}
&\int\limits_\Omega \left(\lambda^2 + |\operatorname{grad} u|^2\right)^{(l-1)/2} \left(\sum_{i=1}^n u_{x_i}\eta_{x_i}\right) d\sigma \\
&\le \int\limits_\Omega \left(\lambda^2 + |\operatorname{grad} u|^2\right)^{(l-1)/2} |\operatorname{grad} u\| \operatorname{grad} \eta| d\sigma \\
&\le \int\limits_\Omega \left(\lambda^2 + |\operatorname{grad} u|^2\right)^{l/2} |\operatorname{grad} \eta| d\sigma \\
&\le \left\{\int\limits_\Omega \left(\lambda^2 + |\operatorname{grad} u|^2\right)^{(l+1)/2} \sigma\right\}^{(l+1)/l} \left\{\int\limits_\Omega |\operatorname{grad} u|^{(l+1)} d\sigma\right\}^{1/(l+1)} < \infty. \qquad (17)
\end{aligned}
$$

Since (10) is valid for $\eta_n \in D(\Omega)$, estimate (17) shows that (10) is now valid for any function $\eta \in \overset{\circ}{W}{}^1_{l+1}(\Omega)$.

Along with the solution u of equation (13), we consider the set $\{u + \alpha\eta\}$, $\alpha \in R$, $\eta \in \overset{\circ}{W}{}^1_{l+1}(\Omega)$. Since $(u + \alpha\eta)|_\Gamma = f$, $u + \alpha\eta \in I_f$. Then (9) implies that

$$
F_\lambda(u + \alpha\eta) = \int\limits_\Omega \left(\lambda^2 + |\operatorname{grad}(u + \alpha\eta)|^2\right)^{(l+1)/2} d\sigma. \qquad (18)
$$

Denote now $\left(\lambda^2 + |\operatorname{grad}(u)|^2\right)^{(l+1)/2}$ by $\phi_l(\alpha)$ and prove that for any $l > 0$ and $\alpha \in R$ the estimate

$$
\phi_l(\alpha) \ge \phi_l(0) + \alpha\phi_l(0) \qquad (19)
$$

is valid. To this end, we prove the following

Lemma 2. *If $\alpha \in R$ and $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ are the points from R_n, then the inequality*

$$
\begin{aligned}
&\left(\lambda^2 + \sum_{i=1}^n (a_i + b_i)^2\right)^{(l+1)/2} \\
&\ge \left(\lambda^2 + \sum_{i=1}^n a_i^2\right)^{(l+1)/2} + (l+1)\left(\lambda^2 + \sum_{i=1}^n a_i^2\right)^{(l-1)/2} \left(\sum_{i=1}^n a_i b_i\right) \qquad (20)
\end{aligned}
$$

is valid for any $l > 0$.

Proof. Let t and s be vectors from R_k, and the length s be equal to unity. Then $|s+t| \geq |1+(t,s)|$, where (t,s) is the scalar product of the vectors t and s. Indeed, if θ is the angle between the vectors $t+s$ and t, then $|s+t| \geq |s+t||s||\cos\theta| = |(s+t,s)| = |(s,s)+(t,s)| = |1+(t,s)|$. The $(1+l)$-th degree of $|1+(t,s)|$ can be estimated from below: $|1+(t,s)|^{(l+1)} \geq 1+(l+1)(t,s)$; this inequality coincides, in fact, with that of [9] (p. 12). Therefore

$$|s+t|^{(l+1)} \geq 1+(l+1)(t,s). \tag{21}$$

Let x be an arbitrary non-zero vector from R_k. Substituting $s = x/|x|$ and $y = |x|t$ into (21), we obtain

$$|x+y|^{(l+1)} \geq |x|^{(l+1)} + (l+1)|x|^{(l-1)}(x,y). \tag{22}$$

Clearly, the obtained inequality is likewise valid for $x = 0$. To estimate (20), we have to substitute $x = (\lambda, a_1, \dots, a_n)$ and $y = (0, b_1, \dots, b_n)$ into (22). $\square$

Applying now Lemma 2 to $a_i = u_{x_i}$ and $b_i = \alpha.\eta_{x_i}$, we see that estimate (19) is valid. Further, according to (18) we have

$$F_\lambda(u+\eta) \geq F_\lambda(u) + (l+1)\int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad} u|^2\Big)^{(l-1)/2} \bigg(\sum_{i=1}^n u_{x_i}\eta_{x_i}\bigg) d\sigma,$$

and by virtue of (13) we make sure that $F_\lambda(u+\eta) \geq F_\lambda u$ for any function $\eta \in \overset{\circ}{W}{}^1_{l+1}(\Omega)$. As we know, inf in (9) is obtained on some element u_0 from I_f. Take $\eta = u_0 - u \in \overset{\circ}{W}{}^1_{l+1}(\Omega)$. Then we have $F_\lambda(u_0) \geq F_\lambda(u)$. Since u_0 is the unique minimizing element, $u = u_0$, i.e., u minimizes the functional F_λ. Thus we proved the following

Theorem 3. *Let the function $f \in L_{l+1}(\Gamma)$, $l > 0$, be given on the boundary Γ of the domain Ω (satisfying the above conditions) which is the trace of a function from the space $W^1_{l+1}(\Omega)$. Then the element u of the class I_f which minimizes functional (9) (u exists by Theorem 2) is a solution of equation (13) in a sense of the theory of generalized functions. Conversely, if there exists a solution u of equation (13) belonging to $W^1_{l+1}(\Omega)$ and having a trace $u|_\Gamma = f$, then it minimizes functional (9) in the class I_f and is unique.*

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(Received 30.01.2002)

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