Georgian Mathematical Journal
Volume 9 (2002), Number 3, 449–459

ON A GENERALIZATION OF THE DIRICHLET INTEGRAL

T. CHANTLADZE, N. KANDELAKI, A. LOMTATIDZE, AND D. UGULAVA

Abstract. Using the theory of spline functions, we investigate the problem of minimization of a generalized Dirichlet integral

$$F_\lambda(u) = \int\limits_\Omega \Big(\lambda^2 + \sum_{i=1}^n u_{x_i}^2\Big)^{p/2}, \quad 1 < p < \infty,$$

where Ω is a bounded domain of an n-dimensional Euclidean space R_n, $\lambda \geq 0$ is a fixed number, and u_{x_i} is a generalized according to Sobolev with respect to x_i derivative of the function u defined on Ω. Minimization is realized with respect to the functions u whose boundary values on Γ form a preassigned function, and for them $F_\lambda(u)$ is finite.

2000 Mathematics Subject Classification: 35J40.
Key words and phrases: Dirichlet integral, spline functions, Sobolev space, generalized derivative, l-Laplacian.

Let X and Z be linear normed spaces and Y be a Banach space possessing properties E and U. Property E means that in any closed subspace there exists an element g, least deviating from the given one $y \in Y$. Property U shows that such an element g is unique. It is known that the linear normed space possesses property E if and only if it is reflexive, and property U if and only if it is strictly convex (i.e., when the norm of a midpoint of the segment connecting two different points on the unit sphere, is less than 1). Suppose that $\mathbf{t}$ and $\mathbf{a}$ are continuous linear mappings from X into Y and from X into Z, respectively. For a fixed element $z \in \operatorname{Im}\mathbf{a}$ let us consider its pre-image in X, i.e., a set $I_z = \{x \in X : \mathbf{a}(x) = z\}$ and then in Y, i.e., a set $\Delta_z = \mathbf{t}(I_z)$. This is the set of those elements in Y which have the form $y = \mathbf{t}(x)$, where $x \in I_z$. Find $\|t(x)\|_Y$ when $x \in I_z$. If this inf is attained on some element σ from the set I_z, then it is called a spline corresponding to $\mathbf{t}$, $\mathbf{a}$ and z. By the definition,

$$\|\mathbf{t}(\sigma)\|_Y = \inf_{X \in I_z} \|\mathbf{t}(X)\|_Y. \tag{1}$$

Such a definition of a spline was introduced first for Hilbert spaces X, Y, Z ([1], Ch. 4, §4.4). The above-given definition of the spline is given in [2].

In the sequel, instead of $\operatorname{Ker}(\cdot)$ the operator kernel will be denoted by $N(\cdot)$.

Theorem 1. *The spline corresponding to* $\mathbf{t}$, $\mathbf{a}$ *and* $\mathbf{z}$ *exists if the set* $\mathbf{t}(N(\mathbf{a}))$ *is closed in* Y. *It is unique if* $N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$.

ISSN 1072-947X / $8.00 /

Proof. Clearly, the set $\Delta_z = \mathbf{t}(I_z)$ is a shift of the subspace $\Delta_0 = \mathbf{t}(I_0) = \mathbf{t}(N(\mathbf{a}))$ by the element $\mathbf{t}(x_0)$, where x_0 is an arbitrary fixed element from I_z. By the condition, any subspace of the space Y possesses properties E and U. Then the shifts of the subspace possess the same properties. (Indeed, let $x \in Y$, $x \notin \Delta_z$ and let x_0 be an arbitrary element from Δ_z. For $x-x_0$ in Δ_0 there exists a best approximation element of g_0, i.e., $\|x-x_0-g_0\|_Y = \inf_{g\in\Delta_0} \|x-x_0-g\|_Y = \inf_{g\in\Delta_0} \|x-(x_0+g)\|_Y = \inf_{\widetilde{g}\in\Delta_z} \|x-\widetilde{g}\|_Y$, i.e., $\widetilde{g_0} = x_0+g_0$ is the element of the best approximation in Δ_z. Analogously, we can see that $\widetilde{g}_0$ is unique.) Since, by the condition, the set $\Delta_0 = \mathbf{t}(I_0)$ is closed, it is a subspace, and hence its shift Δ_z likewise possesses properties E and U. Let $\overline{y}$ be a best approximation element of zero in Δ_z, i.e., $\|\overline{y}\|_Y = \inf_{y\in\Delta_z} \|y\|_Y$. Thus we have proved the existence of the spline.

Let us prove its uniqueness under the condition $N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$. Suppose that there exist two splines σ and $\widetilde{\sigma} \in I_z$, $\mathbf{a}(\sigma) = \mathbf{a}(\widetilde{\sigma}) = z$, $\mathbf{a}(\sigma - \widetilde{\sigma}) = 0$, i.e., $\sigma - \widetilde{\sigma} \in I_0 = N(\mathbf{a})$. Since $\overline{y}$ is a best approximation unique element, $\mathbf{t}(\sigma) = \mathbf{t}(\widetilde{\sigma}) = y$, $\mathbf{t}(\sigma - \widetilde{\sigma}) = 0$, i.e., $\sigma - \widetilde{\sigma} \in N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$ and $\sigma = \widetilde{\sigma}$. □

Remark. Let y_0 be an arbitrary fixed element, and instead of problem (1) let us consider a somewhat modified problem. We are to find that $\inf_{x\in I_z} \|y_0 + \mathbf{t}(x)\|_Y$. If the conditions of Theorem 1 are fulfilled, then there exists an element $\widetilde{\sigma} \in I_z$ for which

$$\|y_0 + \mathbf{t}(\widetilde{\sigma})\|_Y = \inf_{x\in I_z} \|y_0 + \mathbf{t}(x)\|_Y .$$

Now we need establish the sufficient condition for the set $\mathbf{t}(N(\mathbf{a}))$ from Y to be closed.

Lemma 1. *If X is a Banach space,* $\operatorname{Im} \mathbf{t}$ *is closed in Y, and $N(\mathbf{t})$ is finite-dimensional, then $\mathbf{t}(N(\mathbf{a}))$ is a closed subspace in Y.*

Proof. Since $N(\mathbf{t})$ is finite-dimensional, it has a complement, i.e., there exists a subspace $N(\mathbf{t})^\perp$ in X such that $N(\mathbf{t}) \cap N(\mathbf{t})^\perp = \{0\}$, $X = N(\mathbf{t}) \oplus N(\mathbf{t})^\perp$. Every element $x \in X$ is represented uniquely as $x = u + v$, $u \in N(\mathbf{t})$, $v \in N(\mathbf{t})^\perp$. Denote by $\widetilde{\mathbf{t}}$ the restriction of the mapping $\mathbf{t}$ on $N(\mathbf{t})$. It is clear that $\operatorname{Im} \mathbf{t} = \operatorname{Im} \widetilde{\mathbf{t}}$, and $\widetilde{\mathbf{t}}$ is a continuous one-to-one mapping acting from $N(\mathbf{t})^\perp$ to a closed subspace of the space Y. According to the Banach theorem on the continuity of an inverse mapping, $\mathbf{s} = \mathbf{t}^{-1}$ is a continuous mapping. Denote by $N_a(\mathbf{t})$ the projection of $N(\mathbf{a})$ onto $N(\mathbf{t})^\perp$. We have

$$\begin{aligned} \mathbf{t}(N(\mathbf{a})) &= \mathbf{t}(N_a(\mathbf{t})) = \widetilde{\mathbf{t}}(N_a(\mathbf{t})) \\ &= \mathbf{s}^{-1}(N_a(t)) = \mathbf{s}^{-1}((N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t})^\perp). \end{aligned} \tag{2}$$

Here we have taken into account the fact that $N_a(\mathbf{t}) = ((N(\mathbf{a})+N(\mathbf{t}))\cap N(\mathbf{t}))^\perp$. Indeed, if $x \in N_a(\mathbf{t})$, then $x \in N(\mathbf{t})^\perp$, i.e., x is the projection of some element $x_1 \in N(\mathbf{a})$ onto $N(\mathbf{t})$ and $x = x_1 - x_2$, where $x_2 \in N(\mathbf{t})$. This implies that $x \in N(\mathbf{a}) + N(\mathbf{t})$ and hence $x \in (N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t})^\perp$. Conversely, if $x \in (N(\mathbf{a}) + N(\mathbf{t})) \cap N(\mathbf{t})^\perp$, then $x = x_1 + x_2$, where $x_1 \in N(\mathbf{a})$, $x_2 \in N(\mathbf{t})$. Next, $x_1 = x - x_2$, but x belongs to $N(\mathbf{t})^\perp$ and hence it is the projection of x_1 onto

$N(\mathbf{t})^{\perp}$, i.e., $x \in N_a(\mathbf{t})$. $N(\mathbf{a})$ is closed, and since $N(\mathbf{t})$ is finite-dimensional, $N(\mathbf{a}) + N(\mathbf{t})$ is likewise closed. Indeed, let the sequence $x_n \in N(\mathbf{a}) + N(\mathbf{t})$, $n = 1, 2, \dots,$ converge in X. This sequence can be written as follows:

$$x_n = \overline{x}_n + \sum_{k=1}^{r} \alpha_n^{(k)} t_k, \tag{3}$$

where $\overline{x}_n \in N(\mathbf{a})$, and $t_1, \dots, t_r$ are linear independent elements from $N(\mathbf{t})$ such that $\mathbf{a}(t_1), \dots, a(t_r)$ are also linear independent in Z. (Indeed, let $\{t_k\}_{1 \le k \le m}$ be a basis in $N(\mathbf{t})$ such that $\mathbf{a}(t_1), \dots, \mathbf{a}(t_r)$, $r \le m$, is a maximal linear independent subset of the set $\{\mathbf{a}(t_k)\}_{1 \le k \le m}$. Then for t_j, $j = r+1, \dots, m$, we have $\mathbf{a}(t_j) = \sum_{k=1}^{r} \beta_k^{(j)} \mathbf{a}(t_k)$; and $t_j - \sum_{k=1}^{r} \beta_k^{(j)} t_k = \xi_j \in N(\mathbf{a})$, or $t_j = \sum_{k=1}^{r} \beta_k^{(j)} t_k + \xi_j$, $\beta_k^{(j)} \in R$, $\xi_j \in N(\mathbf{a})$, $r+1 \le j \le m$. But, by the definition, $x_n = \widetilde{x}_n + \sum_{k=1}^{m} \gamma_n^{(k)} t_k$, $\widetilde{x}_n \in N(\mathbf{a})$, $\gamma_n^{(k)} \in R$. Taking into account all the above-said, we can see that (3) is valid). By virtue of the convergence of (3) and the continuity of the mapping $\mathbf{a}$, the sequence $\mathbf{a}(x_n) = \sum_{k=1}^{r} \alpha_n^{(k)} \mathbf{a}(t_k)$ converges as well.

Thus we conclude that for all $1 \le k \le r$ the number of sequences $\alpha_n^{(k)}$ converge. Further, according to (3), there exists $\lim \overline{x}_n = \overline{x} \in N(\mathbf{a})$, and if $\lim x_n = x$, then $x \in N(\mathbf{a}) + N(\mathbf{t})$. Hence $N(\mathbf{a}) + N(\mathbf{t})$ is a closed set. $N(\mathbf{t})$ is closed, and since $\mathbf{s}^{-1}$ is a continuous mapping, from (2) it follows that the set $\mathbf{t}(N(\mathbf{a}))$ is closed in Y. $\square$

Let us realize the above-described construction for the following case. Let Ω be a simply-connected bounded domain of an n-dimensional Euclidean space R_n for which Sobolev embedding theorems (see, e.g., [3], pp. 279–280) are valid. For instance, they are valid if Ω is representable in the form of a union of a finite number of star-shaped domains.

Let $X = W_p^1(\Omega)$ be a set of integrable on Ω in the p-th degree, $1 < p < \infty$, functions $u(x) = u(x_1, \dots, x_n)$ having the integrable in the p-th degree generalized, according to Sobolev, partial derivatives of first order u_{x_i}, $1 \le i \le n$. The set of such functions with the norm

$$\|u\|_{W_p^1(\Omega)} = \left\{ \int_{\Omega} (u_{x_1}^2 + \dots + u_{x_n}^2)^{p/2} d\sigma \right\}^{1/p} + \left\{ \int_{\Omega} |u|^p d\sigma \right\}^{1/p}, \ d\sigma = dx_1, \dots, dx_n,$$

is the Banach space, or more precisely, the Sobolev space (its different equivalent norms are well-known (see, e.g., [3]). In the capacity of the space Y we take the Cartesian product of n samples of the space $L_p(\Omega)$ and of one-dimensional space R, i.e., $Y = R \times L_p(\Omega) \times \dots \times L_p(\Omega) = R \times \prod_n L_p(\Omega)$. Introduce in Y the norm of the element $(\lambda, u_1, \dots, u_n)$ as follows:

$$\begin{aligned} \|(\lambda, u_1, \dots, u_n)\|_Y &= \|(\lambda^2 + u_1^2 + \dots + u_n^2)^{1/2}\|_{L_p(\Omega)} \\ &= \left\{ \int_{\Omega} (\lambda^2 + \sum_{i=1}^{n} u_i)^{p/2} dv \right\}^{1/p}. \end{aligned} \tag{4}$$

It is known that $L_p(\Omega)$ is a reflexive and strictly normed space for $1 < p < \infty$. The same will be Y with norm (4). Let us prove that the space Y is strictly convex. To this end, we take in Y different points $u = (\lambda_1, u_1, \dots, u_n)$ and $v = (\lambda_2, v_1, \dots, v_n)$ with norms, equal to unity, and prove that $\|u + v\|_Y < 2$.

By means of the inequality

$$\begin{aligned} &\sqrt{(\lambda_1+\lambda_2)^2 + (u_1+v_1)^2 + \cdots + (u_n+v_n)^2} \\ &\leq \sqrt{\lambda_1^2 + u_1^2 + \cdots + u_n^2} + \sqrt{\lambda_2^2 + v_1^2 + \cdots + v_n^2} \end{aligned} \tag{5}$$

we get

$$\begin{aligned} \|u+v\|_Y &= \left\{ \int_\Omega ((\lambda_1+\lambda_2)^2 + (u_1+v_1)^2 + \cdots + (u_n+v_n)^2)^{p/2} d\sigma \right\}^{1/p} \\ &\leq \left\{ \int_\Omega \left(\sqrt{\lambda_1^2 + u_1^2 + \cdots + u_n^2} + \sqrt{\lambda_2 + v_1^2 + \cdots + v_n^2} \right)^p d\sigma \right\}^{1/p} \\ &\overset{\text{def}}{=} \left\{ \int_\Omega (f_1+f_2)^p d\sigma \right\}^{1/p}, \end{aligned} \tag{6}$$

where

$$\begin{gathered} f_1 = \sqrt{\lambda_1^2 + u_1^2 + \cdots + u_n^2}, \quad f_2 = \sqrt{\lambda_2^2 + v_1^2 + \cdots + v_n^2}, \\ \|f_1\|_{L_p(\Omega)} = \|f_2\|_{L_p(\Omega)} = \|u\|_Y = \|v\|_Y = 1. \end{gathered}$$

Consider the following cases:

(1) $f_1 \neq f_2$. By virtue of the strict convexity of the space $L_p(\Omega)$ for $1 < p < \infty$, from (6) we find that

$$\|u+v\|_Y < \|f_1\|_{L_p(\Omega)} + \|f_2\|_{L_p(\Omega)} = \|u\|_Y + \|v\|_Y = 2;$$

(2) $f_1 = f_2$, i.e., $\lambda_1^2 + u_1^2 + \cdots + u_n^2 = \lambda_2^2 + v_1^2 + \cdots + v_n^2$ a.e. on Ω. Suppose first that at least one of the numbers λ_1 and λ_2 is not equal to zero. The equality in inequality (5) is obtained, when the vectors u and v are proportional a.e., and since the first coordinates are the numbers, the coefficient of proportionality is a positive number C, i.e., $v = Cu$. But $\|u\| = \|v\| = 1$ and hence $C = 1$, $v = u$ which is impossible by our supposition.

If $\lambda_1 = \lambda_2 = 0$, then the equality a.e. in (5) is obtained if a.e. $v = C(x)u$, where $C(x) > 0$ for almost all $x \in \Omega$. In this case $u = 0$ a.e. in Ω if and only if $v = 0$ a.e. in Ω. But $|v| = |u|$ a.e., and hence $C(x) = 1$ a.e. in Ω. We obtain $u = v$ and this contradicts our supposition. Thus the strict convexity of the space Y is proved.

In the sequel, we shall need some of the Sobolev embedding theorems on a structure of traces for elements from Sobolev spaces on manifolds of lesser dimensions. As such a manifold we take the boundary Γ of the domain Ω which is required to consist of a finite number of surfaces of the class C^1. By the

Sobolev embedding theorems, for $p > n$ the function f from the space $W^1p(\Omega)$ is continuous everywhere in Ω, including the boundary Γ. Moreover,

$$\max_{x\in\Gamma} |f(x)| \le c\|f\|_{W_p^1(\Omega)}, \tag{7}$$

where c is the constant depending on Γ. If $1 < p \le n$, the function $f \in W_p^1(\Omega)$ has a trace on Γ, which belongs to $L_q(\Gamma)$, where $q < (n-1)p/(n-p)$ and

$$\|f\|_{L_q(\Gamma)} \le c\|f\|_{W_p^1(\Omega)}. \tag{8}$$

Since $(n-1)p/(n-p) > p$, for $1 < p \le n$ the trace of elements of the space $W_p^1(\Omega)$ on Γ belongs to $L_p(\Gamma)$ (the fact that $f|_\Gamma$ is the trace of the element $f \in W_p^1(\Omega)$ on Γ means that in the class of equivalent functions f we can find a function which possesses limit values in the ordinary, or in the L_p sense. The above-mentioned theorems are proved, for example, in [3] or in [4].

As the Banach space Z we introduce the space of $L_p(\Gamma)$-functions, integrable on Γ in the p-th degree. Denote by $\mathbf{a}$ a linear operator from X to Z which to the element $u \in X$ puts into correspondence its trace on the boundary Γ of the domain Ω. Inequalities (7) and (8) imply that $\mathbf{a}$ is a linear continuous operator. The operator $\mathbf{t}$, acting from X to Y, can be defined by the formula

$$\mathbf{t}(u) = (0, u_{x_1}, \dots, u_{x_n}).$$

Clearly,

$$\|\mathbf{t}(u)\|_Y = \left\{ \int_\Omega \left(\sum_{i=1}^n u_{x_i}^2 \right)^{p/2} dv \right\}^{1/p}.$$

The right-hand side of the latter equality is the summand of the norm $\|u\|_X$, and hence $\mathbf{t}$ is a linear continuous operator.

In the space Z we fix an element $f \in \operatorname{Im} \mathbf{a}$ and denote by I_f a set of elements u of the space X whose trace on Γ is f, i.e., $\mathbf{a}(u) = f$, $\Lambda = (\lambda, 0, \dots, 0)$ belongs to Y, and the set $\Lambda + t(u) = (\lambda, u_{x_1}, \dots, u_{x_n})$ for all possible $u \in I_f$ is a shift of the set $\mathbf{t}(N(\mathbf{a}))$ onto the element $(\lambda, v_{x_1}, \dots, v_{x_n})$, where v is some fixed element from I_f.

Consider the functional

$$F_\lambda(u) = \left\{ \int_\Omega \left(\lambda^2 + \sum_{i=1}^n u_{x_i}^2 \right)^{p/2} d\sigma \right\}^{1/p} = \|\Lambda + t(u)\|_Y \tag{9}$$

defined on I_f and find its inf with respect to all $u \in I_f$. By Theorem 1, the closure of the set $\mathbf{t}(N(\mathbf{a}))$ in Y and the equality $N(\mathbf{a}) \cap N(\mathbf{t}) = \{0\}$ guarantee the existence and uniqueness of the minimizing element $\widetilde{u} \in I_f$. To prove the closure of the set $\mathbf{t}(N(\mathbf{a}))$, it is sufficient, following Lemma 1, to prove finite dimensionality of $N(\mathbf{t})$ and closure of $\operatorname{Im} \mathbf{t}$ in Y. $N(\mathbf{t})$ is the set of functions from X whose all generalized partial derivatives of first order are zeros. This means that $N(\mathbf{t})$ is a one-dimensional space of constants. It remains to prove the closure $\operatorname{Im} \mathbf{t}$. Let $u^{(m)}$ be a sequence in X such that all the sequences $u_{x_i}^{(m)}$, $1 \le i \le n$, composed of generalized partial derivatives of $u^{(m)}$ converge in $L_p(\Omega)$ to the functions α_i. We have to prove that the vector $(\alpha_1,\dots,\alpha_n)$ is the generalized gradient of a function $u\in X$, i.e., $u_{x_i}=\alpha_i$, $1\le i\le n$. For $1\le i,j\le n$ denote by $(u^{(m)})_{x_ix_j}$ the derivative with respect to x_j of the function $(u^{(m)})_{x_i}$ in a sense of the theory of generalized functions. Then for any finite in Ω function f we have

$$(u^{(m)})_{x_ix_j}(f)=\frac{\partial u^{(m)}}{\partial x_i}\Big(\frac{\partial f}{\partial x_j}\Big)\to\int\limits_\Omega \alpha_i\frac{\partial f}{\partial x_j}d\sigma,$$
$$(u^{(m)})_{x_jx_i}(f)=\frac{\partial u^{(m)}}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big)\to\int\limits_\Omega \alpha_j\frac{\partial f}{\partial x_i}d\sigma.$$

The left-hand sides of these relations are equal to $u^{(m)}(f_{x_ix_j})$ (elements $u^{(m)}\in X$ are considered as regular generalized functions). Hence their right-hand sides are likewise equal:

$$\int\limits_\Omega \alpha_i\,\partial f/\partial x_j d\sigma=\int\limits_\Omega \alpha_j\partial f/\partial x_i d\sigma,$$

i.e., $(\alpha_j)_{x_i}=(\alpha_i)_{x_j}$, where the derivatives are considered in a sense of the theory of generalized functions. It is known ([3], p. 160; [5], Ch. 5, 5.9.3, 5.11.1) that in the case, where Ω is simply connected, there exists a locally integrable in Ω function u whose generalized partial derivatives u_{x_i}, $1\le i\le n$, belong to $L_p(\Omega)$, and $\operatorname{grad} u=(\alpha_1,\dots,\alpha_n)$. But by the Sobolev embedding theorem ([3], pp. 280–282), such a function u belongs to $L_p(\Omega)$, and hence $u\in X$, i.e., $\operatorname{Im}\mathbf{t}$ is closed in Y. The condition $N(\mathbf{a})\cap N(\mathbf{t})=\{0\}$ can be easily verified. Thus we have proved the following

Theorem 2. *Let Ω be a bounded simply connected domain from R_n, whose boundary Γ consists of a finite number of surfaces of the class C^1, f be the defined on Γ function for which the set $I_f=\{u\in W_p^1(\Omega):u|_\Gamma=f\}$ is non-empty, and let λ be an arbitrary fixed real number. Then for the functional F_λ, defined by formula* (9), *$\inf_{u\in I_f}F_\lambda(u)$ is obtained on some element $u\in I_f$.*

Remark 1. Theorem 2 is also valid for the domain which is divided by means of surfaces of the class C^1 into a finite number of simply connected domains. Indeed, the functions $u\in I_f$ will have, according to the above-mentioned Sobolev theorems, on such surfaces a trace belonging to the corresponding space L_p, and the problem of minimization of the functional F_λ can be considered in the obtained simply connected domains. Using minimizing elements, we can construct a minimizing function in the initial domain.

Remark 2. Using the method described above, we can investigate a functional of more general type such as

$$F_{\lambda,s}(u)=\Big\{\int\limits_\Omega(\lambda^2+Du)^{p/2}d\sigma\Big\}^{1/p}, \tag{10}$$

where

$$Du = \sum_{s_1+\dots+s_n=s} c(s_1,\dots,s_n)\Big(\frac{\partial^s u}{\partial x_1^{s_1}\dots\partial x_n^{s_n}}\Big)^2, \quad c(s_1,\dots,s_n)>0,$$

and λ is a fixed function belonging to $L_p(\Omega)$, while the derivatives are understood in the Sobolev sense. Not going into details, we note that for functional (10) we obtain a theorem analogous to Theorem 2. Here we restrict ourselves to case (9).

For the element u minimizing functional (9) we derive the so-called Euler's generalized equation. To this end, we consider some function η belonging to $D(\Omega)$, a space of finite, infinitely differentiable in Ω function, and for the real numerical parameter α we compose the set $\{u+\alpha\eta\}$. Let $p=l+1$, $l>0$. The function u minimizes the functional

$$I_\lambda(u) = \int\limits_\Omega \bigg(\lambda^2 + \sum_{i=1}^n u_{x_i}^2\bigg)^{(l+1)/2} d\sigma,$$

and since the trace of the function $u+\alpha\eta$ belonging to $W_{l+1}(\Omega)$ on Γ is f, for any $\alpha\in R$ we have

$$I_\lambda(u+\alpha\eta) \ge I_\lambda(u).$$

Let us prove that

$$I_\lambda(u+\alpha\eta) = \int\limits_\Omega \bigg(\lambda^2 + \sum_{i=1}^n (u_{x_i}+\alpha\eta_{x_i})^2\bigg)^{(l+1)/2} d\sigma \tag{11}$$

has a derivative with respect to α for any $\alpha\in R$, and I'_λ can be calculated by differentiation under the integral sign. For this we write the expression

$$\frac{\Delta I_\lambda}{\Delta\alpha} = \frac{I_\lambda(u+(\alpha+\Delta\alpha))\eta - I_\lambda(u+\alpha\eta)}{\Delta\alpha}$$

($\Delta\alpha$ is an increment of α) and transform it by means of the Lagrange mean value theorem. We obtain

$$\begin{aligned}\frac{\Delta I_\lambda}{\Delta\alpha} = (l+1)\int\limits_\Omega \bigg(\lambda^2 + \sum_{i=1}^n (u_{x_i}+\alpha_\theta\eta_{x_i})^2\bigg)^{(l-1)/2}\\ \times\bigg[\sum_{i=1}^n (u_{x_i}+\alpha_\theta\eta_{x_i})\eta_{x_i}\bigg] d\sigma,\end{aligned} \tag{12}$$

where $\alpha_\theta = \alpha+\theta(u_{x_1},\dots,u_{x_n},\alpha)$, $0<|\theta(u_{x_1},\dots,u_{x_n},\alpha)|<|\Delta\alpha|$.

By the Cauchy–Bunjakovsky inequality we find that the integrand is estimated from the above by the expressions

$$\bigg[\lambda^2 + \sum_{i=1}^n (u_{x_i}+\alpha_\theta\eta_{x_i})^2\bigg]^{l/2} \bigg(\sum_{i=1}^n \eta_{x_i}^2\bigg)^{1/2}$$

and

$$\max\left\{1,\left[\lambda^2+\sum_{i=1}^{n}(u_{x_i}+\alpha_\theta\eta_{x_i})^2\right]^{l+1/2}\right\}\left(\sum_{i=1}^{n}\eta_{x_i}^2\right)^{1/2}.$$

But

$$\lambda^2+\sum_{i=1}^{n}(u_{x_i}+\alpha_\theta\eta_{x_i})^2\leq\left(\lambda^2+\sum_{i=1}^{n}u_{x_i}^2\right)(1+|\alpha_\theta|)+(|\alpha_\theta|+\alpha_\theta^2)\sum_{i=1}^{n}\eta_{x_i}^2,$$

and $u\in W_{l+1}$. Therefore we can easily see that the integrand in (12) has a majorant belonging to $L_1(\Omega)$. By the Lebesgue theorem, we can pass in (12) to the limit under the integral sign. Thus from (11) we find that

$$I'_\lambda(u+\alpha\eta)=(l+1)\int\limits_\Omega\left(\lambda^2+\sum_{i=1}^{n}(u_{x_i}+\alpha\eta_{x_i})^2\right)^{(l-1)/2}\left[\sum_{i=1}^{n}(u_{x_i}+\alpha\eta_{x_i})\eta_{x_i}\right]d\sigma.$$

Since $\alpha=0$ is the minimum point of the function $I_\lambda(u+\alpha\eta)$, we have that $I'_\lambda(u)=0$, and hence the function u, minimizing $I_\lambda(u)$, satisfies the generalized Euler equation

$$\int\limits_\Omega\left(\lambda^2+\sum_{i=1}^{n}u_{x_i}^2\right)^{(l-1)/2}\left(\sum_{i=1}^{n}u_{x_i}\eta_{x_i}\right)d\sigma=0\tag{13}$$

for any function $\eta\in D(\Omega)$. For $\lambda=0$ we get the equation

$$\int\limits_\Omega|\operatorname{grad}u|^{l-1}\left(\sum_{i=1}^{n}u_{x_i}\eta_{x_i}\right)d\sigma=0.\tag{14}$$

It is seen from (14) that u is a solution of the equation

$$\Delta_l(u)\overset{\text{def}}{=}\operatorname{div}\Big(|\operatorname{grad}u|^{l-1}\operatorname{grad}u\Big)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(|\operatorname{grad}u|^{l-1}u_{x_i}\Big)=0,\tag{15}$$

where partial derivatives $\partial/\partial x_i$ are understood in a sense of the theory of generalized functions. The solution of equation (13) satisfies in the same sense equation (14), where $\operatorname{grad}u$ is replaced by an expression of the type $\Big(\lambda^2+\sum_{i=1}^{n}u_{x_i}^2\Big)^{1/2}$. Equation (14) is known in the literature as the l-Laplace equation, and the corresponding operator Δ_l is called the l-Laplacian ([8]). For $\lambda=0$, $l=1$ the functional F_λ, defined by formula (9), coincides with the known Dirichlet integral, and Δ_1 is the known Laplace operator.

Let now the function u belonging to the space $W_{l+1}^1(\Omega)$, $l>0$, satisfy in Ω the equation

$$\{\Delta_l\}_\lambda u=\operatorname{div}\Big(\Big(\lambda^2+|\operatorname{grad}u|^2\Big)^{(l-1)/2}\operatorname{grad}u\Big)=0\tag{16}$$

in a sense of the theory of generalized functions. Suppose that the trace of the function u on the boundary Γ of the domain Ω is the function $f\in L_{l+1}(\Gamma)$. Let us prove that in such a case a solution $u\in W_{l+1}^1(\Omega)$ of equation (16) minimizes functional (9) in the class I_f. As above, I_f denotes the set of functions from $W^1_{l+1}(\Omega)$ whose trace on Γ is f. It is clear that the solution u of equation (16) satisfies equation (13) for any function $\eta \in D(\Omega)$. Note first that (13) is also satisfied for any function $\eta \in W^1_{l+1}(\Omega)$ whose trace on Γ is zero. We denote the set of such functions by $\mathring{W}^1_{l+1}(\Omega)$. Indeed, let η be an arbitrary function from $\mathring{W}^1_{l+1}(\Omega)$. Then there exists in Ω a sequence $\eta_n \in D(\Omega)$ such that $\|\eta - \eta_n\|^1_{W_{l+1}}(\Omega) \to 0$ as $n \to \infty$. For $l = 1$ this proposition is known (see, e.g., [6], p. 119). The proof is analogous for any $l > 0$. The integral in the left-hand side of (10) exists if $u,\ \eta \in W^1_{l+1}(\Omega)$. Indeed, from the Hölder inequality we obtain

$$\begin{aligned}
&\int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad} u|^2\Big)^{(l-1)/2} \Big(\sum_{i=1}^n u_{x_i}\eta_{x_i}\Big) d\sigma \\
&\le \int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad} u|^2\Big)^{(l-1)/2} |\operatorname{grad} u| |\operatorname{grad} \eta| d\sigma \\
&\le \int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad} u|^2\Big)^{l/2} |\operatorname{grad} \eta| d\sigma \\
&\le \Big\{\int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad} u|^2\Big)^{(l+1)/2} \sigma\Big\}^{(l+1)/l} \Big\{\int\limits_\Omega |\operatorname{grad} u|^{(l+1)} d\sigma\Big\}^{1/(l+1)} < \infty. \qquad (17)
\end{aligned}$$

Since (10) is valid for $\eta_n \in D(\Omega)$, estimate (17) shows that (10) is now valid for any function $\eta \in \mathring{W}^1_{l+1}(\Omega)$.

Along with the solution u of equation (13), we consider the set $\{u + \alpha\eta\}$, $\alpha \in R$, $\eta \in \mathring{W}^1_{l+1}(\Omega)$. Since $(u + \alpha\eta)|_\Gamma = f$, $u + \alpha\eta \in I_f$. Then (9) implies that

$$F_\lambda(u + \alpha\eta) = \int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad}(u + \alpha\eta)|^2\Big)^{(l+1)/2} d\sigma. \qquad (18)$$

Denote now $\Big(\lambda^2 + |\operatorname{grad}(u)|^2\Big)^{(l+1)/2}$ by $\phi_l(\alpha)$ and prove that for any $l > 0$ and $\alpha \in R$ the estimate

$$\phi_l(\alpha) \ge \phi_l(0) + \alpha\phi_l(0) \qquad (19)$$

is valid. To this end, we prove the following

Lemma 2. *If $\alpha \in R$ and $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ are the points from R_n, then the inequality*

$$\begin{aligned}
&\Big(\lambda^2 + \sum_{i=1}^n (a_i + b_i)^2\Big)^{(l+1)/2} \\
&\ge \Big(\lambda^2 + \sum_{i=1}^n a_i^2\Big)^{(l+1)/2} + (l+1)\Big(\lambda^2 + \sum_{i=1}^n a_i^2\Big)^{(l-1)/2} \Big(\sum_{i=1}^n a_i b_i\Big) \qquad (20)
\end{aligned}$$

is valid for any $l > 0$.

Proof. Let t and s be vectors from R_k, and the length s be equal to unity. Then $|s+t| \geq |1+(t,s)|$, where (t,s) is the scalar product of the vectors t and s. Indeed, if θ is the angle between the vectors $t+s$ and t, then $|s+t| \geq |s+t||s||\cos\theta| = |(s+t,s)| = |(s,s)+(t,s)| = |1+(t,s)|$. The $(1+l)$-th degree of $|1+(t,s)|$ can be estimated from below: $|1+(t,s)|^{(l+1)} \geq 1+(l+1)(t,s)$; this inequality coincides, in fact, with that of [9] (p. 12). Therefore

$$|s+t|^{(l+1)} \geq 1+(l+1)(t,s). \tag{21}$$

Let x be an arbitrary non-zero vector from R_k. Substituting $s = x/|x|$ and $y = |x|t$ into (21), we obtain

$$|x+y|^{(l+1)} \geq |x|^{(l+1)} + (l+1)|x|^{(l-1)}(x,y). \tag{22}$$

Clearly, the obtained inequality is likewise valid for $x = 0$. To estimate (20), we have to substitute $x = (\lambda, a_1, \dots, a_n)$ and $y = (0, b_1, \dots, b_n)$ into (22). □

Applying now Lemma 2 to $a_i = u_{x_i}$ and $b_i = \alpha.\eta_{x_i}$, we see that estimate (19) is valid. Further, according to (18) we have

$$F_\lambda(u+\eta) \geq F_\lambda(u) + (l+1)\int\limits_\Omega \Big(\lambda^2 + |\operatorname{grad} u|^2\Big)^{(l-1)/2}\Big(\sum_{i=1}^n u_{x_i}\eta_{x_i}\Big)d\sigma,$$

and by virtue of (13) we make sure that $F_\lambda(u+\eta) \geq F_\lambda u$ for any function $\eta \in \overset{\circ}{W}{}^1_{l+1}(\Omega)$. As we know, inf in (9) is obtained on some element u_0 from I_f. Take $\eta = u_0 - u \in \overset{\circ}{W}{}^1_{l+1}(\Omega)$. Then we have $F_\lambda(u_0) \geq F_\lambda(u)$. Since u_0 is the unique minimizing element, $u = u_0$, i.e., u minimizes the functional F_λ. Thus we proved the following

Theorem 3. *Let the function $f \in L_{l+1}(\Gamma)$, $l > 0$, be given on the boundary Γ of the domain Ω (satisfying the above conditions) which is the trace of a function from the space $W^1_{l+1}(\Omega)$. Then the element u of the class I_f which minimizes functional* (9) *(u exists by Theorem* 2*) is a solution of equation* (13) *in a sense of the theory of generalized functions. Conversely, if there exists a solution u of equation* (13) *belonging to $W^1_{l+1}(\Omega)$ and having a trace $u|_\Gamma = f$, then it minimizes functional* (9) *in the class I_f and is unique.*

References

1. P. J. Loran, Approximation et optimization. *Hermann, Paris,* 1975.

2. D. Ugulava, On the notion of a spline in the Banach space. (Russian) *Trudy Inst. Vychisl. Mat. Akad. Nauk Gruzin. SSR* **26**(1986), No. 1, 234–241.

3. S. L. Sobolev, Introduction to the theory of cubature formulas. (Russian) *Nauka, Moscow*, 1974.

4. V. I. Smirnov, Course on higher mathematics, V. (Russian) *FIZMATGIZ, Moscow*, 1959.

5. R. Edwards, Functional analysis. Theory and applications. *Holt, Rinehart and Winston, New York–Toronto–London,* 1965.

6. V. P. Mikhaylov, Partial differential equations. (Russian) *Nauka, Moscow*, 1983.

7. N. Kandelaki, A. Lomtatidze, T. Chantladze, and D. Ugulava, The p-Laplacian and connected pairs of functions. *Mem. Differential Equations Math. Phys.* **20**(2000), 113–126.

8. E. F. Beckenbach and R. Bellman, Inequalities, *Springer, Berlin etc.*, 1983.

(Received 30.01.2002)

Authors' address:

T. Chantladze, N. Kandelaki, and D. Ugulava
N. Muskhelishvili Intitute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093
Georgia

A. Lomtatidze
Department of Mathematical Analysis
Faculty of Science
Masaryk University
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic
E-mail: bacho@math.muni.cz