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SOME UNIQUENESS RESULTS FOR IMPULSIVE SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The Banach contraction principle is used to investigate the existence and uniqueness of solutions for first and second order impulsive semilinear neutral functional differential equations in Banach spaces.

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1. Introduction

This note is concerned with the existence and uniqueness of mild solutions for initial value problems for first and second order semilinear neutral functional differential equations with impulsive effects in Banach spaces. More precisely, in Section 3 we consider first order impulsive semilinear neutral functional differential equations (NFDEs) of the form

$$\frac{d}{dt}[y(t)-g(t,y_t)]=Ay(t)+f(t,y_t), \quad t\in J=[0,T], \ t\neq t_k, \quad k=1,\dots,m, \tag{1}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)), \quad k=1,\dots,m, \tag{2}$$

$$y(t)=\phi(t), \quad t\in[-r,0], \tag{3}$$

where $f, \ g : J\times C([-r,0],E)\to E$ are given functions, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t\geq 0$ in E, $\phi\in C([-r,0],E)$, $(0<r<\infty)$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, $I_k : E\to E$ $(k=1,2,\dots,m)$, $\Delta y|_{t=t_k}=y(t_k^+)-y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and the right limit of $y(t)$ at $t=t_k$, respectively, and E is a real Banach space with norm $|\cdot|$.

For any continuous function y defined on $[-r,T]-\{t_1,\dots,t_m\}$ and any $t\in J$, we denote by y_t the element of $C([-r,0],E)$ defined by

$$y_t(\theta)=y(t+\theta), \quad \theta\in[-r,0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

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In Section 4 we study second order impulsive semilinear neutral functional differential equations of the form

$$\frac{d}{dt}[y'(t) - g(t, y_t)] = Ay(t) + f(t, y_t), \quad t \in J = [0, T], \quad t \neq t_k, \quad k = 1, \ldots, m, \tag{4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{5}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{6}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \; y'(0) = \eta, \tag{7}$$

where $f, g,\ I_k$, and ϕ are as in problem (1)–(3), A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$, of bounded linear operators in E,$\overline{I}_k : E \to E$ and $\eta \in E$.

Impulsive differential and partial differential equations are used to describe various models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. That is why in recent years they have been the object of investigations. We refer to the monographs of Bainov and Simeonov [1], Lakshmikantham *et al* [10], and Samoilenko and Perestyuk [13] where numerous properties of their solutions are studied, and a detailed bibliography is given.

The extension to functional and neutral functional differential equations with impulsive effects has been done by Dong in [4] by using the coincidence degree theory and by Benchohra *et al* [2], [3] with the aid of a nonlinear alternative of Leray–Schauder type and Schaefer's theorem. Other results on functional differential equations without impulsive effect can be found in the monograph of Erbe *et al* [5], Hale [8], Henderson [9], and the survey paper of Ntouyas [11]. The main theorems of this note extend some existence results in the above literature to the impulsive case. Our approach here is based on the Banach contraction principle.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper.

$C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

For each $k = 0, \ldots, m$ let $J_k := [t_k, t_{k+1}]$. By $C(J_k, E)$ we denote the Banach space of all continuous functions from J_k into E with the norm

$$\|y\|_{J_k} = \sup\{|y(t)| : t \in J_k\}.$$

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the properties of the Bochner integral, see, for instance, Yosida [16].)

$L^1(J,E)$ denotes the Banach space of Bochner integrable measurable functions $y : J \longrightarrow E$ with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E with the norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if:

(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,
(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

A strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)yds, \quad y \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families we refer the reader to the books of Goldstein [7], Fattorini [6], and to the papers of Travis and Webb [14], [15]. For the properties of semigroup theory we refer the interested reader to the books of Goldstein [7] and Pazy [12].

In order to define a mild solution of problems (1)–(3) and (4)–(7) we will consider the space

$$\begin{aligned}\Omega \ &= \{y : [-r,T] \longrightarrow E : y_k \in C(J_k,E), k = 0,\dots,m, \ \text{ and there exist } y(t_k^-) \\ &\quad \text{and } y(t_k^+) \text{ with } y(t_k^-) = y(t_k), \ k = 1,\dots,m\}\end{aligned}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k}, k = 0,\dots,m\},$$

where y_k is the restriction of y to J_k, $k = 0,\dots,m$.

Definition 2.1. A map $f : J \times C([-r,0],E) \longrightarrow E$ is said to be L^1-Carathéodory if:

(i) $t \longmapsto f(t,u)$ is measurable for each $u \in C([-r,0],E)$;
(ii) $u \longmapsto f(t,u)$ is continuous for almost all $t \in J$;
(iii) For each $q > 0$ there exists $h_q \in L^1(J,\mathbb{R}_+)$ such that

$$|f(t,u)| \le h_q(t) \quad \text{for all } \ \|u\| \le q \quad \text{and for almost all } \ t \in J.$$

In what follows we will assume that f is an L^1-Carathéodory function.

3. First Order Impulsive Semilinear NFDEs

The aim of this section is to study the existence of mild solutions for the initial value problem (1)–(3).

Definition 3.1. A function $y \in \Omega$ is said to be a mild solution of (1)–(3) if $y(t) = \phi(t)$ on $[-r, 0]$, the function $AT(t-s)g(s, y_s)$, $s \in [0, t)$ is integrable for each $0 \leq t < T$ and

$$y(t) = T(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)\, ds$$
$$+ \int_0^t T(t-s)f(s, y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J.$$

We are now in the position to state and prove our existence result for the problem (1)–(3).

Theorem 3.2. *Assume that the following conditions are satisfied:*

(H1) *there exists a positive constant c such that*

$$|g(t,u) - g(t,\overline{u})| \leq c\|u - \overline{u}\|, \quad t \in J, \ u, \overline{u} \in C([-r, 0], E);$$

(H2) *there exist constants d_k such that*

$$|I_k(y) - I_k(\overline{y})| \leq d_k|y - \overline{y}|, \ k = 1, \dots, m, \ \textit{for each } y, \overline{y} \in E;$$

(H3) *there exists a positive constant d such that*

$$|f(t,u) - f(t,\overline{u})| \leq d\|u - \overline{u}\|, \quad t \in J, \ u, \overline{u} \in C([-r, 0], E);$$

(H4) *A is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$ in E such that*

$$\|T(t)\|_{B(E)} \leq M_1, \ \ M_1 > 0, \quad \|AT(t)\|_{B(E)} \leq M_2, \ \ M_2 > 0, \quad t \in J.$$

If

$$\left(c + M_2Tc + M_1Td + \sum_{k=1}^m d_k\right) < 1,$$

then the problem (1)–(3) *has a unique mild solution.*

Proof. We transform the problem into a fixed point problem. Consider the operator, $N_1 : \Omega \longrightarrow \Omega$ defined by

$$N_1(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ T(t)[\phi - g(0, \phi)] + g(t, y_t) + \int_0^t A(t-s) g(s, y_s)\, ds & \\ + \int_0^t T(t-s) f(s, y_s)\, ds + \sum_{0<t_k<t} [I_k(y(t_k)) & \text{if } t \in J. \end{cases}$$

Remark 3.3. It is clear that the fixed points of N_1 are mild solutions of (1)–(3).

We will show that N_1 is a contraction operator. Indeed, consider $y, \overline{y} \in \Omega$; thus for each $t \in J$

$$\begin{aligned} |N_1(y)(t) - N_1(\overline{y})(t)| &= \Big| \int_0^t AT(t-s)[g(s, y_s) - g(s, \overline{y}_s)] ds \\ &\quad + g(t, y_t) - g(t, \overline{y}_t) \\ &\quad + \int_0^t T(t-s)[f(s, y_s) - f(s, \overline{y}_s)] ds \\ &\quad + \sum_{0<t_k<t} [I_k(y(t_k)) - I_k(\overline{y}(t_k))] \Big| \\ &\le TM_2 c \|y_t - \overline{y}_t\| + c\|y_t - \overline{y}_t\| + TM_1 d \|y_t - \overline{y}_t\| \\ &\quad + \sum_{k=1}^m d_k |y(t_k)) - \overline{y}(t_k)| \\ &\le TM_2 c \|y - \overline{y}\|_\Omega + c\|y - \overline{y}\|_\Omega + TM_1 d \|y - \overline{y}\|_\Omega \\ &\quad + \sum_{k=1}^m d_k \|y - \overline{y}\|_\Omega. \end{aligned}$$

Therefore

$$\|N_1(y) - N_1(\overline{y})\|_\Omega \le \left[c + TM_2 c + TM_1 d + \sum_{k=1}^m d_k \right] \|y - \overline{y}\|_\Omega,$$

showing that N_1 is a contraction and hence it has a unique fixed point which is a mild solution of (1)–(3). □

4. Second Order Impulsive Semilinear NFDEs

In this section we study the second order impulsive semilinear neutral functional differential equations (4)–(7).

We adopt the following definition.

Definition 4.1. A function $y \in \Omega$ is said to be a mild solution of (4)–(7) if $y(t) = \phi(t)$ on $[-r, 0]$, and

$$y(t) = C(t)\phi(0) + S(t)[\eta - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)\,ds$$
$$+ \int_0^t S(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k))], \quad t \in J.$$

Assume that:

(H5) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators from E into itself.

(H6) There exists a constant $\overline{M}_1 > 0$ such that $\|C(t)\|_{B(E)} \leq \overline{M}_1$ for each $t \in \mathbb{R}$;

(H7) there exist positive constants $\overline{d}_k$ such that

$$|\overline{I}_k(y) - \overline{I}_k(\overline{y})| \leq \overline{d}_k|y - \overline{y}|, \ k = 1, \ldots, m, \text{ for each } y, \overline{y} \in E.$$

Now we are in the position to state and prove the main theorem of this section.

Theorem 4.2. *Assume that hypotheses* (H1)–(H3), (H5)–(H7) *hold. If*

$$\left(T\overline{M}_1 c + T^2\overline{M}_1 d + \sum_{k=1}^{m} [d_k + (T - t_k)\overline{d}_k] \right) < 1,$$

then the problem (4)–(7) *has a unique mild solution.*

Proof. Transform the problem into a fixed point problem. This time define an operator $N_2 : \Omega \longrightarrow \Omega$ by

$$N_2(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ C(t)\phi(0) + S(t)[\eta - g(0,\phi)] & \\ + \displaystyle\int_0^t C(t-s)g(s,y_s)\,ds + \int_0^t S(t-s)f(s,y_s)\,ds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))] & \text{if } t \in J. \end{cases}$$

Remark 4.3. Clearly the fixed points of N_2 are mild solutions of (4)–(7).

We will show as in Section 3 that N_2 is a contraction operator. Indeed, consider $y, \overline{y} \in \Omega$, thus for each $t \in J$

$$|N_2(y)(t) - N_2(\overline{y})(t)| = \Big| \int_0^t C(t-s)[g(s,y_s) - g(s,\overline{y}_s)]ds$$

$$
\begin{aligned}
&\quad + \int_0^t S(t-s)[f(s,y_s) - f(s,\overline{y}_s)]ds \\
&\quad + \sum_{0<t_k<t} [I_k(y(t_k)) - I_k(\overline{y}(t_k))] \\
&\quad + \sum_{0<t_k<t} (t-t_k)[\overline{I}_k(y(t_k)) - \overline{I}_k(\overline{y}(t_k))]\Big| \\
&\leq T\overline{M}_1 c\|y_t - \overline{y}_t\| + T^2\overline{M}_1 d\|y_t - \overline{y}_t\| \\
&\quad + \sum_{k=1}^{m} [d_k + (T-t_k)\overline{d}_k]|y(t_k)) - \overline{y}(t_k)| \\
&\leq T\overline{M}_1 c\|y - \overline{y}\|_\Omega + T^2\overline{M}_1 d\|y - \overline{y}\|_\Omega \\
&\quad + \sum_{k=1}^{m} [d_k + (T-t_k)\overline{d}_k]\|y - \overline{y}\|_\Omega.
\end{aligned}
$$

Therefore

$$
\|N_2(y) - N_2(\overline{y})\|_\Omega \leq \left[T\overline{M}_1 c + T^2\overline{M}_1 d + \sum_{k=1}^{m}[d_k + (T-t_k)\overline{d}_k]\right]\|y - \overline{y}\|_\Omega,
$$

showing that N_2 is a contraction and hence it has a unique fixed point which is a mild solution of (4)–(7). $\square$

Remark 4.4. The reasoning used in Sections 3 and 4 can be applied to obtain existence results for the following impulsive semilinear neutral functional integrodifferential equations of Volterra type:

$$
\frac{d}{dt}[y(t) - g(t,y_t)] = Ay(t) + \int_0^t K(t,s)f(s,y_s)ds, \tag{8}
$$

$$
t \in J = [0,T], \quad t \neq t_k, \quad k = 1,\dots,m,
$$

$$
\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, y(t) = \phi(t), \quad t \in [-r,0], \tag{9}
$$

and

$$
\frac{d}{dt}[y'(t) - g(t,y_t)] = Ay(t) + \int_0^t K(t,s)f(s,y_s)ds, \tag{10}
$$

$$
t \in J = [0,T], \quad t \neq t_k, ; \quad k = 1,\dots,m,
$$

$$
\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{11}
$$

$$
\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{12}
$$

$$
y(t) = \phi(t), \quad t \in [-r,0], \ y'(0) = \eta, \tag{13}
$$

where $K : D \to \mathbb{R}$, and $D : \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$.

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