Georgian Mathematical Journal
Volume 9 (2002), Number 3, 413–422

ON OBTAINING DUAL SEQUENCES VIA QUASI-MONOMIALITY

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Abstract. In this paper, we introduce a method to obtain the dual sequence of a given polynomial set using the lowering operator associated with the involved polynomials. As application, we derive polynomial expansions of analytic functions. The particular case corresponding to Boas–Buck polynomials allows us to unify many polynomial expansions of analytic functions in the literature. This method can be useful in studying many problems arising in the theory of polynomials as the so-called connection and linearization problems.

2000 Mathematics Subject Classification: 41A10, 41A58.
Key words and phrases: Polynomial set, dual sequence, quasi-monomiality, expansion theorem, Boas–Buck polynomials, generalized Taylor series.

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle \mathcal{L}, f\rangle$ the effect of the functional $\mathcal{L} \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Let $\{Q_n\}_{n\geq 0}$ be a polynomial set, that is $\deg Q_n = n$. Its dual sequence $\{\mathcal{L}_n\}_{n\geq 0}$ is defined by

$$\langle \mathcal{L}_n, Q_m\rangle = \delta_{nm}, \qquad n, m \geq 0. \tag{1.1}$$

One of the problems related to this notion asks *to express explicitly the dual sequence of a given polynomial set.* Such a problem arises in various fields of mathematics. Among the methods developed to this end, we could mention the one which requires that the polynomial set $\{Q_m\}_{m\geq 0}$ be orthogonal. Then $\langle \mathcal{L}_n, f\rangle$, $n \geq 0$, are the corresponding Fourier coefficients. Another one (see, for instance, [16]), which was deduced from the inversion formula

$$z^k = \sum_{n=0}^{\infty} \pi_{k,n} Q_n(z), \quad k = 0, 1, 2, \dots ,$$

provides

$$\langle \mathcal{L}_n, f\rangle = \sum_{k=0}^{\infty} \pi_{k,n} f^{(k)}(0)/k!.$$

Our purpose in this work is to present a further method to construct the dual sequence of a given polynomial set for the general case.

ISSN 1072-947X / \$8.00 /

In the previous paper [2], we show that every polynomial set $\{Q_n\}_{n\geq 0}$ is quasi-monomial. That is to say, there exist a lowering operator σ and a rising operator τ, independent of n, such that

$$\sigma\left(Q_n\right)(x) = nQ_{n-1}(x) \qquad \text{and} \qquad \tau\left(Q_n(x)\right) = Q_{n+1}(x). \tag{1.2}$$

σ and τ are respectively in $\Lambda^{(-1)}$ and $\Lambda^{(1)}$ where $\Lambda^{(j)}$, j being an integer, denotes the space of operators acting on analytic functions that augment (resp. reduce) the degree of every polynomial by exactly j if $j \geq 0$ (resp. $j \leq 0$). That includes the fact that $\sigma(1) = 0$.

In this paper, starting from a polynomial set, we construct its dual sequence in terms of its lowering operator. We state the following main result.

Theorem 1.1. *Let $\{Q_n\}_{n\geq 0}$ be a polynomial set, let σ be its lowering operator, and let $\{\mathcal{L}_n\}_{n\geq 0}$ be its dual sequence. Then there exists a power series $\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n$, $\alpha_0 \neq 0$, such that*

$$\langle \mathcal{L}_n, f\rangle = \frac{1}{n!}\left[\sigma^n \varphi(\sigma)(f)(x)\right]_{x=0} = \frac{\sigma^n\varphi(\sigma)}{n!} f(0), \quad n=0,1,\ldots, \quad f \in \mathcal{P}. \tag{1.3}$$

The outline of the paper is as follows. In Section 2, we prove Theorem 1.1 from which we deduce an expansion theorem in Section 3 where we also apply the obtained result to Boas–Buck polynomial sets. That leads us to unify many examples of polynomial expansions of analytic functions in the literature. In Section 4, we discuss the possibility of using the results of this paper in studying some problems arising in the theory of polynomials.

2. Proof of the Main Result

Let us introduce firstly the following notion

Definition 2.1. Let $\sigma \in \Lambda^{(-1)}$. A polynomial set $\{P_n\}_{n\geq 0}$ is called a *sequence of basic polynomials* for σ if:

(i) $P_0(x) = 1$,
(ii) $P_n(0) = 0$ whenever $n > 0$,
(iii) $\sigma P_n(x) = nP_{n-1}(x)$.

As a consequence of this definition, we mention the orthogonality relation

$$\sigma^m P_n(0) = n!\delta_{nm}, \qquad n, m = 0, 1, \ldots. \tag{2.1}$$

To prove Theorem 1.1, we need the following two lemmas.

Lemma 2.1. *Every $\sigma \in \Lambda^{(-1)}$ has a sequence of basic polynomials*

Proof. Put $P_0(x) = 1$ and define $P_n(x)$, $n \geq 1$, recurrently by the identity

$$a_n P_n(x) = x^n - \sum_{k=0}^{n-1} \frac{1}{k!}\left[\sigma^k\left(\xi^n\right)\right]_{\xi=0} P_k(x), \qquad n \geq 1, \tag{2.2}$$

where $a_n = \frac{\sigma^n(x^n)}{n!} \neq 0$. Inducing on n, we verify that this sequence satisfies the conditions given by Definition 2.1. $\square$

Lemma 2.2. *Let $\{Q_n\}_{n\geq 0}$ be a polynomial set and let σ be its lowering operator. Let $\{P_n\}_{n\geq 0}$ be a sequence of basic polynomials for σ. Then there exists a power series $\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n$, $\alpha_0 \neq 0$, such that*

$$\varphi(\sigma)(Q_n) = P_n, \qquad n = 0, 1, \dots \tag{2.3}$$

Proof. Since $\{Q_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ are two polynomial sets, it is possible to write

$$P_n(x) = \sum_{k=0}^{n} \alpha_{n,k} \frac{n!}{(n-k)!} Q_{n-k}(x), \qquad n = 0, 1, \dots, \tag{2.4}$$

where the coefficients $\alpha_{n,k}$ depend on n and k and $\alpha_{n,0} \neq 0$. We need to prove that these coefficients are independent of n. Apply the operator σ to each member of (2.4) to obtain

$$P_{n-1}(x) = \sum_{k=0}^{n-1} \alpha_{n,k} \frac{(n-1)!}{(n-1-k)!} Q_{n-1-k}(x), \qquad n = 1, 2, \dots, \tag{2.5}$$

since $\sigma Q_0 = 0$. Shifting the index $n \longrightarrow n+1$ in (2.5), we have

$$P_n(x) = \sum_{k=0}^{n} \alpha_{n+1,k} \frac{(n)!}{(n-k)!} Q_{n-k}(x), \qquad n = 0, 1, 2, \dots \tag{2.6}$$

Compare (2.4) and (2.6) to note that $\alpha_{n,k} = \alpha_{n+1,k}$ for all k and n, which means that $\alpha_{n,k} = \alpha_k$ independent of n. That leads us to write (2.4) in the form

$$P_n(x) = \sum_{k=0}^{n} \alpha_k \sigma^k Q_n(x) = \left(\sum_{k=0}^{\infty} \alpha_k \sigma^k \right) (Q_n)(x), \qquad n = 0, 1, \dots, \tag{2.7}$$

since $\sigma^m Q_n(x) = 0$ for $m > n$, which finishes the proof. $\square$

Proof of Theorem 1.1. Let $\{Q_n\}_{n\geq 0}$ be a polynomial set. From Theorem 2.1 in [2] it follows that there exists a lowering operator $\sigma \in \Lambda^{(-1)}$ such that $\sigma Q_n = n Q_{n-1}$. According to Lemma 2.1, there exists a sequence $\{P_n\}_{n\geq 0}$ of basic polynomials for σ, and according to Lemma 2.2, there exists a power series $\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n$, $\alpha_0 \neq 0$, satisfying (2.3). Define a sequence of linear functionals $\{\mathcal{L}_n\}_{n\geq 0}$ as

$$\langle \mathcal{L}_n, f \rangle = \frac{1}{n!} \left[\sigma^n \varphi(\sigma)(f)(x) \right]_{x=0} = \frac{\sigma^n \varphi(\sigma)}{n!} f(0), \quad n = 0, 1, \dots,$$

where f is a polynomial. From (2.1) and (2.3), we have

$$\langle \mathcal{L}_n, Q_m \rangle = \frac{1}{n!} \sigma^n \varphi(\sigma)(Q_m)(0) = \frac{1}{n!} \sigma^n (P_m)(0) = \delta_{nm},$$

which finishes the proof. $\square$

3. Polynomial Expansions of Analytic Functions

Let $\{Q_n\}_{n\geq 0}$ be a polynomial set. Suppose that we want to represent a given analytic function at the origin $f(z)$ as a series $\sum c_n Q_n(z)$. Let us consider $\{\mathcal{L}_n\}_{n\geq 0}$ the dual sequence of $\{Q_n\}_{n\geq 0}$. By continuity, the linear functionals $\mathcal{L}_n$, $n \geq 0$, can be extended to the space of formal power series $\mathbb{C}[[X]]$. Then the function f can be represented by the formal series

$$f(z) = \sum_{n=0}^{\infty} \langle \mathcal{L}_n, f\rangle Q_n(z). \tag{3.1}$$

There is an extensive literature devoted to the study of the convergence of this type of expansions, in particular [3], where the authors used the so-called *"method of kernel expansion"* to express explicitly $\langle \mathcal{L}_n, f\rangle$. This method consists in choosing a suitable sequence of functions $g_n(\omega)$ to define the kernel

$$K(z,\omega) = \sum_{n=0}^{\infty} Q_n(z) g_n(\omega).$$

If $f(z) = \frac{1}{2i\pi}\int_\Gamma K(z,\omega)F(\omega)d\omega$ for a suitable function F and a closed contour Γ, then

$$\langle \mathcal{L}_n, f\rangle = \frac{1}{2i\pi}\int_\Gamma Q_n(\omega)F(\omega)d\omega.$$

Now, if we apply Theorem 1.1 we obtain a further method to construct the sequence $\{\mathcal{L}_n\}_{n\geq 0}$ for the general case from which we deduce the following expansion theorem.

Theorem 3.1. *Let $\{Q_n\}_{n\geq 0}$ be a polynomial set with lowering operator σ. Then there exists a power series $\varphi(t) = \sum_{k=0}^{\infty} \alpha_n t^n$, $\alpha_0 \neq 0$, such that every analytic function f has the expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{\sigma^n \varphi(\sigma) f(0)}{n!} Q_n(z). \tag{3.2}$$

If, moreover, the translation operator T_a commutates with σ, then

$$f(z+a) = \sum_{n=0}^{\infty} \frac{\sigma^n \varphi(\sigma) f(a)}{n!} Q_n(z). \tag{3.3}$$

Notice that the problem of the convergence of series (3.2) or (3.3) is to be studied separately for each polynomial set $\{Q_n\}_{n\geq 0}$. All the operations we perform in this paper are formal and we pay no attention to convergence problems. Next, we apply Theorem 3.1 to some classes of polynomial sets given by their generating functions.

Corollary 3.1. *Let $\{Q_n\}_{n\geq 0}$ be a Boas–Buck polynomial set generated by the formal relation* [3]

$$G(x,t) = A(t)B\left(xC(t)\right) = \sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} t^n, \tag{3.4}$$

where

$$A(t)=\sum_{k=0}^{\infty} a_k t^k, \quad B(z)=\sum_{k=0}^{\infty} b_k z^k, \quad \textit{and} \quad C(t)=\sum_{k=0}^{\infty} c_k t^{k+1} \tag{3.5}$$

are three formal power series with the condition $a_0 c_0 b_k \neq 0$ for all k.

Let $\nu := \nu_x \in \Lambda^{(-1)}$ such that

$$\nu B(xt) = tB(xt). \tag{3.6}$$

Put $\sigma = C^*(\nu)$ where C^* is the inverse of C, i.e.,

$$C^*(C(t)) = C(C^*(t)) = t, \quad \textit{with} \quad C^*(t)=\sum_{n=0}^{\infty} c_n^* t^{n+1}; \; c_0^* \neq 0.$$

Then every analytic function f has the expansion

$$f(z)=\sum_{n=0}^{\infty} \frac{\sigma^n}{A(\sigma)} f(0) \frac{Q_n(z)}{n!}. \tag{3.7}$$

If, moreover, the translation operator T_a commutates with σ then

$$f(z+a)=\sum_{n=0}^{\infty} \frac{\sigma^n}{A(\sigma)} f(a) \frac{Q_n(z)}{n!}. \tag{3.8}$$

Proof. From Corollary 3.2 in [2], it follows that $\sigma Q_n = nQ_{n-1}$. It is easy to verify that the polynomial set $\{P_n(x)\}_{n\geq 0}$ generated by

$$G_0(x,t) = B\left(xC(t)\right) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \tag{3.9}$$

is a sequence of basic polynomials for σ. From the relations $G_0(x,t)=\frac{1}{A(t)}G(x,t)$ and $\sigma^k Q_n = \frac{n!}{(n-k)!}Q_{n-k}$, we deduce, according to the notation of Theorem 3.1, that $\varphi(\sigma)=\frac{1}{A(\sigma)}$, and consequently, expansion (3.2) is reduced to (3.7). □

Notice that some works, based on umbral calculus, stated results similar to this theorem. The authors took as starting points invariant operators or shift-invariant operators (cf., for instance, [9, 10, 11, 13]).

Two particular cases of Boas–Buck polynomial sets are worth to note. The first one is the Brenke set where $C(t) = t$; then $\sigma = \nu$. The second one corresponds to Sheffer polynomials, where $B(t) = e^t$; then $\nu = D$, the derivative operator. Many examples of polynomial expansions of analytic functions in the literature may be deduced from Corollary 3.2 applied to Sheffer polynomials. Below, we recall some of them.

Example 1: Taylor series. The polynomial set $\{Q_n(x) = x^n\}_{n\geq 0}$ is generated by $G_0(x,t) = e^{xt}$. For this case, we have $\sigma = D$ and $A(\sigma) = 1$; then expansion (3.7) is reduced to the well known Taylor series

$$f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \tag{3.10}$$

Example 2: Newton series. The polynomial set $\left\{Q_n(x) = n!\binom{x}{n}\right\}_{n\geq 0}$ is generated by $G_0(x,t) = \exp\left(x\log(1+t)\right)$. For this case, we have $C(t) = \log(1+t)$ so $C^*(t) = e^t - 1$ and

$$\sigma = C^*(D) = e^D - 1 = T_1 - 1 = \Delta,$$

where Δ is the difference operator defined by $\Delta f(x) = f(x+1) - f(x)$, f being an analytic function. Then expansion (3.7) is reduced to the Newton series [7]

$$f(z) = \sum_{n=0}^{\infty} \Delta^n f(0)\binom{z}{n} = \sum_{n=0}^{\infty} \frac{\Delta^n f(0)}{n!} z(z-1)\dots(z-n+1), \tag{3.11}$$

where

$$\Delta^n f(0) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k).$$

Example 3: Abel series. The polynomial set $\{Q_n(x) = x(x-n\beta)^{n-1}\}_{n\geq 0}$ is generated by $G_0(x,t) = \exp\left(xC(t)\right)$ where $C(t)$ is such that $C^*(t) = te^{\beta t}$. For this case, we have $\sigma = C^*(D) = De^{\beta D} = DT_\beta$. Then expansion (3.7) is reduced to the Abel series [15]

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(n\beta)}{n!} z(z-n\beta)^{n-1}. \tag{3.12}$$

Example 4: Buck series. The polynomial set

$$\left\{Q_n(x) = (n-1)!x\binom{x-\beta n-1}{n-1}\right\}_{n\geq 0}$$

is generated by $G_0(x,t) = \exp\left(xC(t)\right)$ where $C(t)$ is such that $C^*(t) = \left(e^t - 1\right)e^{\beta t}$. For this case, we have

$$\sigma = C^*(D) = \left(e^D - 1\right)e^{\beta D} = \left(T_1 - 1\right)T_\beta = \Delta T_\beta.$$

Then expansion (3.7) is reduced to the Buck series [4]

$$f(z) = f(0) + \sum_{n=1}^{\infty} \Delta^n f(\beta n)\frac{z}{n}\binom{z-\beta n-1}{n-1}, \tag{3.13}$$

where

$$\Delta^n f(\beta n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(\beta n + k).$$

For $\beta = 0$ (resp. $\beta = -\frac{1}{2}$ or $\beta = 1$), we obtain the Newton series (resp. the Stirling series or the Gelfond series).

Example 5: Bernoulli series. The Bernoulli polynomial set $\{Q_n = B_n\}_{n\geq 0}$ is generated by $G(x,t) = \frac{t}{e^t-1}\exp(xt)$. For this case, we have $\sigma = D$ and $\frac{1}{A(\sigma)} = \frac{e^D-1}{D} = \frac{\Delta}{D}$. Then expansion (3.8) is reduced to the Bernoulli series [7]

$$f(z+u) = \int\limits_u^{u+1} f(t)dt + \sum_{n=1}^{\infty} \frac{\Delta D^{n-1} f(u)}{n!} B_n(z). \tag{3.14}$$

Example 6: Boole series. The Boole polynomial set $\{Q_n = \xi_n\}_{n\geq 0}$ is generated by $G(x,t) = \frac{1}{1+\frac{t}{2}}\exp(x\log(1+t))$. For this case, we have $\sigma = \Delta$ (as for the second example) and $\frac{1}{A(\sigma)} = 1 + \frac{1}{2}\Delta = M$ where M designates the operator of the mean that is $Mf(x) = \frac{1}{2}(f(x) + f(x+1))$. Then expansion (3.8) is reduced to the Boole series [7]

$$f(z+u) = \sum_{n=0}^{\infty} \frac{M\Delta^n f(u)}{n!} \xi_n(z). \tag{3.15}$$

Example 7: Frappier series. The Frappier polynomial set $\{Q_n = \Phi_{n,\alpha}\}_{n\geq 0}$ is generated by $G_0(x,t) = \exp(xC(t))$, where $C(t)$ is such that

$$C^*(t) = \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{g_\alpha\left(\frac{it}{2}\right)}$$

where $g_\alpha(z) = 2^\alpha \Gamma(\alpha+1) J_\alpha(z)/z^\alpha$ and

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha+2k}}{2^{\alpha+2k} k! \Gamma(\alpha+k+1)}$$

is the first kind Bessel function of order α. For this case, the lowering operator, denoted by b_α in [6], is given by

$$\sigma_\alpha = \sum_{k=0}^{\infty} \frac{\left((-1)^k - 1\right)}{k!} B_{k,\alpha} D^k,$$

where the coefficients $B_{k,\alpha}$ are defined by the relation

$$\frac{e^{-\frac{t}{2}}}{g_\alpha\left(\frac{it}{2}\right)} = \sum_{k=0}^{\infty} \frac{B_{k,\alpha}}{k!} t^k.$$

Here, expansion (3.8) is reduced to the Frappier series [6]

$$f(z+u) = \sum_{n=0}^{\infty} \frac{\sigma_\alpha^n f(u)}{n!} \Phi_{n,\alpha}(z). \tag{3.16}$$

For $\alpha = \frac{1}{2}$, we have the Taylor series.

4. Concluding Remarks

In this section, we discuss some problems arising in the theory of polynomials, and for which the obtained results in this paper can be useful.

1. Connection and linearization problems. Let $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be two polynomial sets. The so-called *connection problem* between them asks to find the coefficients $C_m(n)$ in the expression

$$R_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x).$$

Let $\{Q_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ be three polynomial sets. The so-called *linearization problem* asks to find the coefficients L_{ijk} in the expression

$$R_i(x)S_j(x) = \sum_{k=0}^{i+j} L_{ijk} Q_k(x).$$

The literature on these topics is extremely vast and a wide variety of methods based on specific properties of the involved polynomials have been devised for computing the linearization and connection coefficients (see, for instance, [1, 12] and the references therein).

From Theorem 3.1 we derive then a simple and general method to compute the coefficients $C_m(n)$ and L_{ijk} which consists in putting in (3.2) $f = R_n$ and $f = R_iS_j$, respectively. This approach does not need particular properties of the polynomials involved in the problem.

2. d-orthogonality and d-dimensional functionals. Let $\{Q_n\}_{n\geq 0}$ be a polynomial set. The corresponding monic polynomial sequence $\{\widehat{Q}_n\}_{n\geq 0}$ is given by $Q_n = \lambda_n \widehat{Q}_n$, $n \geq 0$, where λ_n is the normalization coefficient and let $\{\mathcal{L}_n\}_{n\geq 0}$ be its dual sequence. Let d be an arbitrary positive integer. $\{Q_n\}_{n\geq 0}$ is called a d-orthogonal polynomial set with respect to the d-dimensional functional $\mathcal{L} = {}^t(\mathcal{L}_0, \cdots, \mathcal{L}_{d-1})$ if it fulfils [8,14]

$$\begin{cases} \langle \mathcal{L}_k, Q_m Q_n \rangle = 0\ , \quad m > dn + k\ , n \geq 0, \\ \langle \mathcal{L}_k, Q_n Q_{dn+k} \rangle \neq 0\ ,\ n \geq 0, \end{cases} \tag{4.1}$$

for each integer k belonging to $\{0, 1, \ldots, d-1\}$.

The orthogonality conditions (4.1) are equivalent to the fact that the sequence $\{Q_n\}_{n\geq 0}$ satisfies a $(d+1)$-order recurrence relation [14] which we write in the *monic* form

$$\widehat{Q}_{m+d+1}(x) = (x - \beta_{m+d})\widehat{Q}_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} \widehat{Q}_{m+d-1-\nu}(x)\ ,\ m \geq 0, \tag{4.2}$$

with the initial conditions

$$\begin{cases} \widehat{Q}_0(x) = 1 \quad , \quad \widehat{Q}_1(x) = x - \beta_0 \quad \text{and} \quad \text{if} \quad d \geq 2: \\ \widehat{Q}_n(x) = (x - \beta_{n-1})\widehat{Q}_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} \widehat{Q}_{n-2-\nu}(x)\ ,\ 2 \leq n \leq d, \end{cases} \tag{4.3}$$

and the regularity conditions

$$\gamma_{n+1}^0 \neq 0\,, \quad n \geq 0.$$

One of the problems related to these notions consists *in expressing explicitly the d-dimensional functional* $\mathcal{L} = {}^t(\mathcal{L}_0, \cdots, \mathcal{L}_{d-1})$ if the recurrence relation is given.

This problem, for the classical case (in Hahn's sense) was treated by Douak and Maroni in [5]. A different approach may be deduced from Theorem 1.1 for the general case.

Acknowledgements

The paper was started while the author was visiting University of Roma "La Sapienza". Thanks are due to Professor P. E. Ricci for the hospitality and for many enlightening discussions, and to the University for the support.

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(Received 3.12.2001)

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