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# NONEXPANSIVE MAPPINGS AND ITERATIVE METHODS IN UNIFORMLY CONVEX BANACH SPACES

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**Abstract.** In this paper, most of classical and modern convergence theorems of iterative schemes for nonexpansive mappings are presented and the main results in the paper generalize and improve the corresponding results given by many authors.

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## 1. Introduction and Preliminaries

In this paper, we will present several strong and weak convergence results of successive approximations to fixed points of nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize and improve various ones concerned with constructive techniques for approximations of fixed points of nonexpansive mappings (cf. [1]–[18]).

Let $X$ be a real normed linear space and $C$ be a nonempty subset of $X$. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. Furthermore, $T$ is said to be *quasi-nonexpansive* if the set $F(T)$ of fixed points of $T$ is not empty, for all $x \in D(T)$ and, for $y \in F(T)$,

$$\|Tx - y\| \le \|x - y\|. \tag{1.1}$$

*Remark* 1.1. Nonexpansive mappings with the nonempty fixed point set $F(T)$ are quasi-nonexpansive, and linear quasi-nonxpansive mappings are nonexpansive, but it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive, for example, consider a mapping $T$ defined by $Tx = (\frac{x}{2})\sin(\frac{1}{x})$ with $T(0) = 0$ on $R$.

There are two important connections between the classes of nonexpansive mappings and accretive mappings which give rise to a strong connection between the fixed point theory of nonexpansive mappings and the theory of accretive mappings.

Note that

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(1) If $T$ is a nonexpansive mapping of $D(T)$ into $X$ and if we set $U = I - T$ $D(U) = D(T)$, then $U$ is an accretive mapping of $D(U)$ into $X$.

(2) If $\{U(t) : t \geq 0\}$ is a semigroup of (nonlinear) mappings of $X$ into itself with infinitesimal generator $T$, then all the mappings $U(t)$ are nonexpansive if and only if $(-T)$ is accretive.

Let $X$ be a normed linear space, $C$ be a nonempty convex subset of $X$ and $T : C \to C$ be a given mapping. Then, for arbitrary $x_1 \in C$, the *Ishikawa iterative scheme* $\{x_n\}$ is defined by

$$\begin{cases} y_n = (1 - s_n)x_n + s_n T x_n, & n \geq 1, \\ x_{n+1} = (1 - t_n)x_n + t_n T y_n, & n \geq 1, \end{cases} \tag{IS}$$

where $\{s_n\}$ and $\{t_n\}$ are some suitable sequences in $[0, 1]$. With $X$, $C$, $\{t_n\}$ and $x_1$ as above, the *Mann iterative scheme* $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - t_n)x_n + t_n T x_n, & n \geq 1. \end{cases} \tag{M}$$

We now begin with a serial of lemmas which will be needed in the sequel for the proof of our main theorems:

**Lemma 1.1.** *Let $X$ be a real normed linear space, $C$ be a nonempty convex subset of $X$ and $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is the iterative scheme defined by* (IS)*, then the following holds:*

$$\|x_{n+1} - Tx_{n+1}\| \leq (1 + 2\tau_n)\|x_n - Tx_n\|$$

*for all $n \geq 1$, where $\tau_n = \min\{t_n, 1 - t_n\}s_n$. In particular, $\|x_{n+1} - Tx_{n+1}\| \leq \|x_n - Tx_n\|$ if $s_n \equiv 0$ and $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists.*

*Proof.* By Tan and Xu [15, Lemma 3], we have the following estimation:

$$\|x_{n+1} - Tx_{n+1}\| \leq [1 + 2s_n(1 - t_n)]\|x_n - Tx_n\|. \tag{1.2}$$

Observe that

$$\begin{aligned} \|x_n - Ty_n\| &\leq \|x_n - Tx_n\| + \|x_n - y_n\| \leq \|x_n - Tx_n\| + s_n\|x_n - Tx_n\| \\ &= (1 + s_n)\|x_n - Tx_n\|. \end{aligned} \tag{1.3}$$

It follows that

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq (1 - t_n)\|x_n - Tx_{n+1}\| + t_n\|Ty_n - Tx_{n+1}\| \\ &\leq (1 - t_n)(\|x_n - Tx_n\| + \|x_n - x_{n+1}\|) + t_n\|y_n - x_{n+1}\| \\ &\leq (1 - t_n)\|x_n - Tx_n\| + \|x_n - x_{n+1}\| + t_n s_n\|x_n - Tx_n\| \\ &\leq (1 + 2t_n s_n)\|x_n - Tx_n\|. \end{aligned} \tag{1.4}$$

Combining (1.2) with (1.4), we reach the desired conclusion. This completes the proof. □

**Lemma 1.2.** *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + b_n)a_n \tag{1.5}$$

*for all* $n \geq 1$. *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists. In particular,* $a_n \to 0$ *as* $n \to \infty$ *whenever* $\{a_n\}$ *admits a subsequence which converges to 0.*

*Proof.* For $n, m \geq 1$, we have

$$a_{n+m+1} \leq (1 + b_{n+m})a_{n+m} \leq \cdots \leq \prod_{j=n}^{n+m} (1 + b_j)a_n, \tag{1.6}$$

which infers that $\{a_n\}$ is bounded since $\sum_n b_n$ converges. Set $M = \sup\{a_n : n \geq 1\}$. Then (1.5) reduces to

$$a_{n+1} \leq a_n + Mb_n$$

for all $n \geq 1$. Now the conclusion follows from Lemma 1 of Tan and Xu [15]. This completes the proof. □

The *modulus of convexity* of a real Banach space $X$ is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\|,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. $X$ is said to be *uniformly convex* if $\delta_X(0) = 0$ and $\delta_X(\epsilon) > 0$ for all $0 < \varepsilon \leq 2$.

**Lemma 1.3.** *Let* $X$ *be a uniformly convex Banach space with the modulus of uniform convexity* $\delta_X$. *Then* $\delta_X : [0, 2] \to [0, 1]$ *is a continuous and increasing function with* $\delta_X(0) = 0$ *and* $\delta_X(t) > 0$ *for* $t > 0$ *and, further,*

$$\|cu + (1-c)v\| \leq 1 - 2\min\{c, 1-c\}\delta_X(\|u - v\|) \tag{1.7}$$

*whenever* $0 \leq c \leq 1$ *and* $\|u\|,\ \|v\| \leq 1$.

*Proof.* See Bruck [3]. □

Let $X$ be a real Banach space with a norm $\|\cdot\|$ and $S = \{x \in X : \|x\| = 1\}$ be its unit sphere. The norm of $X$ is said to be *Fréchet differentiable* if, for each $x \in S$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in S$. For $x \neq 0$ and $y \in X$, we denote this limit by $(x, y)$. It is known that $X$ has a Fréchet differentiable norm if and only if $X$ is strongly smooth. In this case, the normalized duality mapping $J : X \to X^*$ is single-valued and continuous from the strong topology of $X$ to the strong topology of $X^*$. As a matter of facts above mentioned, it is clear that

$$(x, c_1y_1 + c_2y_2) = c_1(x, y_1) + c_2(x, y_2)$$

for all $c_1,\ c_2 \in R$, $x \in X$ with $x \neq 0$ and $y_1,\ y_2 \in X$.

**Lemma 1.4 ([11]).** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ with a Frechet differentiable norm and $\{T_n : n \geq 1\}$ be a family of nonexpansive self-mappings of $C$ with the nonempty common fixed point set $F$. If $x_1 \in C$ and define a sequence $\{x_n\}$ in $C$ by $x_{n+1} = T_n x_n$ for all $n \geq 1$, then $\lim_{n\to\infty}(f_1 - f_2, x_n)$ exists for all $f_1, f_2 \in F$ with $f_1 \neq f_2$.*

## 2. The Main Results

Now we prove the following convergence theorems:

**Theorem 2.1.** *Let $X$ be a real uniformly convex Banach space, $C$ be a nonempty closed convex (not necessarily bounded) subset of $X$ and $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\}$ be the iterative scheme defined by* (IS) *with the restrictions that $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$, where $\tau_n$ is as in Lemma* 1.1*. Then, for arbitrary initial value $x_1 \in C$, $\{\|x_n - Tx_n\|\}$ converges to same constant $r_C(T)$, which is independent of the choice of the initial value $x_1 \in C$. In particular, if $\tau_n \equiv 0$, then*

$$r_C(T) = \inf\{\|x - Tx\| : x \in C\}.$$

*Proof.* It follows from Lemmas 1.1 and 1.2 that $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists. We denote this limit by $r(x_1)$. Let $\{x_n^*\}$ be another Ishikawa iterative scheme defined by (IS) with the same restrictions on parameters $\{t_n\}$ and $\{s_n\}$ as the sequence $\{x_n\}$ but with the initial value $x_1^* \in C$. Then $r(x_1^*) = \lim_{n\to\infty} \|x_n^* - Tx_n^*\|$.

It suffices to prove that $r(x_1) = r(x_1^*)$. Since $\|x_{n+1} - x_{n+1}^*\| \leq \|x_n - x_n^*\|$, we have $\lim_{n\to\infty} \|x_n - x_n^*\|$ exists, which denoted this limit by $d$. Without loss of generality, we assume that $d > 0$. Observe that

$$x_{n+1} - x_{n+1}^* = t_n(Ty_n - Ty_n^*) + (1 - t_n)(x_n - x_n^*)$$

and

$$\|Ty_n - Ty_n^*\| \leq \|x_n - x_n^*\|.$$

By using Lemma 1.3, we have

$$\|x_{n+1} - x_{n+1}^*\| \leq \left[1 - 2t_n(1 - t_n)\delta_X\left(\frac{\|x_n - x_n^* - (Ty_n - Ty_n^*)\|}{\|x_n - x_n^*\|}\right)\right]\|x_n - x_n^*\|,$$

which implies that

$$\sum_{n=1}^{\infty} t_n(1 - t_n)\delta_X\left(\frac{\|x_n - x_n^* - (Ty_n - Ty_n^*)\|}{\|x_n - x_n^*\|}\right) < \infty.$$

Since $t_n(1 - t_n)s_n \leq \tau_n$ and $\sum_{n=1}^{\infty} \tau_n < \infty$, we have $\sum_{n=1}^{\infty} t_n(1 - t_n)s_n < \infty$ and hence

$$\sum_{n=1}^{\infty} t_n(1 - t_n)\left[\delta_X\left(\frac{\|x_n - x_n^* - (Ty_n - Ty_n^*)\|}{\|x_n - x_n^*\|}\right) + s_n\right] < \infty.$$

Since $\sum_{n=1}^{\infty} t_n(1-t_n)=\infty$, we conclude

$$\liminf_{n\to\infty}\left[\delta_X\left(\frac{\|x_n-x_n^*-(Ty_n-Ty_n^*)\|}{\|x_n-x_n^*\|}\right)+s_n\right]=0.$$

Since $\delta_X$ is strictly increasing and continuous and $\lim_{n\to\infty}\|x_n-x_n^*\|=d>0$, we have

$$\lim_{k\to\infty}\|x_{n_k}-x_{n_k}^*-(Ty_{n_k}-Ty_{n_k}^*)\|=0$$

and $\lim_{k\to\infty} s_{n_k}=0$. On the other hand, we have

$$\begin{aligned}|\|x_{n_k}-Tx_{n_k}\|-\|x_{n_k}^*-Tx_{n_k}^*\||&\le\|(x_{n_k}-Tx_{n_k})-(x_{n_k}^*-Tx_{n_k}^*)\|\\&\le\|x_{n_k}-x_{n_k}^*-(Ty_{n_k}-Ty_{n_k}^*)\|+\|Tx_{n_k}-Ty_{n_k}\|+\|Tx_{n_k}^*-Ty_{n_k}^*\|\\&\le\|x_{n_k}-x_{n_k}^*-(Ty_{n_k}-Ty_{n_k}^*)\|\\&\quad+s_{n_k}(\|x_{n_k}-Tx_{n_k}\|+\|x_{n_k}^*-Tx_{n_k}^*\|),\end{aligned}\tag{2.1}$$

which converges to 0 as $k\to\infty$. This leads to

$$\lim_{n\to\infty}|\|x_n-Tx_n\|-\|x_n^*-Tx_n^*\||=0.$$

Therefore, we have $r(x_1)=r(x_1^*)$.

If $\tau_n\equiv 0$, then $\|x_{n+1}-Tx_{n+1}\|\le\|x_n-Tx_n\|$ for all $n\ge 1$ and so $r(x_1)\le\|x_1-Tx_1\|$ for every $x_1\in C$, which implies that

$$r_C(T)=\inf\{\|x-Tx\|:x\in C\}.$$

This completes the proof. □

**Theorem 2.2.** *Let $X$, $C$ and $T$ be as in Theorem 2.1. Then the following statements are equivalent:*

(i) *$F(T)\ne\varnothing$.*
(ii) *For any specific $x_1\in C$, the sequence $\{x_n\}$ of Picard iterates defined by $x_n=T^n x_1$ starting at $x_1$ is bounded in $C$.*
(iii) *For every $x_1\in C$, the Ishikawa iterative scheme $\{x_n\}$ defined by* (IS) *with the restrictions that $t_n\to t>0$ and $s_n\to s<1$ as $n\to\infty$ is bounded.*
(iv) *There is a bounded sequence $\{y_n\}\subset C$ such that $y_n-Ty_n\to 0$ as $n\to\infty$.*
(v) *For every $x_1\in C$, the Ishikawa iterative scheme $\{x_n\}$ defined by* (IS) *with the restrictions that $0\le t_n\le t<1$, $\sum_{n=1}^{\infty}t_n=\infty$ and $\sum_{n=1}^{\infty}s_n<\infty$ is bounded.*

*Moreover, let $\{x_n\}$ be the sequence as in Theorem 2.1. Then $\|x_n-Tx_n\|\to 0$ as $n\to\infty$ provided that one of the above conditions* (i)∼(v) *holds.*

*Proof.* For (i)⟺(ii), we may see Browder and Petryshyn [1, Theorem 1, p. 571]. For (i)⟺(v) and (i)⟺(iv), we may consult with the result of Deng [6] and references there.

Now, we will prove the equivalence of (i) and (iii) only. It is sufficient to prove that $\{T^m z\}$ is bounded for all $z \in C$. Now we choose $z \in C$. For arbitrary but fixed $m \geq 1$, using (IS), we have

$$\|y_n - T^{m-1}z\| \leq (1 - s_n)\|x_n - T^{m-1}z\| + s_n\|x_n - T^{m-2}z\| \tag{2.2}$$

and

$$\begin{aligned}\|x_{n+1} - T^m z\| &\leq (1 - t_n)\|x_n - T^m z\| + t_n\|y_n - T^{m-1}z\| \\ &\leq (1 - t_n)\|x_n - T^m z\| + t_n(1 - s_n)\|x_n - T^{m-1}z\| \\ &\quad + t_n s_n\|x_n - T^{m-2}z\|. \end{aligned}\tag{2.3}$$

Since $\{x_n\}$ is bounded, by taking lim sup on the both sides of (2.3), we have

$$\limsup_{n\to\infty} \|x_n - T^m z\| \leq (1 - s) \limsup_{n\to\infty} \|x_n - T^{m-1}z\| + s \limsup_{n\to\infty} \|x_n - T^{m-2}z\|.$$

Set $a_m = \limsup_{n\to\infty} \|x_n - T^m z\|$. Then it follows that

$$a_m \leq (1 - s)a_{m-1} + s a_{m-2}. \tag{2.4}$$

By induction, we can prove that

$$a_m \leq \left[\sum_{j=0}^{m-1}(-1)^j s^j\right]a_1 + \left[s\sum_{j=0}^{m-2}(-1)^j s^j\right]a_0 \tag{2.5}$$

for all $m \geq 2$. Indeed, it is true for $m = 2$. Now we assume that it is true for the integers that are less than $m$ and we have to prove that it is also true for $m$. By induction assumption, we have

$$a_{m-1} \leq \left[\sum_{j=0}^{m-2}(-1)^j s^j\right]a_1 + \left[s\sum_{j=0}^{m-3}(-1)^j s^j\right]a_0 \tag{2.6}$$

and

$$a_{m-2} \leq \left[\sum_{j=0}^{m-3}(-1)^j s^j\right]a_1 + \left[s\sum_{j=0}^{m-4}(-1)^j s^j\right]a_0. \tag{2.7}$$

Substituting (2.6) and (2.7) in (2.4) yields that

$$\begin{aligned} a_m &\leq (1 - s)\left\{\left[\sum_{j=0}^{m-2}(-1)^j s^j\right]a_1 + \left[s\sum_{j=0}^{m-3}(-1)^j s^j\right]a_0\right\} + s\left\{\left[\sum_{j=0}^{m-3}(-1)^j s^j\right]a_1 \right. \\ &\quad \left. + \left[s\sum_{j=0}^{m-4}(-1)^j s^j\right]a_0\right\} \leq \left[\sum_{j=0}^{m-1}(-1)^j s^j\right]a_1 + \left[s\sum_{j=0}^{m-2}(-1)^j s^j\right]a_0, \end{aligned}\tag{2.8}$$

which proves our claim. It follows from (2.8) that

$$a_m \leq \frac{1}{1+s}a_1 + \frac{s}{1+s}a_0 \leq \max\{a_0, a_1\}$$

for all $m \geq 1$, which proves the boundedness of $\{T^m z\}$. This completes the proof. $\square$

Recall that a Banach space $X$ is said to satisfy *Opial's condition* [10] if, for each sequence $\{x_n\}$ in $X$, the condition $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known from [10] that all $l^p$ spaces for $1 < p < \infty$ have this property. However, the $L^p$ space do not have unless $p = 2$.

We say also that a mapping $T : C \to C$ is said to satisfy the *Condition (A)* if $F(T) \neq \varnothing$ and there is a non-decreasing function $f : R^+ \to R^+$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - z\| : z \in F(T)\}$.

**Theorem 2.3.** *Let $X$ be a real uniformly convex Banach space with a Fréchet differentiable norm or which satisfies Opial's condition, $C$ be a closed convex subset of $X$ and $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. Then, for any initial value $x_1 \in C$, the Ishikawa iterative scheme $\{x_n\}$ defined by* (IS) *with the restrictions that $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$, where $\tau_n = \min\{t_n, 1-t_n\}s_n$, converges weakly to a fixed point of $T$.*

*Proof.* Set $T_n x = t_n T(s_n Tx + (1 - s_n)x) + (1 - t_n)x$ for all $x \in C$ and $n \geq 1$. Then the mapping $T_n : C \to C$ is also nonexpansive and the Ishikawa iterative scheme $\{x_n\}$ defined by (IS) can be written as

$$x_{n+1} = T_n x_n$$

for all $n \geq 1$. Furthermore, we have $F(T) \subset F(T_n)$ for all $n \geq 1$. Let $\omega_w(x_n)$ denote the weak $\omega$-lim set of the sequence $\{x_n\}$. Since $F(T) \neq \varnothing$, by Theorem 2.2, we see that

$$x_n - Tx_n \to 0$$

as $n \to \infty$. Without loss of generality, assume that $x_n \rightharpoonup p$ as $n \to \infty$. By Browder's demi-closedness principle, we assert that $Tp = p$, which gives that $\omega_w(x_n) \subset F(T)$.

To show that $\{x_n\}$ converges weakly to a fixed point of $T$, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. To this end, we first assume that $X$ satisfies Opial's condition and suppose that $p \neq q$ are in $\omega_w(x_n)$. Then $x_{n_k} \to p$ and $x_{m_k} \to q$ respectively. Since $\lim_{n\to\infty} \|x_n - z\|$ exists for any $z \in F(T)$, by Opial's condition, we conclude

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\ &= \lim_{j\to\infty} \|x_{m_j} - q\| < \lim_{j\to\infty} \|x_{m_j} - p\| = \lim_{n\to\infty} \|x_n - p\|, \end{aligned}$$

which is a contradiction.

We now assume that $X$ has a Fréchet differentiable norm and suppose that $f_1, f_2 \in \omega_w(x_n)$ with $f_1 \neq f_2$. It follows from Lemma 1.4 that $(f_1 - f_2, f_1) =$

$(f_1 - f_2, f_2)$, which means that $f_1 = f_2$, which is a contradiction again. This completes the proof. □

**Theorem 2.4.** *Let $X$, $C$, $T$ and $\{x_n\}$ be as in Theorem* 2.1*. Suppose, in addition, that $F(T) \neq \varnothing$ and $T$ satisfies the Condition $(A)$. Then the iterative scheme $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* It follows from Theorem 2.2 that $x_n - Tx_n \to 0$ as $n \to \infty$. In view of the Condition (A), we have

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T)))$$

for all $n \geq 1$, which implies that $d(x_n, F(T)) \to 0$ as $n \to \infty$. This leads to $x_n \to p \in F(T)$ as $n \to \infty$. This completes the proof. □

**Corollary 2.1.** *Let $X$, $C$, $T$ and $\{x_n\}$ be as in Theorem* 2.1*. Suppose, in addition, that $T(C)$ is relatively compact. Then the iterative scheme $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* It follows from Schauder's Fixed Point Theorem that $T$ has at least a fixed point in $C$. Thus $F(T) \neq \varnothing$. Also it is easily seen that $T$ is demicompact, i.e., whenever $\{x_n\} \subset C$ is bounded and $\{x_n - Tx_n\}$ converges strongly then there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly. Since $T$ is continuous as well as demicompact, according to Opial [10], the mapping $(I-T)$ maps closed bounded subsets of $C$ onto closed subsets of $X$. In view of Senter and Dotson [14, Lemma 1], we see that $T$ must satisfy the Condition (A) with respect to the sequence $\{x_n\}$. Now Theorem 2.4 can be used to deduce the conclusion of the corollary. This completes the proof. □

*Remark* 2.1. Theorem 2.4 shows that the Condition (A) is a sufficient condition which guarantees convergence of the iterates $\{x_n\}$ defined by (IS). This condition is also necessary. Indeed, suppose that $x_n \to p \in F(T)$. Let $A = I-T$. By Xu and Roach [16, Theorem 2], the mapping $A$ satisfies the Condition (I) (cf. [16]). Thus, from Xu and Roach [16, Theorem 1], there must be a strictly increasing function $f : R^+ \to R^+$ with $f(0) = 0$ and $j(x_n - p) \in J(x_n - p)$ such that

$$\langle x_n - Tx_n, j(x_n - p)\rangle \geq f(\|x_n - p\|)\|x_n - p\|,$$

which implies that

$$\|x_n - Tx_n\| \geq f(\|x_n - p\|) \geq f(d(x_n, F(T))).$$

This establishes the necessity.

For quasi-nonexpansive mappings, we have the following:

**Theorem 2.5.** *Let $X$ be a real uniformly convex Banach space, $C$ be a nonempty closed convex subset of $X$ and $T : C \to C$ be a quasi-nonexpansive mapping. Let $\{x_n\}$ be the Ishikawa iterative scheme defined by* (IS) *with the restriction that $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$ and $Tx_n - Ty_n \to 0$ as $n \to \infty$. Then the iterative scheme $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $T$ satisfies the Condition* (A) *with respect to the sequence $\{x_n\}$.*

*Proof.* (⇒) As shown in Remark 2.1, $A = I - T$ satisfies the Condition (I) and $T$ satisfies the Condition (A) with respect to the sequence $\{x_n\}$ and so the necessity follows.

(⇐) By Tan and Xu [15, Lemma 3], we have

$$\liminf_{n\to\infty} \|x_n - Ty_n\| = 0.$$

By virtue of assumption that $Tx_n - Ty_n \to 0$, we conclude

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

It follows from the Condition (A) that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $d(x_{n_j}, F(T)) \to 0$ as $j \to \infty$. Therefore, we have $d(x_n, F(T)) \to 0$ as $n \to \infty$ since $\lim_{n\to\infty} d(x_n, F(T))$ exists. This leads to the desired conclusion. This completes the proof. □

**Corollary 2.2.** *Let $X$, $C$, $T$ and $\{x_n\}$ be as in Theorem* 2.5*. Suppose in addition that $T$ is uniformly continuous and $s_n \to 0$ as $n \to \infty$. Then the conclusion of Theorem* 2.5 *holds.*

*Proof.* Observe that

$$x_n - y_n = s_n(x_n - Tx_n),$$

while $\{\|x_n - Tx_n\|\}$ is bounded and so $x_n - y_n \to 0$ as $n \to \infty$. By the uniform continuity of $T$, we conclude that $Tx_n - Ty_n \to 0$ as $n \to \infty$. Now the conclusion of the corollary follows from Theorem 2.5. □

*Remark* 2.2. Theorem 2.5 improves and generalizes the corresponding results by Tan and Xu [15, Theorems 2, 3], Xu and Roach [16, Theorem 2], Zhou and Jia [18, Theorem 2] and others.

*Remark* 2.3. All the results in this paper can be extended to the case in which the Ishikawa iterative scheme defined by (IS) admits errors in the senses of Liu [7] and Xu [17].

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