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FRAME CHARACTERIZATIONS OF BESOV AND TRIEBEL–LIZORKIN SPACES ON SPACES OF HOMOGENEOUS TYPE AND THEIR APPLICATIONS

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Abstract. The author first establishes the frame characterizations of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type. As applications, the author then obtains some estimates of entropy numbers for the compact embeddings between Besov spaces or between Triebel–Lizorkin spaces. Moreover, some real interpolation theorems on these spaces are also established by using these frame characterizations and the abstract interpolation method.

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1. Introduction

It is well-known that the spaces of homogeneous type introduced by Coifman and Weiss in [3] include $\mathbb{R}^n$, the n-torus in $\mathbb{R}^n$, the C^∞-compact Riemannian manifolds, and in particular, the Lipschitz manifolds recently introduced by Triebel in [19] and the d-sets in $\mathbb{R}^n$ as special models. It has been proved by Triebel in [17] that the d-sets in $\mathbb{R}^n$ include various kinds of fractals; see also [18].

In [9], the inhomogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type were introduced by the generalized Littlewood-Paley g-functions when $p,\ q \geq 1$. In [10], the inhomogeneous Triebel–Lizorkin spaces were generalized to the cases where $p_0 < p \leq 1 \leq q < \infty$ via the generalized Littlewood-Paley S-functions, where p_0 is a positive number. The inhomogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type when $p_0 \leq p,\ q \leq 1$ were introduced in [14]. The main purpose of this paper is first to establish the frame characterizations of these spaces. Applying the frame characterization, we will then obtain some estimates of entropy numbers for the compact embeddings between Besov spaces or between Triebel–Lizorkin spaces and we will also establish some real interpolation theorems on Besov and Triebel–Lizorkin spaces by use of these frame characterizations and the abstract interpolation method.

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We mention that, recently, some new characterizations on inhomogeneous Besov and Triebel–Lizorkin spaces and their applications were given in [13] and [20]. In particular, in [20], it was proved that the Besov spaces on d-sets introduced by Triebel via traces in [17] and, equivalently, via quarkonial decompositions in [18] are the same as those Besov spaces introduced in [9] by regarding d-sets as spaces of homogeneous type.

Let us now recall some definitions and notation on spaces of homogeneous type. A quasi-metric ρ on a set X is a function $\rho:\ X \times X \to [0,\infty)$ satisfying

(i) $\rho(x, y) = 0$ if and only if $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x,\ y \in X$;
(iii) There exists a constant $A \in [1,\infty)$ such that for all $x,\ y$ and $z \in X$,

$$\rho(x, y) \le A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls

$$B(x, r) = \{y \in X:\ \rho(y, x) < r\}$$

for all $x \in X$ and all $r > 0$ form a basis.

The spaces of homogeneous type defined below, which was first introduced in [13], are the variants of the spaces of homogeneous type introduced by Coifman and Weiss in [3]. In what follows, we set diam $X = \sup\{\rho(x, y):\ x,\ y \in X\}$. We also make the following conventions. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper, we will denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. We denote $\mathbb{N} \cup \{0\}$ simply by $\mathbb{Z}_+$ and for any $q \in [1,\infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. Let A be a set and we will denote by χ_A the characteristic function of A. Also, for two topological spaces, $\mathcal{A}_1$ and $\mathcal{A}_2$, $\mathcal{A}_1 \subset \mathcal{A}_2$ means a linear and continuous embedding.

Definition 1.1. Let $d > 0$ and $0 < \theta \le 1$. A space of homogeneous type $(X,\ \rho,\ \mu)_{d,\theta}$ is a set X together with a quasi-metric ρ and a nonnegative Borel measure μ on X with $\operatorname{supp}\mu = X$ and there exists a constant $C_0 > 0$ such that for all $0 < r <$ diam X and all $x,\ x',\ y \in X$,

$$\mu(B(x, r)) \sim r^d \tag{1.1}$$

and

$$|\rho(x, y) - \rho(x', y)| \le C_0\rho(x, x')^\theta[\rho(x, y) + \rho(x', y)]^{1-\theta}. \tag{1.2}$$

Remark 1.1. From (1.1), it is easy to deduce $\mu(\{x\}) = 0$ for all $x \in X$. This means that the spaces of homogeneous type defined by Definition 1.1 are atomless measure spaces.

When diam $X < \infty$, spaces of homogeneous type in Definition 1.1 cover the boundaries of bounded Lipschitz domains, Lipschitz manifolds of compact case in [19], and compact d-sets; see [17], [18] and [20]; while when diam $X = \infty$, spaces of homogeneous type in Definition 1.1 specifically include Euclidean spaces and Lipschitz manifolds of non-compact case in [19]. Moreover, in Definition 1.1, if we choose $d = 1$, then Macias and Segovia in [15] have proved that the spaces $(X,\ \rho,\ \mu)_{d,\theta}$ are just the spaces of homogeneous type in the sense of Coifman and Weiss, whose definitions only require that ρ is a quasi-metric without (1.2) and μ satisfies the following doubling condition which is weaker than (1.1): there is a constant $C' > 0$ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \le C'\mu(B(x, r)).$$

However, in [15], Macias and Segovia have shown that for the spaces of homogeneous type in the sense of Coifman and Weiss, one can replace the original quasi-metric ρ by another quasi-metric $\bar{\rho}$, which yields the same topology on X as ρ, such that

$$\bar{\rho}(x, y) \sim \inf\{\mu(B) : B \ \text{ is a ball containing } \ x \ \text{ and } \ y\}$$

and (1.2) holds with ρ, C_0 and θ replaced, respectively, by $\bar{\rho}$, some $\bar{C}_0 > 0$ and some $\bar{\theta} \in (0, 1]$. Moreover, μ satisfies (1.1) with $d = 1$ for the balls corresponding to $\bar{\rho}$.

We now recall the definition of the spaces of test functions on X in [12]; see also [8].

Definition 1.2. Fix $\gamma > 0$ and $\theta \ge \beta > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \le C\dfrac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$;

(ii) $|f(x) - f(y)| \le C\Big(\dfrac{\rho(x, y)}{r + \rho(x, x_0)}\Big)^\beta \dfrac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$ for $\rho(x, y) \le \dfrac{1}{2A}[r + \rho(x, x_0)]$.

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by $\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C : \text{(i) and (ii) hold}\}$.

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that $\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$ with the equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Also, let the dual space $(\mathcal{G}(\beta, \gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists a finite constant $C \ge 0$ such that for all $f \in \mathcal{G}(\beta, \gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta, \gamma))'$ and $f \in \mathcal{G}(\beta, \gamma)$. It is also easy to see that $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$ if and only if $f \in \mathcal{G}(\beta, \gamma)$. Thus, for all $h \in (\mathcal{G}(\beta, \gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well-known that even when $X = \mathbb{R}^n$, $\mathcal{G}(\beta_1, \gamma)$ is not dense in $\mathcal{G}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows, we let $\mathring{\mathcal{G}}(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\theta, \theta)$ in $\mathcal{G}(\beta, \gamma)$ when $0 < \beta,\ \gamma < \theta$.

To state the definition of the inhomogeneous Besov spaces $B^s_{pq}(X)$ and the inhomogeneous Triebel–Lizorkin spaces $F^s_{pq}(X)$ studied in [14], we need the following approximations to the identity which were first introduced in [8].

Definition 1.3. A sequence $\{S_k\}_{k=0}^\infty$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0, \theta]$ if there exist $C_1,\ C_2 > 0$ such that for all $k \in \mathbb{Z}_+$ and all $x,\ x',\ y$ and $y' \in X$, $S_k(x, y)$, the kernel of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $S_k(x, y) = 0$ if $\rho(x, y) \geq C_1 2^{-k}$ and $\|S_k\|_{L^\infty(X)} \leq C_2 2^{dk}$;

(ii) $|S_k(x, y) - S_k(x', y)| \leq C_2 2^{k(d+\epsilon)} \rho(x, x')^\epsilon$;

(iii) $|S_k(x, y) - S_k(x, y')| \leq C_2 2^{k(d+\epsilon)} \rho(y, y')^\epsilon$;

(iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C_2 2^{k(d+2\epsilon)} \rho(x, x')^\epsilon \rho(y, y')^\epsilon$;

(v) $\int\limits_X S_k(x, y)\, d\mu(y) = 1$;

(vi) $\int\limits_X S_k(x, y)\, d\mu(x) = 1$.

Remark 1.2. By a similar Coifman's construction in [4], one can construct an approximation to the identity with compact supports as in Definition 1.3 for those spaces of homogeneous type in Definition 1.1.

We also need the following construction of Christ in [2], which provides an analogue of the grid of Euclidean dyadic cubes on a space of homogeneous type.

Lemma 1.1. *Let X be a space of homogeneous type. Then there exists a collection $\{Q^k_\alpha \subset X : \ k \in \mathbb{Z}_+,\ \alpha \in I_k\}$ of open subsets, where I_k is some (possibly finite) index set, and constants $\delta \in (0, 1)$ and $C_4,\ C_5 > 0$ such that*

(i) *$\mu(X \setminus \cup_\alpha Q^k_\alpha) = 0$ for each fixed k and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$;*

(ii) *for any $\alpha,\ \beta,\ k,\ l$ with $l \geq k$, either $Q^l_\beta \subset Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$;*

(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q^k_\alpha \subset Q^l_\beta$;*

(iv) $\mathrm{diam}(Q^k_\alpha) \leq C_4 \delta^k$*;*

(v) *each Q^k_α contains some ball $B(z^k_\alpha, C_5 \delta^k)$, where $z^k_\alpha \in X$.*

In fact, we can think of Q^k_α as being essentially a cube of diameter rough δ^k with center z^k_α. In what follows, we always suppose $\delta = 1/2$. See [12] for how to remove this restriction. Also, we will denote by $Q^{k,\nu}_\tau$, $\nu = 1,\ 2,\ \ldots, N(k, \tau)$, the set of all cubes $Q^{k+j}_{\tau'} \subset Q^k_\tau$, where j is a fixed large positive integer. Denote by $y^{k,\nu}_\tau$ a point in $Q^{k,\nu}_\tau$. For any dyadic cube Q and any $f \in L^1_{\mathrm{loc}}(X)$, we set

$$m_Q(f) = \frac{1}{\mu(Q)} \int\limits_Q f(x)\, d\mu(x),$$

and we also let $a_+ = \max(a, 0)$.

Definition 1.4. Let $s \in (-\theta, \theta)$, $\{S_k\}_{k=0}^\infty$ be as in Definition 1.3 with order θ, $D_0 = S_0$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. Suppose β and γ satisfying

$$\max(0, -s + d(1/p - 1)_+) < \beta < \theta \quad \text{and} \quad 0 < \gamma < \theta. \tag{1.3}$$

Let $j \in \mathbb{N}$ be fixed and large enough and $\{Q_\tau^{0,\nu} : \tau \in I_0,\ \nu = 1, \ldots, N(0,\tau)\}$ be as above. The inhomogeneous Besov space $B_{pq}^s(X)$ for $\max(d/(d+\theta), d/(d+\theta+s)) < p \le \infty$ and $0 < q \le \infty$ is the collection of all $f \in \left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$ such that

$$\|f\|_{B_{pq}^s(X)} = \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \left[m_{Q_\tau^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\{ \sum_{k=1}^{\infty} \left[2^{ks} \|D_k(f)\|_{L^p(X)} \right]^q \right\}^{1/q} < \infty;$$

The inhomogeneous Triebel–Lizorkin space $F_{pq}^s(X)$ for $\max(d/(d+\theta), d/(d+\theta+s)) < p < \infty$ and $\max(d/(d+\theta), d/(d+\theta+s)) < q \le \infty$ is the collection of all $f \in \left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$ such that

$$\|f\|_{F_{pq}^s(X)} = \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \left[m_{Q_\tau^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\| \left\{ \sum_{k=1}^{\infty} \left[2^{ks} |D_k(f)| \right]^q \right\}^{1/q} \right\|_{L^p(X)} < \infty.$$

Here, for $k \in \mathbb{Z}_+$ and a suitable f,

$$D_k(f)(x) = \int_X D_k(x,y) f(y)\, d\mu(y).$$

It was proved in [14] that Definition 1.4 is independent of the choices of large positive integers j, approximations to the identity and the pairs (β, γ) as in (1.3).

2. Frame Characterizations

In this section, we will establish the frame characterizations of the Besov spaces $B_{pq}^s(X)$ and the Triebel–Lizorkin spaces $F_{pq}^s(X)$ in Definition 1.4. These results were given in [13] when $p,\ q > 1$. However, our proof here is quite different from that in [13]. In [13], the proof strongly depends on the dual argument. The new ingredient in the current proof is the application of the inhomogeneous Plancherel-Pôlya inequality in [5]. We also point that in this section, we have no restriction on $\mu(X)$, namely, $\mu(X)$ can be finite or infinite. Let us now give some basic properties of the spaces $B_{pq}^s(X)$ and $F_{pq}^s(X)$.

Lemma 2.1. *Let $|s| < \theta$.*

(i) *If* $\max(d/(d+\theta), d/(d+\theta+s)) < p < \infty$ *and* $\max(d/(d+\theta), d/(d+\theta+s)) < q \le \infty$, *then* $B^s_{p,\min(p,q)}(X) \subset F^s_{pq}(X) \subset B^s_{p,\max(p,q)}(X)$.

(ii) *If* $f \in \mathcal{G}(\beta,\gamma)$ *with* $\max(0,s) < \beta$ *and* $\max(0, d(1/p-1)_+) < \gamma$, *then* $f \in B^s_{pq}(X)$ *when* $\max(d/(d+\theta), d/(d+\theta+s)) < p \le \infty$ *and* $0 < q \le \infty$, *and* $f \in F^s_{pq}(X)$ *when* $\max(d/(d+\theta), d/(d+\theta+s)) < p < \infty$ *and*

$$\max(d/(d+\theta), d/(d+\theta+s)) < q \le \infty.$$

Proof. The proof of (i) is trivial; see Proposition 2.3 in [18]. To prove (ii), let us first prove the following claim: for all $k \in \mathbb{Z}_+$ and all $x \in X$,

$$|D_k(f)(x)| \le C2^{-k\beta}\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}. \tag{2.1}$$

In fact, we have

$$\begin{aligned}
|D_0(f)(x)| &\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \int_{\{y:\ \rho(x,y)\le C_1\}} \frac{1}{(1+\rho(y,x_0))^{d+\gamma}}\,d\mu(y)\\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)}\Big\{\chi_{\{x:\rho(x,x_0)\le 2AC_1\}}(x)\int_{\{y:\rho(x,y)\le C_1\}}\frac{1}{(1+\rho(y,x_0))^{d+\gamma}}d\mu(y)\\
&\quad+\chi_{\{x:\ \rho(x,x_0)>2AC_1\}}(x)\int_{\{y:\ \rho(x,y)\le C_1\}}\frac{1}{(1+\rho(y,x_0))^{d+\gamma}}\,d\mu(y)\Big\}\\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)}\Big\{\chi_{\{x:\rho(x,x_0)\le 2AC_1\}}(x)+\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}\chi_{\{x:\rho(x,x_0)>2AC_1\}}(x)\Big\}\\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}},
\end{aligned}$$

which is a desired estimate. For $k \in \mathbb{N}$, we write

$$\begin{aligned}
|D_k(f)(x)| &= \Big|\int_X D_k(x,y)[f(y)-f(x)]\,d\mu(y)\Big|\\
&\le C2^{kd}\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\beta+\gamma}}\int_{\{y:\ \rho(x,y)\le 2C_1 2^{-k}\}}\rho(x,y)^\beta\,d\mu(y)\\
&\le C2^{-k\beta}\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}},
\end{aligned}$$

which is also a desired estimate. Thus, (2.1) holds. From (2.1), it follows that

$$\chi_{Q^{0,\nu}_\tau}(x)|D_0(f)(x)| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}\inf_{x\in Q^{0,\nu}_\tau}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}. \tag{2.2}$$

By (2.1), (2.2) and Definition 1.4, we obtain

$$\|f\|_{B^s_{pq}(X)} = \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q^{0,\nu}_\tau)\Big[m_{Q^{0,\nu}_\tau}(|D_0(f)|)\Big]^p\Big\}^{1/p}+\Big\{\sum_{k=1}^\infty\Big[2^{ks}\|D_k(f)\|_{L^p(X)}\Big]^q\Big\}^{1/q}$$

$$\leq C\|f\|_{\mathcal{G}(\beta,\gamma)}\Big\{\Big[\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\int\limits_{Q_\tau^{0,\nu}}\frac{1}{(1+\rho(x,x_0))^{(d+\gamma)p}}\,d\mu(x)\Big]^{1/p}$$
$$+\Big[\sum_{k=1}^{\infty}2^{k(s-\beta)q}\Big]^{1/q}\Big\|\frac{1}{(1+\rho(\cdot,x_0))^{d+\gamma}}\Big\|_{L^p(X)}\Big\}\leq C\|f\|_{\mathcal{G}(\beta,\gamma)},$$

since $\beta > s$ and $\gamma > d(1/p-1)$. This proves (ii) with the spaces $B^s_{pq}(X)$. On the spaces $F^s_{pq}(X)$, we can deduce a desired concusion by this and (i). We finish the proof of Lemma 2.1. □

Before we state our main theorem, we recall the discrete Calderón reproducing formulas in [11], which is the key of the whole theory.

Lemma 2.2. *Suppose that* $\{D_k\}_{k=0}^{\infty}$ *is as in Definition* 1.4. *Then there exist functions* $\widetilde{D}_{Q_\tau^{0,\nu}}$ *with* $\tau\in I_0$ *and* $\nu=1,\dots,N(0,\tau)$ *and* $\widetilde{D}_k(x,y)$ *with* $k\in\mathbb{N}$ *such that for any fixed* $y_\tau^{k,\nu}\in Q_\tau^{k,\nu}$ *with* $k\in\mathbb{N}$, $\tau\in I_k$ *and* $\nu\in\{1,\dots,N(k,\tau)\}$ *and all* $f\in(\mathcal{G}(\beta_1,\gamma_1))'$ *with* $0<\beta_1<\theta$ *and* $0<\gamma_1<\theta$,

$$f(x)=\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})m_{Q_\tau^{0,\nu}}(D_0(f))\widetilde{D}_{Q_\tau^{0,\nu}}(x)$$
$$+\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu}),\tag{2.3}$$

where the series converge in $(\mathcal{G}(\beta_1',\gamma_1'))'$ *for* $\beta_1<\beta_1'<\theta$ *and* $\gamma_1<\gamma_1'<\theta$; $\widetilde{D}_k(x,y)$ *with* $k\in\mathbb{N}$ *satisfies that for any given* $\epsilon\in(0,\theta)$,

(i) $\left|\widetilde{D}_k(x,y)\right|\leq C\dfrac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$,

(ii) $\left|\widetilde{D}_k(x,y)-\widetilde{D}_k(x',y)\right|\leq C\Big(\dfrac{\rho(x,x')}{2^{-k}+\rho(x,y)}\Big)^{\epsilon}\dfrac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ *for*

$$\rho(x,x')\leq\frac{1}{2A}(2^{-k}+\rho(x,y)),$$

(iii) $\displaystyle\int\limits_X\widetilde{D}_k(x,y)\,d\mu(x)=\int\limits_X\widetilde{D}_k(x,y)\,d\mu(y)=0$;

$\operatorname{diam}(Q_\tau^{0,\nu})\sim 2^{-j}$ *for* $\tau\in I_0$ *and* $\nu=1,\dots,N(0,\tau)$ *with some* $j\in\mathbb{N}$; $\widetilde{D}_{Q_\tau^{0,\nu}}(x)$ *for* $\tau\in I_0$ *and* $\nu=1,\dots,N(0,\tau)$ *satisfies that*

(iv) $\displaystyle\int\limits_X\widetilde{D}_{Q_\tau^{0,\nu}}(x)\,d\mu(x)=1$,

(v) *for any given* $\epsilon\in(0,\theta)$, *there is a constant* $C>0$ *such that*

$$|\widetilde{D}_{Q_\tau^{0,\nu}}(x)|\leq C\frac{1}{(1+\rho(x,y))^{d+\epsilon}}$$

for all $x\in X$ *and* $y\in Q_\tau^{0,\nu}$ *and*

(vi) $$|\widetilde{D}_{Q_\tau^{0,\nu}}(x) - \widetilde{D}_{Q_\tau^{0,\nu}}(z)| \le C\Big(\frac{\rho(x,z)}{1+\rho(x,y)}\Big)^\epsilon \frac{1}{(1+\rho(x,y))^{d+\epsilon}}$$

for all $x,\ z \in X$ *and all* $y \in D_\tau^{0,\nu}$ *satisfying*

$$\rho(x,z) \le \frac{1}{2A}(1+\rho(x,y)).$$

Moreover, j *can be any fixed large positive integer and the constant* C *in* (v) *and* (v*i*) *is independent of* j.

The following lemma is an obvious corollary of Theorem 1 in [5].

Lemma 2.3. *Let* $s \in (-\theta, \theta)$. *Let* $\{D_k\}_{k=0}^\infty$ *be as in Lemma 2.2. Then, if*

$$\max\left(d/(d+\theta), d/(d+\theta+s)\right) < p \le \infty$$

and $0 < q \le \infty$, *for all* $f \in (G(\beta,\gamma))'$ *with* $0 < \beta,\ \gamma < \theta$, *we have*

$$\begin{aligned}
&\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\Big\}^{1/p} + \Big\{\sum_{k=1}^\infty \left[2^{ks}\|D_k(f)\|_{L^p(X)}\right]^q\Big\}^{1/q}\\
&\quad\sim \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\Big\}^{1/p}\\
&\quad\quad + \Big\{\sum_{k=1}^\infty 2^{ksq}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\Big[\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\Big]^p\Big)^{q/p}\Big\}^{1/q}\\
&\quad\sim \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\Big\}^{1/p}\\
&\quad\quad + \Big\{\sum_{k=1}^\infty 2^{ksq}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\Big[\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\Big]^p\Big)^{q/p}\Big\}^{1/q}; \qquad (2.4)
\end{aligned}$$

If $\max(d/(d+\theta), d/(d+\theta+s)) < p < \infty$ *and* $\max(d/(d+\theta), d/(d+\theta+s)) < q \le \infty$, *for all* $f \in (\mathcal{G}(\beta,\gamma))'$ *with* $0 < \beta,\ \gamma < \theta$, *we have*

$$\begin{aligned}
&\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\Big\}^{1/p} + \Big\|\Big\{\sum_{k=1}^\infty \left[2^{ks}|D_k(f)|\right]^q\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\quad\sim \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\Big\}^{1/p}\\
&\quad\quad + \Big\|\Big\{\sum_{k=1}^\infty\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \Big[2^{ks}\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\chi_{Q_\tau^{k,\nu}}(\cdot)\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\quad\sim \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\Big\}^{1/p}
\end{aligned}$$

$$+\left\|\left\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\left[2^{ks}\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\chi_{Q_\tau^{k,\nu}}(\cdot)\right]^q\right\}^{1/q}\right\|_{L^p(X)}. \tag{2.5}$$

On the estimates relative to the spaces $F^s_{pq}(X)$, we need the following useful lemma which can be found in [12, p. 93] and [7, pp. 147-148].

Lemma 2.4. *Let* $1 \le p \le \infty$, $0 < r \le 1$, μ, $\eta \in \mathbb{Z}_+$ *with* $\eta \le \mu$ *and for "dyadic cubes"* Q^μ_τ,

$$|f_{Q^\mu_\tau}(x)| \le (1 + 2^\eta \rho(x, z^\mu_\tau))^{-d-\sigma},$$

where z^μ_τ *is the "center" of* Q^μ_τ *as in Lemma* 1 *and* $\sigma > d(1/r - 1)$ (*recall that* $\mu(Q^\mu_\tau) \sim 2^{-\mu d}$). *Then*

$$\sum_\tau |\lambda_{Q^\mu_\tau}||f_{Q^\mu_\tau}(x)| \le C 2^{(\mu-\eta)d/r}\left[M\Big(\sum_\tau |\lambda_{Q^\mu_\tau}|^r \chi_{Q^\mu_\tau}\Big)(x)\right]^{1/r},$$

where C *is independent of* x, μ *and* η, *and* M *is the Hardy-Littlewood maximal operator on* X.

The following theorem is the main theorem of this section which will play a fundamental role in the estimates of entropy numbers between Besov spaces or between Triebel–Lizorkin spaces in next section.

Theorem 2.1. *With the notation of Lemma* 2.2, *let*

$$\lambda = \left\{\lambda^{k,\nu}_\tau :\ k \in \mathbb{Z}_+,\ \tau \in I_k,\ \nu = 1, \dots, N(k, \tau)\right\}$$

be a sequence of complex numbers. Let $|s| < \theta$.

(i) *If* $\max(d/(d+\theta), d/(d+\theta+s)) < p \le \infty$, $0 < q \le \infty$ *and*

$$\|\lambda\|_{b^s_{pq}(X)} = \left\{\sum_{k=0}^{\infty} 2^{ksq}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)\left|\lambda^{k,\nu}_\tau\right|^p\Big]^{q/p}\right\}^{1/q} < \infty, \tag{2.6}$$

then the series

$$\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q^{0,\nu}_\tau)\lambda^{0,\nu}_\tau\widetilde{D}_{Q^{0,\nu}_\tau}(x)+\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)\lambda^{k,\nu}_\tau\widetilde{D}_k(x, y^{k,\nu}_\tau) \tag{2.7}$$

converge to some $f \in B^s_{pq}(X)$ *both in the norm of* $B^s_{pq}(X)$ *and in* $(\mathcal{G}(\beta,\gamma))'$ *with* β *and* γ *as in* (1.3) *when* p, $q < \infty$ *and only in* $(\mathcal{G}(\beta,\gamma))'$ *with* β *and* γ *as in* (1.3) *when* $\max(p, q) = \infty$. *Moreover,*

$$\|f\|_{B^s_{pq}(X)} \le C\|\lambda\|_{b^s_{pq}(X)}. \tag{2.8}$$

(ii) *If* $\max(d/(d+\theta), d/(d+\theta+s)) < p < \infty$, $\max(d/(d+\theta), d/(d+\theta+s)) < q \le \infty$ *and*

$$\|\lambda\|_{f^s_{pq}(X)} = \left\|\left\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\left[2^{ks}\left|\lambda^{k,\nu}_\tau\right|\chi_{Q^{k,\nu}_\tau}(\cdot)\right]^q\right\}^{1/q}\right\|_{L^p(X)} < \infty, \tag{2.9}$$

then the series in (2.7) *converge to some* $f \in F^s_{pq}(X)$ *both in the norm of* $F^s_{pq}(X)$ *and in* $(\mathcal{G}(\beta,\gamma))'$ *with* β *and* γ *as in* (1.3) *when* $q < \infty$ *and only in* $(\mathcal{G}(\beta,\gamma))'$ *with* β *and* γ *as in* (1.3) *when* $q = \infty$. *Moreover,*

$$\|f\|_{F^s_{pq}(X)} \le C\|\lambda\|_{f^s_{pq}(X)}. \tag{2.10}$$

Proof. Let us first show the series in (2.7) converge in $(\mathcal{G}(\beta,\gamma))'$ with β and γ as in (1.3). It is easy to see that for all $k \in \mathbb{Z}_+$ and $\tau \in I_k$, $N(k,\tau)$ is a finite set. Let us suppose $I_k = \mathbb{N}$ for all $k \in \mathbb{Z}_+$; the other cases are easier. With this assumption, for $L \in \mathbb{N}$, we define

$$f_L(x) = \sum_{\tau=1}^{L}\sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau)\lambda^{0,\nu}_\tau \widetilde{D}_{Q^{0,\nu}_\tau}(x) + \sum_{k=1}^{L}\sum_{\tau=1}^{L}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\lambda^{k,\nu}_\tau \widetilde{D}_k(x, y^{k,\nu}_\tau).$$

Then $f_L \in \mathcal{G}(\epsilon,\epsilon)$ and $f_L \in (\mathcal{G}(\beta,\gamma))'$ with any $\beta,\ \gamma \in (0,\theta)$, where ϵ can be any positive number in $(0,\theta)$. In what follows, we will choose $\epsilon > \max(\beta,\gamma)$ such that $p > \max(d/(d+\epsilon), d/(d+\epsilon+s))$ for the spaces $b^s_{pq}(X)$ and

$$p,\ q > \max(d/(d+\epsilon), d/(d+\epsilon+s))$$

for the spaces $f^s_{pq}(X)$.

For any $\psi \in \mathcal{G}(\beta,\gamma)$ with (β,γ) as in (1.3), $L_1,\ L_2 \in \mathbb{N}$ and $L_1 < L_2$, we have

$$\begin{aligned}\left|\langle f_{L_2} - f_{L_1}, \psi\rangle\right| &\le \sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau)\left|\lambda^{0,\nu}_\tau\right|\left|\langle \widetilde{D}_{Q^{0,\nu}_\tau}, \psi\rangle\right| \\ &\quad + \sum_{k=L_1+1}^{L_2}\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\left|\lambda^{k,\nu}_\tau\right|\left|\langle \widetilde{D}_k(\cdot, y^{k,\nu}_\tau), \psi\rangle\right| \\ &\quad + \sum_{k=1}^{L_1}\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\left|\lambda^{k,\nu}_\tau\right|\left|\langle \widetilde{D}_k(\cdot, y^{k,\nu}_\tau), \psi\rangle\right| = D_1 + D_2 + D_3.\end{aligned}$$

To estimate D_1, D_2 and D_3, let us first establish the following estimates: for $\tau \in I_0$ and $\nu = 1, \dots, N(0,\tau)$,

$$\left|\langle \widetilde{D}_{Q^{0,\nu}_\tau}, \psi\rangle\right| \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \inf_{x\in Q^{0,\nu}_\tau} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}, \tag{2.11}$$

and for $k \in \mathbb{N}$, $\tau \in I_k$, $\nu = 1, \dots, N(k,\tau)$,

$$\left|\langle \widetilde{D}_k(\cdot, y^{k,\nu}_\tau), \psi\rangle\right| \le C2^{-k\beta}\|\psi\|_{\mathcal{G}(\beta,\gamma)} \inf_{x\in Q^{k,\nu}_\tau} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}. \tag{2.12}$$

For (2.11), we have that for any $x \in Q^{0,\nu}_\tau$,

$$\left|\langle \widetilde{D}_{Q^{0,\nu}_\tau}, \psi\rangle\right| \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\Bigg\{\int_{\{y:\rho(y,x_0)\ge\frac{1}{2A}\rho(x,x_0)\}} \left|\widetilde{D}_{Q^{0,\nu}_\tau}(y)\right| \frac{1}{(1+\rho(y,x_0))^{d+\gamma}}\, d\mu(y)$$

$$+\int\limits_{\{y:\ \rho(x,y)\ge\frac{1}{2A}\rho(x,x_0)\}}\frac{1}{(1+\rho(y,x))^{d+\epsilon'}}\frac{1}{(1+\rho(y,x_0))^{d+\gamma}}\,d\mu(y)\Big\}$$
$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}.$$

Thus, (2.11) holds.

To show (2.12), we write

$$\begin{aligned}\left|\langle\widetilde{D}_k(\cdot,y_\tau^{k,\nu}),\psi\rangle\right| &= \left|\int\limits_X \widetilde{D}_k(y,y_\tau^{k,\nu})\left[\psi(y)-\psi(y_\tau^{k,\nu})\right]\,d\mu(y)\right|\\ &\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(y_\tau^{k,\nu},x_0))^{d+\gamma+\beta}}\int\limits_X\left|\widetilde{D}_k(y,y_\tau^{k,\nu})\right|\rho(y,y_\tau^{k,\nu})^\beta\,d\mu(y)\\ &\le C2^{-k\beta}\|\psi\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(y_\tau^{k,\nu},x_0))^{d+\gamma}}\\ &\le C2^{-k\beta}\|\psi\|_{\mathcal{G}(\beta,\gamma)}\inf_{x\in Q_\tau^{k,\nu}}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}.\end{aligned}$$

That is, (2.12) also holds.

From (2.11) and the Hölder inequality, it follows that

$$|D_1|\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\begin{cases}\left\{\sum\limits_{\tau=L_1+1}^{L_2}\sum\limits_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})^p\left|\lambda_\tau^{0,\nu}\right|^p\right\}^{1/p}, & p<1,\\ \left\{\sum\limits_{\tau=L_1+1}^{L_2}\sum\limits_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\lambda_\tau^{0,\nu}\right|^p\right\}^{1/p} & \\ \quad\times\left\{\int\limits_{X_{L_1}^{L_2}}\frac{1}{(1+\rho(x,x_0))^{(d+\gamma)p'}}\,d\mu(x)\right\}^{1/p'}, & 1\le p\le\infty,\end{cases}$$
$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\left\{\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})^p\left|\lambda_\tau^{0,\nu}\right|^p\right\}^{1/p}, \tag{2.13}$$

where

$$X_{L_1}^{L_2}=\bigcup_{\tau=L_1+1}^{L_2}\bigcup_{\nu=1}^{N(0,\tau)}Q_\tau^{0,\nu},$$

and when $p\le 1$, we used the following well-known inequality:

$$\Big(\sum_j|a_j|\Big)^p\le\sum_j|a_j|^p \tag{2.14}$$

with $a_j\in\mathbb{C}$ for all j.

For D_2, by (2.12), (2.14) and the Hölder inequality, we obtain

$$|D_2| \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \begin{cases} \sum_{k=L_1+1}^{L_2} 2^{-k\beta}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})^p \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{1/p}, \ p<1, \\ \sum_{k=L_1+1}^{L_2} 2^{-k\beta}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{1/p} \\ \quad \times\Big\{\int_X \frac{1}{(1+\rho(x,x_0))^{(d+\gamma)p'}}\,d\mu(x)\Big\}^{1/p'}, \ 1\le p\le\infty, \end{cases}$$

$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \begin{cases} \sum_{k=L_1+1}^{L_2} 2^{-k[\beta+s-d(1/p-1)]}2^{ks}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{1/p}, \ p<1, \\ \sum_{k=L_1+1}^{L_2} 2^{-k(\beta+s)}2^{ks}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{1/p}, \\ \qquad 1\le p\le\infty, \end{cases}$$

$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \begin{cases} \Big\{\sum_{k=L_1+1}^{L_2} 2^{ksq}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{q/p}\Big\}^{1/q}, \ p,q<1, \\ \Big\{\sum_{k=L_1+1}^{L_2} 2^{ksq}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{q/p}\Big\}^{1/q} \\ \quad \times\Big\{\sum_{k=L_1+1}^{L_2} 2^{-k[\beta+s-d(1/p-1)]q'}\Big\}^{1/q'}, \\ \qquad p<1, \ 1\le q\le\infty, \\ \Big\{\sum_{k=L_1+1}^{L_2} 2^{ksq}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{q/p}\Big\}^{1/q}, \\ 1\le p\le\infty, \ q<1, \\ \Big\{\sum_{k=L_1+1}^{L_2} 2^{ksq}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{q/p}\Big\}^{1/q} \\ \quad \times\Big\{\sum_{k=L_1+1}^{L_2} 2^{-k(\beta+s)q'}\Big\}^{1/q'}, \ 1\le p, \ q\le\infty, \end{cases}$$

$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\Big\{\sum_{k=L_1+1}^{L_2} 2^{ksq}\Big[\sum_{\tau=1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{q/p}\Big\}^{1/q}, \tag{2.15}$$

where we used the fact that $\beta > \max(0, d(1/p-1)_+ - s)$.

Similarly, by (2.12), the Hölder inequality and (2.14), we can verify

$$|D_3| \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\Big\{\sum_{k=1}^{L_1} 2^{ksq}\Big[\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p\Big]^{q/p}$$
$$\times\Big[\int_{X_{L_1}^{L_2}} \frac{1}{(1+\rho(x,x_0))^{(d+\gamma)p'}}\,d\mu(x)\Big]^{q/p'}\Big\}^{1/q}$$

$$\leq C\|\psi\|_{\mathcal{G}(\beta,\gamma)}\left\{\sum_{k=1}^{L_1} 2^{ksq}\left[\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\left|\lambda_\tau^{k,\nu}\right|^p\right]^{q/p}\right\}^{1/q}. \tag{2.16}$$

Combining (2.13), (2.15) and (2.16), by (2.6),

$$\int_X \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}\,d\mu(x) < \infty$$

when $p = \infty$,

$$\sum_{k=1}^{\infty} 2^{-k[\beta+s-d(1/p-1)]} < \infty$$

when $p < 1$ and $q = \infty$, and

$$\sum_{k=1}^{\infty} 2^{-k(\beta+s)} < \infty$$

when $1 \leq p \leq \infty$ and $q = \infty$, it is easy to see that $\{\langle f_L, \psi\rangle\}_{L\in\mathbb{N}}$ is a Cauchy sequence. This just means that the series in (2.7) converge to some $f \in (\mathcal{G}(\beta,\gamma))'$ with $\beta,\ \gamma$ satisfying (1.3) if λ satisfies (2.6). If λ satisfies (2.9), by this fact and

$$b^s_{p,\min(p,q)}(X) \subset f^s_{pq}(X) \subset b^s_{p,\max(p,q)}(X) \tag{2.17}$$

(see Proposition 2.3 in [18]), we also obtain that the series in (2.7) converge in $(\mathcal{G}(\beta,\gamma))'$ with β and γ as in (1.3).

Let us now show that the series in (2.7) converge in the norm of $B^s_{pq}(X)$ when $p,\ q < \infty$, if λ satisfies (2.6). Let f be the series in (2.7). We estimate the norm in $B^s_{pq}(X)$ of $f - f_L$ by writing

$$\begin{aligned} f - f_L = &\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\lambda_\tau^{0,\nu}\widetilde{D}_{Q_\tau^{0,\nu}}(x) \\ &+\sum_{k=1}^{\infty}\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\lambda_\tau^{k,\nu}\widetilde{D}_k(x,y_\tau^{k,\nu}) \\ &+\sum_{k=L+1}^{\infty}\sum_{\tau=1}^{L}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\lambda_\tau^{k,\nu}\widetilde{D}_k(x,y_\tau^{k,\nu}) = G_1 + G_2 + G_3. \end{aligned}$$

To estimate G_1, G_2 and G_3, we first recall the following known estimates: for $k' \in \mathbb{Z}_+$, $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$,

$$\left|D_{k'}\left(\widetilde{D}_{Q_\tau^{0,\nu}}\right)(z)\right| \leq C2^{-k'\epsilon}\frac{1}{(1+\rho(z,y_\tau^{0,\nu}))^{d+\epsilon}}; \tag{2.18}$$

and for $k' \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $\tau \in I_k$ and $\nu = 1, \ldots, N(k,\tau)$,

$$\left|D_{k'}\left(\widetilde{D}_k\right)(z,y_\tau^{k,\nu})\right| \leq C2^{-|k-k'|\epsilon}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+\rho(z,y_\tau^{k,\nu}))^{d+\epsilon}}, \tag{2.19}$$

where $\epsilon \in (0, \theta)$ satisfies $p > \max(d/(d+\epsilon), d/(d+\epsilon+s))$ for the spaces $b^s_{pq}(X)$ and $p,\ q > \max(d/(d+\epsilon), d/(d+\epsilon+s))$ for the spaces $f^s_{pq}(X)$, $a \wedge b = \min(a, b)$ and

$$D_{k'}\left(\widetilde{D}_k\right)(z, y_\tau^{k,\nu}) = \int\limits_X D_{k'}(z,x)\widetilde{D}_k(x, y_\tau^{k,\nu})\, d\mu(x);$$

see [5], (3.9) in [8] and (1.6) in [9] for their proofs.

By (2.18), (2.14) and the Hölder inequality, we have

$$\begin{aligned}
&\left\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\mu(Q_{\tau'}^{0,\nu'})\Big[m_{Q_{\tau'}^{0,\nu'}}(|D_0(G_1)|)\Big]^p\right\}^{1/p}\\
&\quad\le\left\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\mu(Q_{\tau'}^{0,\nu'})\Big[\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\lambda_\tau^{0,\nu}\right|\right.\\
&\qquad\left.\times\sup_{z\in Q_{\tau'}^{0,\nu'}}\left|D_0\left(\widetilde{D}_{Q_\tau^{0,\nu}}\right)(z)\right|\Big]^p\right\}^{1/p}\\
&\quad\le C\begin{cases}\left\{\displaystyle\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\lambda_\tau^{0,\nu}\right|^p\int\limits_X\frac{1}{(1+\rho(z,y_\tau^{0,\nu}))^{(d+\epsilon)p}}\,d\mu(z)\right\}^{1/p}, & p\le 1,\\
\left\{\displaystyle\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\mu(Q_{\tau'}^{0,\nu'})\Big[\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\lambda_\tau^{0,\nu}\right|^p\right.\\
\quad\times\displaystyle\frac{1}{(1+\rho(y_{\tau'}^{0,\nu'},y_\tau^{0,\nu}))^{d+\epsilon}}\Big]\\
\quad\times\left.\Big[\displaystyle\int\limits_X\frac{1}{(1+\rho(y_{\tau'}^{0,\nu'},y))^{d+\epsilon}}\,d\mu(y)\Big]^{p/p'}\right\}^{1/p}, & 1<p<\infty,\end{cases}\\
&\quad\le C\left\{\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\lambda_\tau^{0,\nu}\right|^p\right\}^{1/p}.
\end{aligned}\tag{2.20}$$

From (2.19), (2.14) and the Hölder inequality, it follows that

$$\begin{aligned}
&\left\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\mu(Q_{\tau'}^{0,\nu'})\left[m_{Q_{\tau'}^{0,\nu'}}(|D_0(G_2)|)\right]^p\right\}^{1/p}\\
&\quad\le C\left\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\mu(Q_{\tau'}^{0,\nu'})\right.\\
&\qquad\left.\times\left[\sum_{k=1}^{\infty}2^{-k\epsilon}\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\left|\lambda_\tau^{k,\nu}\right|\frac{1}{(1+\rho(y_{\tau'}^{0,\nu'},y_\tau^{k,\nu}))^{d+\epsilon}}\right]^p\right\}^{1/p}
\end{aligned}$$

$$
\leq C \begin{cases} \Bigg\{ \sum_{k=1}^{\infty} 2^{-k\epsilon p} \sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \\ \quad \times \int\limits_X \frac{1}{(1+\rho(z, y_\tau^{k,\nu}))^{(d+\epsilon)p}} \, d\mu(z) \Bigg\}^{1/p}, \; p \leq 1, \\ \Bigg\{ \sum_{k=1}^{\infty} 2^{-k\epsilon_1 p} \sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \\ \quad \times \Big[\sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau')} \mu(Q_{\tau'}^{0,\nu'}) \frac{1}{(1+\rho(y_{\tau'}^{0,\nu'}, y_\tau^{k,\nu}))^{d+\epsilon}} \Big] \\ \quad \times \Big[\sum_{k=1}^{\infty} 2^{-k\epsilon_2 p} \int\limits_X \frac{1}{(1+\rho(y_{\tau'}^{0,\nu'}, y))^{d+\epsilon}} \, d\mu(y) \Big]^{p/p'} \Bigg\}^{1/p}, \\ 1 < p < \infty, \end{cases}
$$

$$
\leq C \begin{cases} \Bigg\{ \sum_{k=1}^{\infty} 2^{-k[\epsilon+s-d(1/p-1)]p} 2^{ksp} \sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Bigg\}^{1/p}, \; p \leq 1, \\ \Bigg\{ \sum_{k=1}^{\infty} 2^{-k(\epsilon_1+s)p} 2^{ksp} \sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Bigg\}^{1/p}, \; 1 < p < \infty, \end{cases}
$$

$$
\leq C \Bigg\{ \sum_{k=1}^{\infty} 2^{ksq} \Big[\sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Big]^{q/p} \Bigg\}^{1/p}, \tag{2.21}
$$

where we choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\epsilon = \epsilon_1 + \epsilon_2$ and $\epsilon_1 > -s$.

Similarly, by (2.19), (2.14) and the Hölder inequality, we can show

$$
\Bigg\{ \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau')} \mu(Q_{\tau'}^{0,\nu'}) \Big[m_{Q_{\tau'}^{0,\nu'}}(|D_0(G_3)|) \Big]^p \Bigg\}^{1/p}
$$
$$
\leq C \Bigg\{ \sum_{k=L+1}^{\infty} 2^{ksq} \Big[\sum_{\tau=1}^{L} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Big]^{q/p} \Bigg\}^{1/p}. \tag{2.22}
$$

By (2.18), (2.14), the Hölder inequality and the fact that $\mu(Q_\tau^{0,\nu})$ can be regarded as a constant, we have

$$
\Bigg\{ \sum_{k'=1}^{\infty} 2^{k'sq} \Big(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) \Big[\inf_{z \in Q_{\tau'}^{k',\nu'}} |D_{k'}(G_1)(z)| \Big]^p \Big)^{q/p} \Bigg\}^{1/q}
$$
$$
\leq C \Bigg\{ \sum_{k'=1}^{\infty} 2^{k'(s-\epsilon)q} \Big(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'})
$$
$$
\times \Big[\sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(0,\tau)} \left|\lambda_\tau^{0,\nu}\right| \frac{1}{(1+\rho(y_{\tau'}^{k',\nu'}, y_\tau^{0,\nu}))^{d+\epsilon}} \Big]^p \Big)^{q/p} \Bigg\}^{1/q}
$$

$$
\leq C\begin{cases}\Bigg\{\sum_{k'=1}^{\infty}2^{k'(s-\epsilon)q}\Big[\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\left|\lambda_{\tau}^{0,\nu}\right|^{p} \\ \quad\times\int_{X}\frac{1}{(1+\rho(z,y_{\tau}^{0,\nu}))^{(d+\epsilon)p}}\,d\mu(z)\Big]^{q/p}\Bigg\}^{1/q}, \\ \qquad\qquad p\leq 1, \\ \Bigg\{\sum_{k'=1}^{\infty}2^{k'(s-\epsilon)q}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\Big[\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\left|\lambda_{\tau}^{0,\nu}\right|^{p} \\ \quad\times\frac{1}{(1+\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{0,\nu}))^{d+\epsilon}}\Big] \\ \quad\times\Big[\int_{X}\frac{1}{(1+\rho(y_{\tau'}^{k',\nu'},y))^{d+\epsilon}}\,d\mu(y)\Big]^{p/p'}\Big)^{q/p}\Bigg\}^{1/q},\ 1<p<\infty\end{cases}
$$

$$
\leq C\Big[\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_{\tau}^{0,\nu})\left|\lambda_{\tau}^{0,\nu}\right|^{p}\Big]^{1/p}. \tag{2.23}
$$

From (2.19), (2.14) and the Hölder inequality, it follows that

$$
\Bigg\{\sum_{k'=1}^{\infty}2^{k'sq}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\Big[\inf_{z\in Q_{\tau'}^{k',\nu'}}|D_{k'}(G_3)(z)|\Big]^{p}\Big)^{q/p}\Bigg\}^{1/q}
$$

$$
\leq C\Bigg\{\sum_{k'=1}^{\infty}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\Big[\sum_{k=L+1}^{\infty}\sum_{\tau=1}^{L}\sum_{\nu=1}^{N(k,\tau)}2^{(k'-k)s-|k'-k|\epsilon}\mu(Q_{\tau}^{k,\nu})
$$

$$
\times 2^{ks}\left|\lambda_{\tau}^{k,\nu}\right|\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{k,\nu}))^{d+\epsilon}}\Big]^{p}\Big)^{q/p}\Bigg\}^{1/q}
$$

$$
\leq C\begin{cases}\Bigg\{\sum_{k'=1}^{\infty}\Big[\sum_{k=L+1}^{\infty}\sum_{\tau=1}^{L}\sum_{\nu=1}^{N(k,\tau)}2^{(k'-k)sp-|k'-k|\epsilon p}\mu(Q_{\tau}^{k,\nu})^{p}2^{ksp}\left|\lambda_{\tau}^{k,\nu}\right|^{p} \\ \quad\times\int_{X}\frac{2^{-(k\wedge k')\epsilon p}}{(2^{-(k\wedge k')}+\rho(z,y_{\tau}^{k,\nu}))^{(d+\epsilon)p}}\,d\mu(z)\Big]^{q/p}\Bigg\}^{1/q},\ p\leq 1, \\ \Bigg\{\sum_{k'=1}^{\infty}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\Big[\sum_{k=L+1}^{\infty}\sum_{\tau=1}^{L}\sum_{\nu=1}^{N(k,\tau)}2^{(k'-k)sp-|k'-k|\epsilon_1 p}\mu(Q_{\tau}^{k,\nu}) \\ \quad\times 2^{ksp}\left|\lambda_{\tau}^{k,\nu}\right|^{p}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{k,\nu}))^{d+\epsilon}}\Big]\Big[\sum_{k=L+1}^{\infty}2^{-|k-k'|\epsilon_2 p'} \\ \quad\times\int_{X}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+\rho(y_{\tau'}^{k',\nu'},y))^{d+\epsilon}}\,d\mu(y)\Big]^{p/p'}\Big)^{q/p}\Bigg\}^{1/q},\ 1<p<\infty,\end{cases}
$$

$$\leq C \begin{cases} \Big\{ \sum_{k'=1}^{\infty} \Big[\sum_{k=L+1}^{\infty} 2^{(k'-k)sp-|k'-k|\epsilon p+kd(1-p)-(k\wedge k')d(1-p)} 2^{ksp} \\ \quad \times \Big(\sum_{\tau=1}^{L} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Big) \Big]^{q/p} \Big\}^{1/q}, \ p \leq 1, \\ \Big\{ \sum_{k'=1}^{\infty} \Big[\sum_{k=L+1}^{\infty} 2^{(k'-k)sp-|k'!k|\epsilon_1 p} 2^{ksp} \Big(\sum_{\tau=1}^{L} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Big) \Big]^{q/p} \Big\}^{1/q}, \\ \qquad 1 < p < \infty, \end{cases}$$

$$\leq C \Big\{ \sum_{k=L+1}^{\infty} 2^{ksq} \Big[\sum_{\tau=1}^{L} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Big]^{q/p} \Big\}^{1/q}, \tag{2.24}$$

where we choose $\epsilon_1, \ \epsilon_2 > 0$ such that $\epsilon = \epsilon_1 + \epsilon_2$ and $\epsilon_1 > |s|$.

Similarly, by (2.19), (2.14) and the Hölder inequality, we can prove

$$\Big\{ \sum_{k'=1}^{\infty} 2^{k'sq} \Big(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) \Big[\inf_{z \in Q_{\tau'}^{k',\nu'}} |D_{k'}(G_2)(z)| \Big]^p \Big)^{q/p} \Big\}^{1/q}$$

$$\leq C \Big\{ \sum_{k=1}^{\infty} 2^{ksq} \Big[\sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left|\lambda_\tau^{k,\nu}\right|^p \Big]^{q/p} \Big\}^{1/q}. \tag{2.25}$$

Combining (2.20), (2.21), (2.22), (2.23), (2.24) and (2.25), by Lemma 2.3, we have

$$\|f - f_L\|_{B^s_{pq}(X)} \to 0$$

as $L \to \infty$, if λ satisfies (2.6). Moreover, by Lemma 2.1, we know that $f_L \in B^s_{pq}(X)$. Thus, it follows $f \in B^s_{pq}(X)$ when λ satisfies (2.6). The same arguments as those for (2.20), (2.21), (2.22), (2.23), (2.24) and (2.25) also give (2.8). This finishes the proof of (i) of Theorem 2.1.

To finish the proof of (ii) of Theorem 2.1, we still need to show $\|f - f_L\|_{F^s_{pq}(X)} \to 0$ as $L \to \infty$ and (2.10), if λ satisfies (2.9). To see this, by (2.18), (2.14) and the Hölder inequality, we have

$$\Big\| \Big\{ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \Big[2^{k's} \inf_{z \in Q_{\tau'}^{k',\nu'}} |D_{k'}(G_1)(z)| \, \chi_{Q_{\tau'}^{k',\nu'}}(\cdot) \Big]^q \Big\}^{1/q} \Big\|_{L^p(X)}^p$$

$$\leq C \Big\| \Big\{ \sum_{k'=1}^{\infty} 2^{k'(s-\epsilon)} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \Big[\sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(0,\tau)} \left|\lambda_\tau^{0,\nu}\right| \chi_{Q_{\tau'}^{k',\nu'}}(\cdot)$$

$$\times \frac{1}{(1+\rho(y_{\tau'}^{k',\nu'}, y_\tau^{0,\nu}))^{d+\epsilon}} \Big]^q \Big\}^{1/q} \Big\|_{L^p(X)}^p$$

$$\leq C \int_X \sum_{k'=1}^{\infty} 2^{k'(s-\epsilon)pa} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \chi_{Q_{\tau'}^{k',\nu'}}(\cdot) \Big[\sum_{\tau=L+1}^{\infty} \sum_{\nu=1}^{N(0,\tau)} \left|\lambda_\tau^{0,\nu}\right|$$

$$\times \frac{1}{(1+\rho(y_{\tau'}^{k',\nu'}, y_\tau^{0,\nu}))^{d+\epsilon}} \Big]^p d\mu(x)$$

$$\leq C\begin{cases}\displaystyle\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\left|\lambda_{\tau}^{0,\nu}\right|^p\int_X\frac{1}{(1+\rho(z,y_{\tau}^{0,\nu}))^{(d+\epsilon)p}}\,d\mu(z),\ p\leq 1,\\ \displaystyle\sum_{k'=1}^{\infty}2^{k'(s-\epsilon)pa}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\Big[\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\left|\lambda_{\tau}^{0,\nu}\right|\\ \quad\displaystyle\times\frac{1}{(1+\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{0,\nu}))^{d+\epsilon}}\Big]\Big[\int_X\frac{1}{(1+\rho(y_{\tau'}^{k',\nu'},y))^{d+\epsilon}}\,d\mu(y)\Big]^{p/p'},\\ \qquad 1<p<\infty,\end{cases}$$

$$\leq C\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_{\tau}^{0,\nu})\left|\lambda_{\tau}^{0,\nu}\right|^p, \tag{2.26}$$

where $a=1$ if $p\leq q$ and $a=1/2$ if $p>q$. From (2.19), Lemma 2.4, (2.14), the Hölder inequality and the Fefferman–Stein vector-valued maximal function inequality in [6], it follows that

$$\begin{aligned}&\left\|\left\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\left[2^{k's}\inf_{z\in Q_{\tau'}^{k',\nu'}}|D_{k'}(G_2)(z)|\,\chi_{Q_{\tau'}^{k',\nu'}}(\cdot)\right]^q\right\}^{1/q}\right\|_{L^p(X)}\\ &\quad\leq C\left\|\left\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\chi_{Q_{\tau'}^{k',\nu'}}(\cdot)\Big[\sum_{k=1}^{\infty}\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(k,\tau)}2^{-|k-k'|\epsilon}\mu(Q_{\tau}^{k,\nu})\left|\lambda_{\tau}^{k,\nu}\right|\right.\right.\\ &\qquad\left.\left.\times\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+\rho(\cdot,y_{\tau}^{k,\nu}))^{d+\epsilon}}\Big]^q\right\}^{1/q}\right\|_{L^p(X)}\\ &\quad\leq C\left\|\left\{\sum_{k'=1}^{\infty}\Big(\sum_{k=1}^{\infty}2^{(k'-k)s-|k-k'|\epsilon-kd+(k\wedge k')d+[k-(k\wedge k')]d/r}\right.\right.\\ &\qquad\left.\left.\times\Big[M\Big(\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(k,\tau)}2^{ksr}\left|\lambda_{\tau}^{k,\nu}\right|^r\chi_{Q_{\tau}^{k,\nu}}\Big)(\cdot)\Big]^{1/r}\Big)^q\right\}^{1/q}\right\|_{L^p(X)}\\ &\quad\leq C\left\|\left\{\sum_{k=1}^{\infty}\Big[M\Big(\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(k,\tau)}2^{ksr}\left|\lambda_{\tau}^{k,\nu}\right|^r\chi_{Q_{\tau}^{k,\nu}}\Big)(\cdot)\Big]^{q/r}\right\}^{1/q}\right\|_{L^p(X)}\\ &\quad\leq C\left\|\left\{\sum_{k=1}^{\infty}\sum_{\tau=L+1}^{\infty}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\left|\lambda_{\tau}^{k,\nu}\right|^q\chi_{Q_{\tau}^{k,\nu}}(\cdot)\right\}^{1/q}\right\|_{L^p(X)},\end{aligned} \tag{2.27}$$

where we choose r satisfying $\max(d/(d+\epsilon),d/(d+\epsilon+s))<r<\min(p,q,1)$.

Similarly, by (2.19), Lemma 2.4, (2.14), the Hölder inequality and the Fefferman-Stein vector-valued maximal function inequality in [6], we can show

$$\begin{aligned}&\left\|\left\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\left[2^{k's}\inf_{z\in Q_{\tau'}^{k',\nu'}}|D_{k'}(G_3)(z)|\,\chi_{Q_{\tau'}^{k',\nu'}}(\cdot)\right]^q\right\}^{1/q}\right\|_{L^p(X)}\\ &\quad\leq C\left\|\left\{\sum_{k=L+1}^{\infty}\sum_{\tau=1}^{L}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\left|\lambda_{\tau}^{k,\nu}\right|^q\chi_{Q_{\tau}^{k,\nu}}(\cdot)\right\}^{1/q}\right\|_{L^p(X)}.\end{aligned} \tag{2.28}$$

Combining (2.20), (2.21), (2.22), (2.26), (2.27) and (2.28), we see that if λ satisfies (2.9), then $\|f - f_L\|_{F^s_{pq}(X)} \to 0$ as $L \to \infty$. Moreover, by Lemma 2.1, $f_L \in F^s_{pq}(X)$. Thus, it follows that $f \in F^s_{pq}(X)$ if λ satisfies (2.9). The same arguments as those for (2.20), (2.21), (2.22), (2.26), (2.27) and (2.28) also give (2.10). This finishes the proof of Theorem 2.1. □

From Theorem 2.1, Lemma 2.3, Lemma 2.2 and Definition 1.4, it is easy to deduce the following frame characterizations of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$.

Theorem 2.2. *Let all the notation be as in Lemma2.2. For all* $f \in \left(\mathring{\mathcal{G}}(\beta_1, \gamma_1)\right)'$ *with* $0 < \beta_1,\ \gamma_1 < \theta$, *then* (2.3) *holds in* $\left(\mathring{\mathcal{G}}(\beta_1', \gamma_1')\right)'$ *for* $\beta_1 < \beta_1' < \epsilon$ *and* $\gamma_1 < \gamma_1' < \theta$. *Moreover,*

(i) *if* $f \in B^s_{pq}(X)$ *with* $s \in (-\theta, \theta)$, $\max(d/(d+\theta), d/(d+\theta+s)) < p \le \infty$ *and* $0 < q \le \infty$, *then*

$$\|f\|_{B^s_{pq}(X)} \sim \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \left[m_{Q_\tau^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\{ \sum_{k=1}^{\infty} 2^{ksq} \left[\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left| D_k(f)(y_\tau^{k,\nu}) \right|^p \right]^{q/p} \right\}^{1/q}$$

and the series in (2.3) *also converge in the norm of* $B^s_{pq}(X)$ *if* $\max(p,q) < \infty$;

(ii) *if* $f \in F^s_{pq}(X)$ *with* $s \in (-\theta, \theta)$, $\max(d/(d+\theta), d/(d+\theta+s)) < p < \infty$ *and*

$$\max(d/(d+\theta), d/(d+\theta+s)) < q \le \infty,$$

then

$$\|f\|_{F^s_{pq}(X)} \sim \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \left[m_{Q_\tau^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\| \left\{ \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \left| D_k(f)(y_\tau^{k,\nu}) \right|^q \chi_{Q_\tau^{k,\nu}}(\cdot) \right\}^{1/q} \right\|_{L^p(X)}$$

and the series in (2.3) *also converge in the norm of* $F^s_{pq}(X)$ *if* $q < \infty$.

We should remark that Theorem 2.2 is established in [13] when $p,\ q > 1$ by a different method.

3. Some Applications

In this section, we will give two applications of the frame characterizations of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ established in Section 2. By using these characterizations, we will first obatin the estimates for the entropy numbers of the compact embeddings between the spaces $B^s_{pq}(X)$ or between the spaces $F^s_{pq}(X)$ when $\mu(X) < \infty$. It has been proved that the entropy numbers play an extremely key role in the study on the spectra of various differential operators; see [17]. Secondly, we will establish some real interpolation theorems via the abstract interpolation method in [16] and [1] by using these characterizations.

Let us now recall the definition of the entropy numbers; see [17] and [13]. In the following, if $\mathcal{A}$ is a quasi-Banach space, then $\mathcal{U}_{\mathcal{A}} = \{b \in \mathcal{A} : \|b\|_{\mathcal{A}} \le 1\}$ stands for the unit ball in $\mathcal{A}$.

Definition 3.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two quasi-Banach spaces and T be a linear continuous operator from $\mathcal{A}_1$ to $\mathcal{A}_2$. Then for all $k \in \mathbb{N}$, the kth entropy number, $e_k(T)$, of T is defined by

$$e_k(T) = \inf\left\{\epsilon > 0 :\ T(\mathcal{U}_{\mathcal{A}_1}) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \epsilon\mathcal{U}_{\mathcal{A}_2}) \ \text{ for some } \ b_1, \ldots, b_{2^{k-1}} \in \mathcal{A}_2\right\}.$$

The theorem below is proved in [13] when $p,\ q > 1$ and can be proved by the same procedure as there when $p,\ q \le 1$ by replacing Theorem 4.1 and Proposition 4.1 in [13] by Theorem 2.1 and Theorem 2.2. We omit the details.

Theorem 3.1. *Let $\mu(X) < \infty$ and $-\theta < s_2 < s_1 < \theta$.*

(i) *If $\max(d/(d+\theta), d/(d+\theta+s_i)) < p_i \le \infty$ and $0 < q_i \le \infty$ for $i = 1,\ 2$, and*

$$\delta_+ = s_1 - s_2 - d(1/p_1 - 1/p_2)_+ > 0,$$

then the embedding of $B^{s_1}_{p_1,q_1}(X)$ into $B^{s_2}_{p_2,q_2}(X)$ is compact and for the related entropy numbers holds

$$e_k\left(id :\ B^{s_1}_{p_1,q_1}(X) \to B^{s_2}_{p_2,q_2}(X)\right) \sim k^{-(s_1-s_2)/d},$$

where $k \in \mathbb{N}$.

(ii) *If $\max(d/(d+\theta), d/(d+\theta+s_i)) < p_i < \infty$ and $\max(d/(d+\theta), d/(d+\theta+s_i)) < q_i \le \infty$ for $i = 1,\ 2$, and $\delta_+ > 0$, then the embedding of $F^{s_1}_{p_1,q_1}(X)$ into $F^{s_2}_{p_2,q_2}(X)$ is compact and for the related entropy numbers holds*

$$e_k\left(id :\ F^{s_1}_{p_1,q_1}(X) \to F^{s_2}_{p_2,q_2}(X)\right) \sim k^{-(s_1-s_2)/d},$$

where $k \in \mathbb{N}$.

We remark that if X is a compact d-set, in this case, we have $\theta = 1$, and our Theorem 3.1 on the Besov spaces $B^s_{pq}(X)$ is covered by Theorem 20.6 in [17] and the other cases are new.

Let us now consider the real interpolations of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$. We first recall the general background of the real interpolation method; see [16, pp. 62–64] and [1]. Let $\mathcal{H}$ be a linear complex Hausdorff space, and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two complex quasi-Banach spaces such that $\mathcal{A}_0 \subset \mathcal{H}$ and $\mathcal{A}_1 \subset \mathcal{H}$. Let $\mathcal{A}_0 + \mathcal{A}_1$ be the set of all elements $a \in \mathcal{H}$ which can be represented as $a = a_0 + a_1$ with $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$. If $0 < t < \infty$ and $a \in \mathcal{A}_0 + \mathcal{A}_1$, then Peetre's celebrated K-functional is given by

$$K(t, a) = K(t, a; \mathcal{A}_0, \mathcal{A}_1) = \inf\left(\|a_0\|_{\mathcal{A}_0} + t\|a_1\|_{\mathcal{A}_1}\right),$$

where the infimum is taken over all representations of a of the form $a = a_0 + a_1$ with $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$.

Definition 3.2. Let $0 < \sigma < 1$. If $0 < q < \infty$, then

$$(\mathcal{A}_0, \mathcal{A}_1)_{\sigma,q} = \left\{ a : \ a \in \mathcal{A}_0 + \mathcal{A}_1, \ \|a\|_{(\mathcal{A}_0,\mathcal{A}_1)_{\sigma,q}} = \left\{ \int_0^\infty \left[t^{-\sigma} K(t,a) \right]^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}.$$

If $q = \infty$, then

$$(\mathcal{A}_0, \mathcal{A}_1)_{\sigma,\infty} = \left\{ a : \ a \in \mathcal{A}_0 + \mathcal{A}_1, \ \|a\|_{(\mathcal{A}_0,\mathcal{A}_1)_{\sigma,\infty}} = \sup t^{-\sigma} K(t,a) < \infty \right\}.$$

Using Theorem 2.1, Theorem 2.2 and the method of retraction and coretraction as in the proofs of Theorem 2.4.1 and Theorem 2.4.2 in [16], we can obtain the below theorems on the real interpolations of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$, where there is no restriction on $\mu(X)$, namely, $\mu(X)$ can be finite or infinite.

Theorem 3.2. *Let* $\sigma \in (0,1)$.

(i) *Let* $-\theta < s_0, \ s_1 < \theta$, $s_0 \neq s_1$, $1 \leq p \leq \infty$, *and* $0 < q_0, \ q_1, \ q \leq \infty$. *Then*

$$\left(B^{s_0}_{p,q_0}(X), B^{s_1}_{p,q_1}(X) \right)_{\sigma,q} = B^s_{pq}(X),$$

where $s = (1-\sigma)s_0 + \sigma s_1$.

(ii) *Let* $-\theta < s < \theta$, $1 \leq p \leq \infty$, $0 < q_0, \ q_1 \leq \infty$ *and* $q_0 \neq q_1$. *Then*

$$\left(B^{s}_{p,q_0}(X), B^{s}_{p,q_1}(X) \right)_{\sigma,q} = B^s_{pq}(X),$$

where $1/q = (1-\sigma)/q_0 + \sigma/q_1$.

(iii) *Let* $-\theta < s_0, \ s_1 < \theta$ *and* $1 \leq p_0, \ p_1 \leq \infty$. *Then*

$$\left(B^{s_0}_{p_0,p_0}(X), B^{s_1}_{p_1,p_1}(X) \right)_{\sigma,p} = B^s_{p,p}(X),$$

where $1/p = (1-\sigma)/p_0 + \sigma/p_1$.

Theorem 3.3. *Let* $-\theta < s_0, \ s_1 < \theta$, $\max(d/(d+\theta), d/(d+\theta+s_0)) < p_0 < \infty$, $\max(d/(d+\theta), d/(d+\theta+s_1)) < p_1 < \infty$, $1 \leq q_0, \ q_1 \leq \infty$, $\sigma \in (0,1)$, $s = (1-\sigma)s_0 + \sigma s_1$, $1/p = (1-\sigma)/p_0 + \sigma/p_1$ *and* $1/q = (1-\sigma)/q_0 + \sigma/q_1$.

(i) *If* $s_0 \neq s_1$, *then*

$$\left(F^{s_0}_{p_0,q_0}(X), F^{s_1}_{p_1,q_1}(X) \right)_{\sigma,p} = B^s_{p,p}(X) \left(= F^s_{p,p}(X) \right).$$

(ii) *If* $s_0 = s_1 = s$, $p_0 = q_0$, $p_1 = q_1$ *and* $q_0 \neq q_1$, *then*

$$\left(F^{s}_{p_0,p_0}(X), F^{s}_{p_1,p_1}(X) \right)_{\sigma,p} = B^s_{p,p}(X).$$

(iii) *If* $s_0 = s_1 = s$, $q_0 = q_1 = q$ *and* $p_0 \neq p_1$, *then*

$$\left(F^{s}_{p_0,q}(X), F^{s}_{p_1,q}(X) \right)_{\sigma,p} = F^s_{pq}(X).$$

Proofs of Theorems 3.2 *and* 3.3. Since the definitions of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ are independent of the pair (β,γ) as in (1.3). We can suppose

$$B^{s_i}_{p_i,q_i}(X),\ F^{s_i}_{p_i,q_i}(X)\subset\left(\mathring{\mathcal{G}}(\beta_i,\gamma_i)\right)',$$

where $\max(0,-s_i+d(1/p_i-1)_+)<\beta_i<\theta$ and $0<\gamma_i<\theta$, and $i=0,\ 1$. We then let $\beta=\max(\beta_0,\beta_1)$ and $\gamma=(\gamma_0,\gamma_1)$. Then

$$B^{s_i}_{p_i,q_i}(X),\ F^{s_i}_{p_i,q_i}(X)\subset\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'.$$

In this sense, $\left\{B^{s_0}_{p_0,q_0}(X),B^{s_1}_{p_1,q_1}(X)\right\}$ and $\left\{F^{s_0}_{p_0,q_0}(X),F^{s_1}_{p_1,q_1}(X)\right\}$ are interpolation couples in the sense of §1.2.1 in [16]. Now, for $f\in\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$, with the notation of Lemma 2.2, we can define the coretraction operator S by

$$S(f)(x)=\{S(f)_k(x)\}_{k=0}^\infty,$$

where

$$S(f)_0(x)=\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}m_{Q^{0,\nu}_\tau}(D_0(f))\chi_{Q^{0,\nu}_\tau}(x)$$

and for $k\in\mathbb{N}$,

$$S(f)_k(x)=\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}D_k(f)(y^{k,\nu}_\tau)\chi_{Q^{k,\nu}_\tau}(x),$$

and the corresponding retraction operator R by

$$\begin{aligned}R\Big(\{f_k\}\Big)(x)=&\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\Big[\int_{Q^{0,\nu}_\tau}f_0(y)\,d\mu(y)\Big]\widetilde{D}_{Q^{0,\nu}_\tau}(x)\\&+\sum_{k=1}^\infty\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\Big[\int_{Q^{k,\nu}_\tau}f_k(y)\,d\mu(y)\Big]\widetilde{D}_k(x,y^{k,\nu}_\tau).\end{aligned}$$

By Lemma 2.2, for any $f\in\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$, we have

$$RS(f)(x)=f(x).$$

In what follows, for $s\in\mathbb{R}$, $0<q\le\infty$ and $0<p\le\infty$, we say $\{f_k\}_{k=0}^\infty\in l^s_q(L^p)$, if

$$\Big\|\{f_k\}_{k=0}^\infty\Big\|_{l^s_q(L^p)}=\Big\{\sum_{k=0}^\infty 2^{ksq}\|f_k\|^q_{L^p(X)}\Big\}^{1/q}<\infty;$$

and we say $\{f_k\}_{k=0}^\infty\in L^p(l^s_q)$, if

$$\Big\|\{f_k\}_{k=0}^\infty\Big\|_{L^p(l^s_q)}=\Big\|\Big\{\sum_{k=0}^\infty 2^{ksq}|f_k(x)|^q\Big\}^{1/q}\Big\|_{L^p(X)}<\infty,$$

where the usual modifications are made when $p = \infty$ or $q = \infty$. If F is an interpolation functor, then one obtains by Theorem 1.2.4 in [16] that

$$\|f\|_{F\left(\left\{B^{s_0}_{p_0,q_0}(X),B^{s_1}_{p_1,q_1}(X)\right\}\right)} \sim \|S(f)\|_{F\left(\left\{l^{s_0}_{q_0}(L^{p_0}),l^{s_1}_{q_1}(L^{p_1})\right\}\right)}$$

and

$$\|f\|_{F\left(\left\{F^{s_0}_{p_0,q_0}(X),F^{s_1}_{p_1,q_1}(X)\right\}\right)} \sim \|S(f)\|_{F\left(\left\{L^{p_0}\left(l^{s_0}_{q_0}\right),L^{p_1}\left(l^{s_1}_{q_1}\right)\right\}\right)}.$$

Using Theorem 2.1 and Theorem 2.2, we can then finish the proofs of Theorem 3.2 and Theorem 3.3 by the same procedures as those in [16, pp. 182-183] and [16, pp. 185-186]. We omit the details.

This finishes the proofs of Theorems 3.2 and 3.3. □

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