Georgian Mathematical Journal
Volume 9 (2002), Number 3, 549–566

ON A MULTIPLICATIVITY UP TO HOMOTOPY OF THE GUGENHEIM MAP

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Abstract. In the category of differential algebras with strong homotopy there is a Gugenheim's map $\{\rho_i\} : A^* \to C^*$ from Sullivan's commutative cochain complex to the singular cochain complex of a space, which induces a differential graded coalgebra map of appropriate Bar constructions. Both $(BA^*, d_{BA^*}, \triangle,)$ and $(BC^*, d_{BC^*}, \triangle,)$ carry multiplications. We show that the Gugenheim's map

$$B\{\rho_i\} : (BA^*, d_{BA^*}, \triangle) \to (BC^*, d_{BC^*}, \triangle)$$

is *multiplicative up to homotopy* with respect to these structures.

2000 Mathematics Subject Classification: 55N10, 57T05, 57T30, 58A10.
Key words and phrases: De Rham theorem, twisting cochain, deformation functor, Bar-construction.

Let $\mathcal{J}$ denote either the category of C^∞ manifolds (possibly with boundaries and corners) and C^∞-maps, or that of simplicial complexes and simplicial maps. Let $\mathcal{V}$ denote the category of graded vector spaces over the reals or rationals, and let $A^* : \mathcal{J} \to \mathcal{V}$ denote either the classical de Rham functor of differential forms, or Sullivan's functor of rational differential forms; see [11]. By $C^* : \mathcal{J} \to \mathcal{V}$ we denote the functor of either normalized singular differentiable cochains or normalized simplicial cochains. The differentiable cochain (chain) functor is homology equivalent to usual (continuous) singular functors. This is a well-known fact; for the proof see, for example, [23].

The transformation of functors $\rho : A^* \to C^*$ is defined by

$$\langle \rho\omega, c\rangle = \int\limits_c \omega,$$

where c is any singular (or simplicial) chain. This transformation was first constructed in the 1930's in the context of smooth manifolds and ordinary differential forms. The de Rham theorem says, in either case, that ρ induces a homology isomorphism. Additionally, although ρ itself is not a map of algebras, the theorem asserts that the isomorphism

$$\rho^* : H(A^*) \to H(C^*)$$

is a map of algebras [8].

ISSN 1072-947X / $8.00 /

Gugenheim in [12] proved that $\rho = \rho_1 : A^* \to C^*$ can be extended to a map in *DASH* (the category of differential algebras with strong homotopy) [13], [21], [22]. This means that there exists a whole family of "higher homotopies"

$$\rho_i : A^{*\otimes i} \to C^* \quad (i \geq 1),$$

where $A^{*\otimes i} = A^* \otimes \cdots \otimes A^*$ (i times), such that

$$D\rho_i = \sum_{j=1}^{i-1} (-1)^j \Big\{ \phi(\rho_j \otimes \rho_{i-j}) - \rho_{i-1}(1^{(i-j)} \otimes \phi \otimes 1^{(i-j-1)}) \Big\}.$$

Here $D\rho_i = d_{C^*} \cdot \rho_i + (-1)^i \rho_i \cdot d^{\otimes}_{A^*}$, where $d^{\otimes}_{A^*}$ is a usual differential on $A^{*\otimes i}$, ρ_i has degree $-i+1$ and ϕ stands for multiplication. Thus

$$\begin{aligned} \phi(\alpha \otimes \beta) &= \alpha \wedge \beta \quad \text{(exterior product) if } \alpha, \beta \in A^*(X) \\ &= \alpha \smile \beta \quad \text{(cup product) if } \alpha, \beta \in C^*(X). \end{aligned}$$

For $i = 1, 2$ we get

$$D\rho_1 = 0, \qquad D\rho_2 = \rho_1 \phi - \phi(\rho_1 \otimes \rho_1).$$

The latter statement contains, of course, the classical result that $H(\rho)$ is multiplicative.

It is well known that any *DASH* map $\{\rho_i\} : A^* \to C^*$ induces a coalgebra map of appropriate Bar constructions

$$B\{\rho_i\} : (BA^*, d_{BA^*}, \triangle) \to (BC^*, d_{BC^*}, \triangle);$$

both $(BA^*, d_{BA^*}, \triangle)$ equipped with shuffle μ_{sh} and $(BC^*, d_{BC^*}, \triangle)$ equipped with product μ_E introduced by Baues in [3], [4] are Hopf algebras. In this paper we shall prove the following

Theorem 1. *Gugenheim's map*

$$B\{\rho_i\} : (BA^*, d_{BA^*}, \triangle, \mu_{sh}) \to (BC^*, d_{BC^*}, \triangle, \mu_E)$$

is multiplicative up to homotopy.

1. Review of the Notation and Known Results

In this section we recollect the definitions and facts we need; most can be found in [14], [15], [21]. Since these results are not new, we omit their proofs.

1.1. DG module. Let R be a commutative ring with identity 1_R. Let $\{M_i\}_{i\in\mathbb{Z}}$ be a sequence of R-modules; $M = \sum M_i$ is called a graded R-module (throughout this paper we assume $M_i = 0$ for $i < 0$). An element $x \in M_i$ is said to be homogeneous of degree i, in which case we write $|x| = i$. A differential graded module (DGM) is a graded module M together with an endomorphism $d \in End(M)$ of degree $+1$ and square zero. Given DGMs M, N and a homomorphism $f : (M, d_M) \to (N, d_N)$ of degree k, define the differential of f by $Df = d_N f - (-1)^k f d_M$. The category of DGMs is denoted by DM; its morphisms are chain maps, i.e., maps f of degree 0 with $Df = 0$.

1.2. DG algebra. DA denotes the category of augmented DGAs with unit. If $(A, d, \mu, \eta, \varepsilon) \in DA$, then the multiplication $\mu : A \otimes A \to A$, the unit $\eta : R \to A$, the augmentation $\varepsilon : A \to R$ and the differential $d : A \to A$ satisfy the usual requirements. We say that A is connected if $\eta : R \to A_0$ is an isomorphism. A morphism in DA is a chain map preserving all structure. Let $IA = \ker(\varepsilon)$ be the augmentation ideal. The exact sequence $0 \to IA \xrightarrow{i} A \xrightarrow{\varepsilon} R \to 0$ defines a functor $I : DA \to DM$; the multiplication μ induces a map $I\mu : IA \otimes IA \to IA$ given by $i(I\mu) = \mu(i \otimes i)$.

1.3. DG coalgebra. DC denotes the category of coaugmented differential graded coalgebras (DGC's) with counit. If $(C, \Delta, d, \eta, \varepsilon) \in DC$, then the comultiplication $\Delta : C \to C \otimes C$, the counit $\varepsilon : C \to R$, the coaugmentation $\eta : R \to C$ and the differential $d : C \to C$ satisfy the usual requirements. We say that C is connected if $\varepsilon : C_0 \to R$ is an isomorphism. A morphism in DC is a chain map preserving all structure. Let $JC = co\ker(\eta)$ be the coaugmentation ideal. The exact sequence $0 \to R \xrightarrow{\eta} C \xrightarrow{p} JC \to 0$ defines a functor $J : DC \to DM$; the comultiplication Δ induces a map $J\Delta : JC \to JC \otimes JC$ given by $(J\Delta)p = (p \otimes p)\Delta$.

1.4. DG Hopf algebra. A differential graded Hopf algebra (DGHA) is a connected DGA $(A, d, \mu, \eta, \varepsilon)$ together with a coassociative comultiplication $\Delta : A \to A \otimes A$ such that $(A, \Delta, d, \eta, \varepsilon)$ is a DGC and Δ is a map in DA. The DGHA A is cocommutative if $T\Delta = \Delta$ and commutative if $\mu T = \mu$, where $T : A \otimes A \to A \otimes A$ is the twisting involution given by $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$. Let HA denote the category of DGHA's; morphisms in HA are the coalgebra maps f in DA, i.e., $(f \otimes f)\Delta = \Delta f$.

1.5. Suspension. The suspension (resp. desuspension) functor $s : DM \to DM$ (resp. $s^{-1} : DM \to DM$) is given by $(sM)_n = M_{n-1}$ and $d_{sM} = -d_M$. On an object M, the map $s : M \to sM$ has degree $+1$, is the identity in each dimension and $Ds = 0$. If $f : M \to N$ has degree k, so does $sf : sM \to sN$ and the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "s"'] & N \arrow[d, "s"'] \\ sM \arrow[r, "sf"] & sN. \end{tikzcd}$$

1.6. Tensor algebra. The free tensor algebra $T(M)$ on a graded module M is the direct sum $T(M) = \sum_{i \geq 0} M^{\otimes i}$, where $M^{\otimes 0} = R$, with multiplication given by

$$\mu(a_1 \cdots a_i) \otimes (a_{i+1} \cdots a_n) = (a_1 \cdots a_n).$$

Let $i_1 : M \to T(M)$ be the injection. $T(M)$ has the following universal property: If A is any DGA and $\alpha : M \to IA$ is any homomorphism, then there is a unique algebra map $f_\alpha : T(M) \to A$ commuting the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{i_1} & T(M) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle f_\alpha}\downarrow \\ IA & \xrightarrow{i} & A \end{array}$$

An explicit formula for f_α is given by

$$f_\alpha = \mu_A^n(\alpha^{\otimes n} \otimes i^{\otimes n}),$$

where $\mu_A^1 = Id$, $\mu_A^2 = \mu_A$ and $\mu_A^n = \mu_A(\mu_A^{n-1} \otimes Id)$.

Let $(C, d, \triangle, \eta, \varepsilon) \in DC$. The Cobar construction $\Omega(C, d, \triangle, \eta, \varepsilon)$ due to Adams [1] is defined as a free tensor algebra $T\left(s^{-1}(JC)\right)$ whose differential d_Ω is defined on generators $s^{-1}JC$ by

$$d_\Omega \cdot i^1 = -i^1(s^{-1}ds) + i^2(s^{-1} \otimes s^{-1})(J\Delta)s,$$

where $i^k : (s^{-1}JC)^{\otimes k} \to T(s^{-1}JC)$ is an obvious inclusion.

Thus the Cobar construction of a DGC is a DGA with $1 : R \to R$ as unit and augmentation.

1.7. Tensor coalgebra. The cofree tensor coalgebra $T^c(M)$ on a graded module M is the direct sum $T^c(M) = \sum_{i \geq 0} M^{\otimes i}$, where $M^{\otimes 0} = R$, with the diagonal given by

$$\Delta(a_1 \cdots a_n) = \sum_{i=0}^{n} (a_1 \cdots a_i) \otimes (a_{i+1} \cdots a_n).$$

Let $P_1 : T^c(M) \to M$ be the projection. $T^c(M)$ has the following universal property: If C is any DGC and $\alpha : JC \to M$ is any homomorphism, then there is a unique coalgebra map $f_\alpha : C \to T^c(M)$ commuting the following diagram:

$$\begin{array}{ccc} T^c(M) & \xrightarrow{P_1} & M \\ {\scriptstyle f_\alpha}\uparrow & & {\scriptstyle\alpha}\uparrow \\ C & \xrightarrow{p} & JC \end{array}$$

An explicit formula for f_α is given by

$$f_\alpha = \sum_{i=1}^{\infty} (p^{\otimes i} \otimes \alpha^{\otimes i})\Delta_C^i,$$

where $\Delta_C^1 = Id$, $\Delta_C^2 = \Delta_C$ and $\Delta_C^i = (\Delta_C^{i-1} \otimes Id)\Delta_C$.

Let $(A, d, \mu, \eta, \varepsilon) \in DA$. The Bar construction on DGA $(A, d, \mu, \eta, \varepsilon)$ is a DGC $T^c\left(s(IA)\right)$ with differential (coderivation)

$$\begin{aligned} d_{BA}[a_1|\cdots|a_n] &= \sum_{i=0}^{n-1} [\overline{a}_1|\cdots|\overline{a}_i|da_{i+1}|a_{i+2}|\cdots|a_n] \\ &+ \sum_{i=0}^{n-2} [\overline{a}_1|\cdots|\overline{a}_i|\mu(\overline{a}_{i+1} \otimes a_{i+2})|a_{i+3}|\cdots|a_n], \end{aligned} \tag{1.7.1}$$

where $\overline{a} = (-1)^{|a|+1}a$, and degree of $[a_1|\cdots|a_n] \in BA$ is $\sum_{i=1}^{n}|a_i| + n$.

Thus the Bar construction of DGA is a DGC with $1 : R \to R$ as counit and coaugmentation.

Let $S_{p,q}$ denote shuffle permutations in the set of all permutations $\sum_{p+q}$, i.e., $\sigma \in S_{p,q}$ implies $\sigma(1) < \cdots < \sigma(p)$, $\sigma(p+1) < \cdots < \sigma(p+q)$. The shuffle product $\mu_{sh} : BA \otimes BA \to BA$ is given by

$$\mu_{sh}\left([a_1|\cdots|a_p] \otimes [a_{p+1}|\cdots|a_{p+q}]\right) = \sum_{\sigma\in S_{p,q}} \varepsilon(\sigma)[a_{\sigma^{-1}(1)}|\cdots|a_{\sigma^{-1}(p+q)}],$$

where $\varepsilon(\sigma) = +1$ if the number of transpositions is even, and $\varepsilon(\sigma) = -1$ if the number of transpositions is odd. If DGA $(A, d, \mu, \eta, \varepsilon)$ is a commutative algebra, then differential $d_{BA} : BA \to BA$ on the Bar construction is a derivation of μ_{sh}, in which case $(BA, \Delta, d_m, \mu_{sh})$ is a DGHA.

1.8. Twisting cochain. In this section we recall briefly the definition of twisting cochains of Brown [7] and some notions and known facts connected with it.

Definition 2. Let $(C, d_C, \triangle, \varepsilon_C, \eta_C)$ be a DGC and $(A, d_A, \mu, \varepsilon_A, \eta_A)$ DGA. The complex $\mathrm{Hom}(C, A)$ is defined as a DGA with differential

$$Df = d \cdot f - (-1)^{|f|} f \cdot d_C$$

and multiplication

$$f \smile g = \mu(f \otimes g)\triangle, \qquad f, g \in \mathrm{Hom}(C, A).$$

A neutral element in $\mathrm{Hom}(C, A)$ is a composition $e : C \xrightarrow{\varepsilon_C} R \xrightarrow{\eta_A} A$.

It is easy to check that

$$D(f \smile g) = Df \smile g + (-1)^{|f|} f \smile Dg.$$

From now on, throughout the paper, we shall work only with the augmentation and coaugmentation ideals of DGA and DGC. Thus we consider the complex $\mathrm{Hom}(JC,\, IA)$.

Definition 3. An element $t \in \mathrm{Hom}(JC,\, IA)$ of degree -1 is a twisting cochain in the sense of Brown [7] if it satisfies

$$Dt = -t \smile t.$$

It is easy to show that for any DGA $(A, d_A, \mu, \varepsilon_A, \eta_A)$ (resp. DGC $(C, d_C, \triangle, \varepsilon_C, \eta_C)$) the natural map

$$p_1 : BA \to A \quad (\text{resp. } i_1 : C \to \Omega C)$$

is a twisting cochain.

Let $\alpha : JC \to IA$ be a map of degree -1. By the universal properties, mentioned in Sections 1.6. and 1.7, any such homomorphism uniquely defines an algebra map $f_\alpha : \Omega C \to A$ and a coalgebra map $g_\alpha : C \to BA$. Homomorphisms $f_\alpha : \Omega C \to A$ and $g_\alpha : C \to BA$ are chain maps, i.e., maps in DA and DC

respectively, if and only if $\alpha : JC \to IA$ is a twisting cochain; see [15], for example. Thus there are bijections

$$\mathrm{Hom}_{DC}(C, BA) \leftrightarrow T(JC, IA) \leftrightarrow \mathrm{Hom}_{DA}(\Omega C, A), \qquad (1.8.3)$$

where by $T(JC, IA)$ we denote the set of all twisting cochains from JC to IA.

Berikashvili in [5] introduced the following equivalence relation on the set $T(JC, IA)$.

Definition 4. Two twisting cochains $\alpha, \alpha' : JC \to IA$ are equivalent (we write $\alpha \sim \alpha'$) if there exists a homomorphism

$$\beta : JC \to IA$$

of degree 0 satisfying the identity

$$\alpha - \alpha' = \beta d_C - d_A\beta + \beta \smile \alpha' - \alpha \smile \beta.$$

The set $T(C, A)$ factored by this equivalence relation is denoted by $D(C, A)$.

Let $f : A \to B$ be a map in DA. It is trivial to check that for any $\alpha \in T(JC, IA)$ the composition $If\alpha$, where $If : IA \to IB$, is a twisting cochain in $T(JC, IB)$. Moreover, if $\alpha \sim \alpha'$, then $If\alpha \sim If\alpha'$. Thus for any DGC C a DGA map $f : A \to B$ induces a set map

$$f_* : D(JC, IA) \to D(JC, IB)$$

defined by $[\alpha] \longmapsto [If\alpha]$. Moreover, Berikashvili in [6] proved the following result.

Theorem 5. *Let $f : A \to B$ be a map in DA inducing an isomorphism in homology. Then*

$$f_* : D(JC, IA) \to D(JC, IB)$$

is a bijection.

Let $f = \{f_i\} : A \to B$ be a $DASH$ map. Kadeishvili in [16] showed that if $\alpha \in T(JC, IA)$, then

$$\phi = \sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha)\triangle^i$$

belongs to $T(JC, IB)$. Thus for any DGC C a $DASH$ map $f = \{f_i\} : A \to B$ induces a set map

$$T(f) : T(JC, IA) \to T(JC, IB)$$

($\alpha \longmapsto \sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha)J\triangle^i$), which itself induces a set map

$$f_* : DT(JC, IA) \to D(JC, IB)$$

defined by $[\alpha] \longmapsto [\sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha)J\triangle^i]$. Kadeishvili in [16] generalized Theorem 5 in the following way.

Theorem 6. *Let $f = \{f_i\} : A \to B$ be a DASH map, such that its first component $f_1 : A \to B$ induces an isomorphism in homology. Then*

$$f_* : D(JC, IA) \to DT(JC, IB)$$

is a bijection.

Remark 7. Note that the surjectivity of $f_* : D(JC, IA) \to DT(JC, IB)$ implies that for any twisting cochain $\beta : JC \to IB$ there exists a twisting cochain $\alpha : JC \to IA$ making the diagram

$$\begin{tikzcd} JC \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & IA \arrow[d, "\{If_i\}"] \\ & IB \end{tikzcd}$$

homotopy commutative; i.e., $\beta \sim \sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha)J\triangle^i$ in $T(JC, IB)$.

It is known that if $\alpha, \alpha' : JC \to IA$ are equivalent twisting cochains, then $f_\alpha, f_{\alpha'} : \Omega C \to A$ (resp. $g_\alpha, g_{\alpha'} : C \to BA$) are homotopic in DA (resp. DC); see [17], for example.

We complete this section with

Theorem 8 ([18])**.** *Let K be a simplicial set.*

a) *Then in DGA $A = \mathrm{Hom}(C_*(K), \Omega C_*(K) \otimes \Omega C_*(K))$ there exists an element E of degree -1, satisfying the following conditions*

I) *$DE = -E \smile E$ (i.e., E is a twisting cochain)*

II) *The components $E^{1,0} : C \to C \otimes R$ and $E^{0,1} : C \to R \otimes C$ are given by*

$$E^{1,0}(x) = -x \otimes 1,$$
$$E^{0,1}(x) = -1 \otimes x.$$

III) *$E^{n,0} = E^{0,n} = 0$ when $n \neq 1$.*

b) *If E and E' are elements of A of degree -1, satisfying* I) *and having $E^{1,0} = E'^{1,0}$ and $E^{0,1} = E'^{0,1}$, then there exists $P \in A$ of degree 0 having $P^{0,0} = 0$ and satisfying*

$$E' = E + P \smile E - E' \smile P - DP.$$

In the next section we shall prove the dual theorem for Sullivan's functor A^* [10], [11], [19].

2. Sullivan Functor and Some Key Lemmas

We begin this section with a short sketch of the Sullivan functor $A(K)$ mainly following [10], [19]. For more details see [11].

A *simplicial object K with values in a category $\mathcal{C}$* is a sequence $\{K_n\}_{n\geq 0}$ of objects in $\mathcal{C}$, together with $\mathcal{C}$-morphisms

$$d_i : K_{n+1} \to K_n, \;\; 0 \leq i \leq n+1 \quad \text{and} \quad s_j : K_n \to K_{n+1}, \;\; 0 \leq j \leq n,$$

called face and degeneracy operators, satisfying certain identities, see [20] for example.

A *simplicial morphism* $f : K \to L$ between two such simplicial objects is a sequence of $\mathcal{C}$-morphisms $\varphi : K \to L$ commuting with d_i and s_j.

A *simplicial set* K is a simplicial object in the category of sets. Thus it consists of a sequence of sets $\{K_n\}_{n\geq 0}$ together with set maps.

A *simplicial cochain algebra* A is a simplicial object in the category of cochain algebras: it consists of a sequence of cochain algebras $\{A_n\}_{n\geq 0}$ with an appropriate face and degeneracy operators. Similarly, *simplicial cochain complexes, simplicial vector spaces* ... are simplicial objects in the category of cochain complexes, vector spaces ... .

2.1. The construction of $A(K)$**.** Let K be a simplicial set, and let A be a simplicial cochain complex or a simplicial cochain algebra. Then

$$A(K) = \{A^p(K)\}_{p\geq 0}$$

is an "ordinary" cochain complex (or cochain algebra) defined as follows:

• $A^p(K)$ is the set of simplicial set morphisms from K to A^p.

Thus an element $\Phi \in A^p(K)$ is a mapping that to each n-simplex $\sigma \in K_n$ $(n \geq 0)$ assigns an element $\Phi_\sigma \in (A^p)_n$ such that $\Phi_{d_i\sigma} = d_i\Phi_\sigma$ and $\Phi_{s_i\sigma} = s_i\Phi_\sigma$.

• Addition, scalar multiplication and differential are given by

$$(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma, \quad (\lambda \cdot \Phi)_\sigma = \lambda \cdot \Phi_\sigma \quad \text{and} \quad (d\Phi)_\sigma = d\Phi_\sigma.$$

• If A is a simplicial cochain algebra, multiplication in $A(K)$ is given by

$$(\Phi \cdot \Psi)_\sigma = \Phi_\sigma \cdot \Psi_\sigma.$$

• If $\varphi : K \to L$ is a morphism of simplicial sets, then $A(\varphi) : A(L) \to A(K)$ is a morphism of cochain complexes (or cochain algebras) defined by

$$(A(\varphi)\Phi)_\sigma = \Phi_{\varphi\sigma}.$$

• If $\Theta : A \to B$ is a morphism of simplicial cochain complexes (or simplicial cochain algebras), then $\Theta(K) : A(K) \to B(K)$ is a morphism defined by

$$(\Theta(K)\Phi)_\sigma = \Theta(\Phi_\sigma).$$

• When X is a topological space, we write $A(X)$ for $A(S_*(X))$ where $S_\star(X) = \{S_n(X)\}_{n\geq 0}$ is a simplicial set of singular simplices $\sigma : \triangle^n \to X$ on the topological space X.

Remark 9. Note that the construction $A(K)$ is covariant in A and contravariant in K.

Let us denote by $\triangle^n$ the Euclidean n-simplex $(\sum\limits_{i=0}^{n} t_i = 1,\ t_i \geq 0)$ in $R^{n+1}$and by $\triangle^n \underset{\sigma_i}{\overset{\mathfrak{d}_i}{\rightleftarrows}} \triangle^{n+1}$ the ordinary coface and codegeneracy operators given by

$$\mathfrak{d}_i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n) \quad \text{for} \quad 0 \leq i \leq n+1,$$

and

$$\sigma_i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}) \quad \text{for} \quad 0 \leq i \leq n.$$

Let $\triangle$ be a category whose objects are sets $\triangle^n$, while morphisms are compositions of $\mathfrak{d}_i$ and σ_i. A *simplicial object* in a category $\mathcal{B}$ we call any contravariant functor $K : \triangle \to \mathcal{B}$

Let us denote by $A_{DR}(\triangle^n)$ the classical cochain algebra of the C^∞ real differential forms on $\triangle^n$. The (differentiable) maps $\mathfrak{d}_i$ and σ_i induce face and degeneracy operators

$$A_{DR}(\triangle^{n-1}) \xleftarrow{d_i} A_{DR}(\triangle^n) \xrightarrow{s_i} A_{DR}(\triangle^{n+1})$$

such that the family $(A_{DR}(\triangle^n), d_i, s_i)$ is a commutative simplicial DGA, i.e., the simplicial object in the category of commutative differential algebras over the reals. For the sake of simplicity we remove $\triangle^n$ from the notation by writing A_{DR} for $A_{DR}(\triangle^n)$. The *Poincaré lemma allows us to prove that each* $A_{DR}(\triangle^n) = (A_{DR})_n$ *is acyclic* (*see* [19], *for example*).

Let us denote by A_Q a commutative simplicial DGA (over rationals), where $(A_Q)_n$ is a subspace in $(A_{DR})_n$ consisting of those differential forms

$$\sum \omega_{i_1 \cdots i_p} dt^{i_1} \wedge \cdots \wedge dt^{i_p}$$

coefficients $\omega_{i_1 \cdots i_p}$ of which relative in the barycentric coordinates $(t_0, t_1, \dots t_n)$ are rational polynomials.

For a classical homotopy operator we have

$$h : A^p_{DR}(\triangle^n) \to A^{p-1}_{DR}(\triangle^n),$$

with $1_{(A_{DR})^p_n} = dh + hd$, $h(A_Q)^p_n \subset (A_Q)^{p-1}_n$. Thus $(A_Q)_n$ is acyclic as well (see [19], for example).

Definition 10. The Sullivan functor on a topological space X is defined as a DGA $A^*(X, Q)$ with

$$A^p(X, Q) = \mathrm{Hom}(S_*(X), (A_Q)^p),$$

i.e., an element $\Phi \in A^p(X, Q)$ is a function assigning to each singular n-simplex of X a polynomial p-form on $\triangle^n$, $n \geq 0$, compatible with the face and degeneracy maps (see the construction of $A(K)$ at the beginning of this subsection). The Sullivan functor $A^*(X, Q)$ is referred to as the cochain algebra of polynomial differential forms on X over rationals.

By analogy we have $A^p(X, R) = \mathrm{Hom}(S_*(X), (A_{DR})^p)$ over the reals.

2.2. Acyclicity of the complex $\mathbf{Hom(A(X)^{\otimes \mathit{i}}, A(X))}$. In this section we'll prove two lemmas which will be used to prove the main theorem of the paper announced in Introduction.

Lemma 11. *The complex* $\mathrm{Hom}(A(X)^{\otimes i}, A(X))$ *is acyclic.*

We will use the machinery of contravariant acyclic models used by Gugenheim in [12]; the machinery of acyclic models was introduced (covariantly) by Eilenberg and MacLane in [9].

For any contravariant functor $K : \mathcal{J} \to$ {vector spaces} a new contravariant functor $\hat{K} : \mathcal{J} \to$ {vector spaces} is defined by

$$\hat{K}(X) = \prod_{u:M\to X} \{K(M), u\},$$

where the product is over all maps $u : M \to X$ of $\mathcal{J}$, and M is a "model", i.e., one of the standard simplices $\triangle^n$. If $f : X \to Y$ is a map, then $\hat{K}(f) : \hat{K}(Y) \to \hat{K}(X)$ is defined by

$$\hat{K}(f)\{m_v, v\} = \{m_{f\cdot u}, u\},$$

where $u : M \to X$, $v : M_v \to Y$, and $m_v \in K(M_v)$.

If $K, L : \mathcal{J} \to \mathcal{V}$ are contravariant functors, and $\Theta : K \to L$ is a transformation of functors, then there is a transformation of functors $\hat{\Theta} : \hat{K} \to \hat{L}$ defined by

$$\hat{\Theta}(X)\{m, u\} = \{\Theta(M)m, u\}.$$

For any contravariant functor $K : \mathcal{J} \to \mathcal{V}$, the transformation $\Phi : K \to \hat{K}$ is defined by

$$\Phi(X)h = \{K(u)h, u\} \qquad (h \in K(X)).$$

We easily verify that

$$\hat{\Theta}\Phi = \Phi\Theta.$$

In particular, for the functors A^* and C^* the functors $\hat{A}^*$ and $\hat{C}^*$ are defined by

$$(\hat{A})^n = \widehat{(A^n)}, \quad (\hat{C})^n = \widehat{(C^n)},$$

and we get differentials

$$\hat{d} : \hat{A}^* \to \hat{A}^*, \quad \hat{d} : \hat{C} \to \hat{C}^*.$$

To prove Lemma 11 we need the following

Proposition 12. *The Functor A^* is corepresentable, i.e., there is a transformation of functors $\Psi : \hat{A}^* \to A^*$ such that $\Psi\Phi =$ the identity.*

Proof. We define $\Psi(X) : \hat{A}^*(X) \to A^*(X)$ by

$$\langle \Psi(X)\{m_u, u\}, v\rangle = \langle m_v, 1_{M_v}\rangle$$

where $m_u \in A^*(M_u)$, $m_v \in A^*(M_v)$, $v : M_v \to X$, and 1_{M_v} is an identity map of M_v regarded as a singular chain. Let $h \in A^*(X)$, then

$$\langle \Psi\Phi(h), v\rangle = \langle \Psi\{A^*(u)h, u\}, v\rangle = \langle A^*(v)h, 1_{M_v}\rangle = \langle h, v(1_{M_v})\rangle = \langle h, v\rangle,$$

i.e., $\Psi\Phi =$ the identity. $\square$

We have mentioned in Subsection 2.1 that the functor A^* is acyclic on models M. Moreover there is a homotopy operator (contraction)

$$h_M : A^*(M) \to A^*(M)$$

with $1_{A^*(M)} = dh_M + h_M d$. It follows from the definition of tensor product that the functors

$$A^{*\otimes i} : A^{*\otimes i}(M) \to A^{*\otimes i}(M)$$

are acyclic on models as well. Let fix for each $i \geq 1$ and a model M , once and for all, a contraction

$$h^i_M : A^{*\otimes i}(M) \to A^{*\otimes i}(M).$$

For such cases Gugenheim [12] defined the transformation of functors

$$\hat{h}^{\,i} : \hat{A}^{*\otimes i}(X) \to \otimes^i \hat{A}^{*\otimes i}(X)$$

by $\hat{h}^{\,i}_M\{\omega_u, u\} = \{h^i_M \omega_u, u\}$, where $u : M \to X$. Clearly, $D\hat{h}^{\,i}_M = 1_{\otimes^i \hat{A}^*(M)}$ (see [12], for instance).

Proof of Lemma 11. Thus we have to show that if $Df = 0$ for any $f \in \mathrm{Hom}(A^{\otimes i}(X), A(X))$, $\deg(f) = k$, then there exists $g \in \mathrm{Hom}(A^{\otimes i}(X), A(X))$ of degree $k-1$ satisfying $Dg = f$.

Let us assume that g is defined in dimension $< n$, i.e., on $(A^{\otimes i}(X))_{<\,n}$, and satisfies $dg_{j-1} - (-1)^{k-1} g_j d = f_{j-1}$ for any $j < n$. We write the defining equation for g_n as

$$g_n d = (-1)^{k-1}(dg_{n-1} - f_{n-1}).$$

For simplicity, we denote $\Theta_i = (-1)^{k-1}(dg_{n-1} - f_{n-1})$. Thus Θ_i is defined in dimension $n-1$ and we have the following diagram:

$$\begin{array}{ccccc} & & (A^*)_{n-1+k} & & \\ & & {\scriptstyle \Theta_i} \nearrow & & \\ (A^{*\otimes i})_{n-2} & \underset{d}{\longrightarrow} & (A^{*\otimes i})_{n-1} & \underset{d}{\longrightarrow} & (A^{*\otimes i})_n\,. \end{array}$$

By the inductive hypothesis, $\Theta_i d = 0$; indeed,

$$\begin{aligned} (-1)^{k-1}\{dg_{n-1} - f_{n-1}\}d &= (-1)^{k-1} dg_{n-1} d + df_{n-2} \\ &= d\{(-1)^{k-1} g_{n-1} d + f_{n-2}\} = ddg_{n-2} = 0. \end{aligned}$$

We have used here $Df = 0$. Since in the above diagram is functorial, we get

$$\begin{array}{ccccc} & & (\hat{A}^*)_{n-1+k} & & \\ & & {\scriptstyle \hat{\Theta}_i} \nearrow \qquad \nwarrow {\scriptstyle k_i} & & \\ (\hat{A}^{*\otimes i})_{n-2} & \underset{\hat{d}}{\longrightarrow} & (\hat{A}^{*\otimes i})_{n-1} & \underset{\hat{d}}{\overset{\hat{h}^{\,i}}{\longleftrightarrows}} & (\hat{A}^{*\otimes i})_n\,, \end{array}$$

where $\hat{\Theta}_i \hat{d} = 0$. Now we define $k_i = \hat{\Theta}_i \hat{h}^{\,i}$ and obtain

$$k_i \hat{d} = \hat{\Theta}_i \hat{h}^{\,i} \hat{d} = \hat{\Theta}_i (1 - \hat{d}\hat{h}^{\,i}) = \hat{\Theta}_i.$$

Next we define g_n on $(A^{*\otimes i}(X))_n$ by

$$g_n = \Psi k_i \Phi,$$

and verify

$$g_n d = \Psi k_i \Phi d = \Psi k_i \hat{d} \Phi = \Psi \hat{\Theta}_i \Phi = \Psi \Phi \Theta_i = \Theta_i,$$

as required.

2.3. Twisting cochains on Hom(BA*⊗BA*,A*). Sometimes it is be useful to represent the homomorphism $E : BA^* \otimes BA^* \to A^*$ by the collection of its components

$$\Big\{E^{p,q} = E|_{s^{-1}A^{\otimes p}\otimes s^{-1}A^{\otimes q}}((s^{-1})^{\otimes p} \otimes (s^{-1})^{\otimes q}) : (A^*)^{\otimes p} \otimes (A^*)^{\otimes q} \to A^*\Big\}.$$

Note that if $\deg E = k$, then $\deg E^{p,q} = k - (p+q)$.

In the set of all twisting cochains from $BA^* \otimes BA^*$ to A^* there is one remarkable element. This is the "generator" $E_{sh} = p_1 \mu_{sh}$ of the shuffle product $\mu_{sh} : BA^* \otimes BA^* \to BA^*$. Indeed, since the multiplication on A^* is commutative, the Bar construction $(BA^*, d_{BA^*}, \mu_{sh}, \triangle)$ of A^* is a DGHA (see Subsection 1.7), i.e., $\mu_{sh} \in DC$, by bijection (1.8.3)

$$E_{sh} = p_1 \mu_{sh} : BA^* \otimes BA^* \to A^*$$

is a twisting cochain. As we easily see E_{sh} consists only of two nontrivial components: $E^{0,1}_{sh} = 1 : R \otimes A \to A$ and $E^{0,1}_{sh} = 1 : A \otimes R \to A$ to which we will return in the next section.

Let $F \in \mathrm{Hom}(BA^* \otimes BA^*, A^*)$. Let F^n denote the sum of components $F^{p,q}$ of F with $p+q = n$:

$$F^n = \sum_{p+q=n} F^{p,q}.$$

We call a number $p+q$ filtration of F.

Remark 13. It is easy to see that the product of two homogenous elements F^{p_1,q_1} and F^{p_2,q_2} in $\mathrm{Hom}(BA^* \otimes BA^*, A^*)$ is a homogenous element of type $F^{p_1+p_2,q_1+q_2}$. Thus we see that the multiplication in $\mathrm{Hom}(BA^* \otimes BA^*, A^*)$ is compatible with filtration.

For the sake of simplicity we write $\overline{A}$ for $\mathrm{Hom}(BA^* \otimes BA^*, A^*)$; if $F \in \overline{A}$ has degree m, we write $F \in \overline{A}^m$.

Definition 14. Let n be a given number. We say the elements $F_1, F_2 \in \overline{A}^m = \mathrm{Hom}^m(BA^* \otimes BA^*, A^*)$ coincide up to n (write $F_1 \overset{n}{=} F_2$) if

$$F_1^k = F_2^k$$

for each $k \leq n$.

Let X be a simplicial set. Then the differential D on $\mathrm{Hom}(BA^*(X)\otimes BA^*(X), A^*(X))$ can be expressed as a sum

$$D = D_1 + D_2,$$

where

$$D_1F = F(d_\wedge \otimes 1 + 1 \otimes d_\wedge) \quad D_2F = F(d_\otimes \otimes 1 + 1 \otimes d_\otimes) - (-1)^{|F|} d_{A^*(X)}F,$$

$d_\wedge$ and $d_\otimes$ are given by the formula

$$d_\wedge[\Phi_1|\cdots|\Phi_n] = \sum_{i=0}^{n-2} [\overline{\Phi}_1|\cdots|\overline{\Phi}_i|(\overline{\Phi}_{i+1}\cdot\Phi_{i+2})|\Phi_{i+3}|\cdots|\Phi_n],$$
$$d_\otimes[\Phi_1|\cdots|\Phi_n] = \sum_{i=0}^{n-1} [\overline{\Phi}_1|\cdots|\overline{\Phi}_i|d\Phi_{i+1}|\Phi_{i+2}|\cdots|\Phi_n],$$

where $\Phi_i \in A^*(X)$, $\overline{\Phi}_i = (-1)^{|\Phi_i|+1}\Phi_i$ (see Subsection 1.7).

Proposition 15. *Let F be an element in $\overline{A}$ having $F^{n-1} = 0$. Then the equality $DF \overset{n}{=} 0$ guarantees that $D_2F^n = 0$, i.e., F^n is a chain map of (chain) complexes*

$$\sum_{p+q=n} (A^*(X) \otimes \overset{p}{\cdots} \otimes A^*(X)) \otimes (A^*(X) \otimes \overset{q}{\cdots} \otimes A^*(X)) \to A^*(X).$$

Proof. Clear. □

Now we are ready to prove

Lemma 16. *Let the elements $E, E' \in \overline{A}^1$ satisfy the following conditions*
I) *$DE = -E \smile E$ and $DE' = -E' \smile E'$ (i.e., E and E' are twisting cochains);*
II) *$E^{1,0} = E'^{1,0}$ and $E^{0,1} = E'^{0,1}$.*
Then there exists $P \in \overline{A}^0$, $P^{0,0} = 0$, satisfying

$$E' = E + P \smile E - E' \smile P - DP$$

(*i.e., E and E' are equivalent twisting cochains*).

Proof. Note that it suffices to construct the sequence of elements in $\overline{A}^0$

$$P^{(0)}, P^{(1)}, P^{(2)}, \ldots$$

satisfying the following conditions
I) $P^{(0)} = P^{(1)} = 0$,
II) $P^{(i)} \overset{i}{=} P^{(i+1)}$, $\quad i = 0, 1, \ldots,$
III) $(P^{(i)})^k = 0 \quad$ when $k = 0, 1$ or $k > i$,
IV) $DP^{(i)} \overset{i}{=} E - E' + P^{(i-1)} \smile E - E' \smile P^{(i-1)}$.
After the construction of such a sequence we define P by

$$P \overset{i}{=} P^{(i)}, \quad i = 0, 1, \ldots.$$

Clearly, P thus defined satisfies the required identity $E' = E + P \smile E - E' \smile P - DP$.

Assume that $P^{(0)}, P^{(1)}, \dots, P^{(n-1)}$ have already been constructed. Now we want to construct $P^{(n)}$ with required conditions I–IV. Consider the following element in $\overline{A}^1$:

$$\beta(n-1) = E - E' + P^{(n-1)} \smile E - E' \smile P^{(n-1)} - DP^{(n-1)}.$$

According to conditions I–IV and Remark 13 we get

$$\begin{aligned} D\beta(n-1) &= DE - DE' + DP^{(n-1)} \smile E + P^{(n-1)} \smile DE - DE' \smile P^{(n-1)} \\ &\quad + E' \smile DP^{(n-1)} \overset{n}{=} -E \smile E + E' \smile E' + (E - E' \\ &\quad + P^{(n-2)} \smile E - E' \smile P^{(n-2)}) \smile E - P^{(n-1)} \smile E \smile E \\ &\quad + E' \smile E' \smile P^{(n-1)} + E' \smile (E - E' + P^{(n-2)} \smile E \\ &\quad - E' \smile P^{(n-2)}) = P^{(n-2)} \smile E \smile E - P^{(n-1)} \smile E \smile E \\ &\quad + E' \smile E' \smile P^{(n-1)} - E' \smile E' \smile P^{(n-2)} \overset{n}{=} 0. \end{aligned}$$

(We have used here equalities like $P^{(n-2)} \smile E \smile E \overset{n}{=} P^{(n-1)} \smile E \smile E$; indeed, since $P^{(n-2)}$, $P^{(n-1)}$ and E have no nontrivial components with filtration of 0, multiplication is compatible with filtration (Remark 13) and $P^{(n-2)} \overset{n-2}{=} P^{(n-1)}$ (condition II), elements $P^{(n-2)} \smile E \smile E$ and $P^{(n-1)} \smile E \smile E$ coincide up to n.) Since $\beta(n-1)$ has no nontrivial components with filtration $\leq n-1$, by Proposition 15

$$D_2\beta(n-1)^n = 0,$$

or, equivalently, $\beta(n-1)^n$ is a cycle in the chain complex

$$\operatorname{Hom}\Big(\sum_{p+q=n} (A^*(X) \otimes \overset{p}{\cdots} \otimes A^*(X)) \otimes (A^*(X) \otimes \overset{q}{\cdots} \otimes A^*(X)), A^*(X)\Big).$$

Since the complex above is acyclic (see Lemma 11) there exists an element P^n such that

$$D_2P^n = \beta(n-1)^n.$$

Now we define $P^{(n)}$ by

$$P^{(n)} = P^{(n-1)} + P^n.$$

Thus it remains to show that $P^{(n)}$ together with $P^{(0)}, P^{(1)}, \dots, P^{(n-1)}$ satisfies condition I–IV. The first three conditions are checked trivially. Let us show the validity of IV. Since $D_2P^n = \beta(n-1)^n$ and $\beta(n-1)$ has no nontrivial components with filtration $\leq n-1$, we have

$$DP^n \overset{n}{=} \beta(n-1).$$

Then

$$\begin{aligned} DP^{(n)} &= DP^{(n-1)} + DP^n \overset{n}{=} DP^{(n-1)} + \beta(n-1) \\ &= DP^{(n-1)} - DP^{(n-1)} + E - E' + P^{(n-1)} \smile E - E' \smile P^{(n-1)} \end{aligned}$$

$$= E - E' + P^{(n-1)} \smile E - E' \smile P^{(n-1)}.$$

Thus Lemma 16 is completely proved. $\square$

3. Multiplicativity up to Homotopy

In this section we will prove the main theorem of this paper. We use all the lemmas and theorems which we have proved or mentioned in the preceding sections.

Theorem 17. *Gugenheim's map*

$$B\{\rho_i\} : (BA^*(X), d_{BA^*}, \triangle, \mu_{sh}) \to (BC^*(X), d_{BC^*}, \triangle, \mu_E)$$

is multiplicative up to homotopy.

Proof. As we have mentioned in Introduction, Baues introduced a multiplication

$$\mu_E : BC^*(X) \otimes BC^*(X) \to BC^*(X)$$

on $BC^*(X)$ so that $(BC^*(X), d_{BC^*(X)}, \triangle, \mu_E)$ is a DGHA, see [4]. Moreover, it is shown that this product is strict associative and homotopy commutative. Let consider the twisting cochain

$$BA^*(X) \otimes BA^*(X) \overset{B\{\rho_i\}\otimes B\{\rho_i\}}{\longrightarrow} BC^*(X) \otimes BC^*(X) \overset{\mu_E}{\to} BC^*(X) \overset{P_1}{\to} C^*(X)$$

which is the composition of coalgebra maps and the twisting cochain P_1, see Subsection 1.8.

Since the first component $\rho = \rho_1 : A^*(X) \to C^*(\mathrm{X})$ ($\langle \rho\omega, c\rangle = \int_c \omega$) of Gugenheim's ($DASH$) map

$$\rho_i : \otimes^i A^*(X) \to C^*(X) \quad ; (i \geq 1)$$

induces an isomorphism in homology, Theorem 6 says that there exists a twisting cochain (see Remark 7)

$$\overline{E} : BA^*(X) \otimes BA^*(X) \to A^*(X)$$

satisfying $P_1\mu_E(B\{\rho_i\} \otimes B\{\rho_i\}) \sim \sum_i \rho_i(\overline{E} \otimes \cdots \otimes \overline{E})\triangle^i = P_1 B\{\rho_i\}\mu_{\overline{E}}$, where

$$\mu_{\overline{E}} : BA^*(X) \otimes BA^*(X) \to BA^*(X)$$

is the coalgebra map induced by $\overline{E}$ (see the universal property of the tensor coalgebra, Subsection 1.7). As we know, equivalent twisting cochains on $T(C, A)$ induce homotopic DG coalgebra maps on $\mathrm{Hom}(C, BA)$ (see Subsection 1.8); thus $\mu_E(B\{\rho_i\} \otimes B\{\rho_i\}) \sim B\{\rho_i\}\mu_{\overline{E}}$ in DC, i.e.,

$$B\{\rho_i\} : (BA^*(X), d_{BA^*}, \triangle, \mu_{\overline{E}}) \to (BC^*(X), d_{BC^*}, \triangle, \mu_E)$$

is multiplicative up to homotopy, but there is no guarantee that such randomly constructed $\mu_{\overline{E}}$ coincides with μ_{sh}.

Let us assume that the components $\overline{E}^{0,1}$ and $\overline{E}^{1,0}$ of $\overline{E}$ are identity maps (we show below that this is always achievable for our $\overline{E}$). Then we have two twisting cochains E_{sh} (E_{sh} is the "generator" of the shuffle product μ_{sh}, see Subsection

2.3) and $\overline{E}$ on $\mathrm{Hom}(BA^*(X)\otimes BA^*(X),A^*(X)$ having $\overline{E}^{0,1}=E_{sh}^{1,0}=1$ and $\overline{E}^{1,0}=E_{sh}^{1,0}=1$; according to Lemma 16 there exists $P\in\mathrm{Hom}(BA^*(X)\otimes BA^*(X),A^*(X))$ such that $E_{sh}\overset{P}{\sim}\overline{E}$, hence $\mu_{sh}\sim\mu_{\overline{E}}$ or, equivalently,

$$1:(BA^*(X),d_{BA^*},\triangle,\mu_{sh})\to(BA^*(X),d_{BA^*},\triangle,\mu_{\overline{E}})$$

is multiplicative up to homotopy. Taking the composition

$$\begin{aligned}B\{\rho_i\}\cdot 1:(BA^*(X),d_{BA^*},\triangle,\mu_{sh})&\\ \to(BA^*(X),d_{BA^*},\triangle,\mu_{\overline{E}})&\to(BC^*(X),d_{BC^*},\triangle,\mu_E),\end{aligned}$$

we have that Gugenheim's map

$$B\{\rho_i\}:(BA^*(X),d_{BA^*},\triangle,\mu_{sh})\to(BC^*(X),d_{BC^*},\triangle,\mu_E)$$

is multiplicative up to homotopy.

Thus, it remains to show that the components $\overline{E}^{0,1}$ and $\overline{E}^{1,0}$ are the identity maps. First of all, note that $\mu_{\overline{E}}$ is a homotopy commutative, homotopy associative and homotopy compatible with unit. Indeed,

$$\begin{aligned}\{\rho_i\}_*(\overline{E})&=P_1B\{\rho_i\}\mu_{\overline{E}}\sim P_1\mu_E(B\{\rho_i\}\otimes B\{\rho_i\})\sim P_1\mu_ET(B\{\rho_i\}\otimes B\{\rho_i\})\\ &=P_1\mu_E(B\{\rho_i\}\otimes B\{\rho_i\})T\sim P_1B\{\rho_i\}\mu_{\overline{E}}T=P_1B\{\rho_i\}\mu_{\overline{E}T}=\{\rho_i\}_*(\overline{E}T),\end{aligned}$$

where we have used the fact that $\mu_E\sim\mu_ET$ on $BC^*(X)$ [4]. Since $\{\rho_i\}_*(\overline{E})\sim\{\rho_i\}_*(\overline{E}T)$, according to the bijection of Theorem 6

$$\{\rho_i\}_*:D(BA^*(X)\otimes BA^*(X),A^*(X))\to D(BA^*(X)\otimes BA^*(X),C^*(X)),$$

$\overline{E}\sim\overline{E}T$, hence $\mu_{\overline{E}}\sim\mu_{\overline{E}}T$; i.e., $\mu_{\overline{E}}$ is homotopy commutative. By analogy, we can easily show that it is homotopy associative and has a homotopy unit. Then, according to Anick's fundamental theory [2], $\mu_{\overline{E}}$ can be replaced by the new product

$$\mu_F\sim\mu_{\overline{E}}:BA^*(X)\otimes BA^*(X)\to BA^*(X)$$

which is strictly commutative, associative and compatible with unit (see Definition 5.3, Lemma 5.4 and Proposition 5.5 in [2]). The latter means that the twisting cochain

$$F=P_1\mu_F:BA^*(X)\otimes BA^*(X)\to BA^*(X)\to A^*(X)$$

has $F^{0,1}=1$, $F^{1,0}=1$ and, moreover, $F^{0,k}=F^{k,0}=0$ when $k\neq 1$. This completes the proof. $\square$

Acknowledgement

The research described in this publication was made possible in part by Award No. GM1–2083 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

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(Received 7.11.2001)

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