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SOLVABILITY AND THE UNIQUE SOLVABILITY OF A PERIODIC TYPE BOUNDARY VALUE PROBLEM FOR FIRST ORDER SCALAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Nonimprovable in a certain sense, sufficient conditions for the solvability and unique solvability of the problem

$$u'(t) = F(u)(t), \qquad u(a) - \lambda u(b) = h(u)$$

are established, where $F : C([a,b];R) \to L([a,b];R)$ is a continuous operator satisfying the Carathéodory condition, $h : C([a,b];R) \to R$ is a continuous functional, and $\lambda \in R_+$.

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Introduction

The following notation is used throughout.

R is the set of all real numbers, $R_+ = [0,+\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$B^i_{\lambda c}([a,b];R)$ is the set of functions $u \in C([a,b];R)$ satisfying the condition $(-1)^{i+1}(u(a) - \lambda u(b)) \operatorname{sgn}((2-i)u(a) + (i-1)u(b)) \le c$, where $c \in R$, $i = 1,2$.

$\widetilde{C}([a,b];D)$, where $D \subseteq R$, is the set of absolutely continuous functions $u : [a,b] \to D$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \ge 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{L}_{ab}$ is the set of linear operators $\ell : C([a,b];R) \to L([a,b];R)$ for which there is a function $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \quad \text{for} \quad t \in [a,b], \quad v \in C([a,b];R).$$

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ mapping the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

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K_{ab} is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Carathéodory condition, i.e., for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \le q_r(t) \quad \text{for} \quad t \in [a,b], \quad \|v\|_C \le r.$$

$K([a,b] \times A; B)$, where $A \subseteq R^2$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \le q_r(t) \quad \text{for} \quad t \in [a,b], \ x \in A, \ \|x\| \le r.$$

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

By a solution of the equation

$$u'(t) = F(u)(t), \tag{0.1}$$

where $F \in K_{ab}$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (0.1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (0.1) satisfying the boundary condition

$$u(a) - \lambda u(b) = h(u), \tag{0.2}$$

where $h : C([a,b];R) \to R$ is a continuous functional such that for each $r > 0$ there exists $M_r \in R_+$ such that

$$|h(v)| \le M_r \quad \text{for} \quad \|v\|_C \le r,$$

and $\lambda \in R_+$.

In the case where F is the so–called Nemitsky operator, the problem (0.1), (0.2) and analogous problems for systems of linear and nonlinear ordinary differential equations have been studied in detail (see [7, 15–18] and the references therein). The foundation of the theory of general boundary value problems for functional differential equations were laid in the monographs [1] and [30] (see also [2, 3, 8, 14, 19–27, 29]). In spite of a large number of papers devoted to boundary value problems for functional differential equations, at present only a few efficient sufficient solvability conditions are known even for the linear problem

$$u'(t) = \ell(u)(t) + q_0(t), \tag{0.3}$$

$$u(a) - \lambda u(b) = c_0, \tag{0.4}$$

where $\ell \in \mathcal{L}_{ab}$, $q_0 \in L([a,b];R)$, $\lambda \in R_+$, and $c_0 \in R$ (see [5, 6, 8–13, 19–26, 29]). Here we attempt to fill this gap in a certain way. More precisely, in Sections 1 and 2, nonimprovable efficient sufficient conditions are established for the solvability and unique solvability of the problems (0.3), (0.4) and (0.1), (0.2), respectively. Section 3 is devoted to the examples showing the optimality of our main results.

Finally, all results are concretized for the differential equations with deviating arguments of the forms

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))) \tag{0.5}$$

and

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q_0(t), \tag{0.6}$$

where $p, g \in L([a, b]; R_+)$, $q_0 \in L([a, b]; R)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, and $f \in K([a, b] \times R^2; R)$.

1. Linear Problem

We need the following well–known result from the general theory of linear boudary value problems for functional differential equations (see, e.g., [4, 21, 30]).

Theorem 1.1. *The problem* (0.3), (0.4) *is uniquely solvable iff the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{0.3$_0$}$$

$$u(a) - \lambda u(b) = 0 \tag{0.4$_0$}$$

has only a trivial solution.

Remark 1.1. It follows from the Riesz–Schauder theory that if $\ell \in \mathcal{L}_{ab}$ and the problem $(0, 3_0)$, $(0, 4_0)$ has a nontrivial solution, then there exist $q_0 \in L([a, b]; R)$ and $c_0 \in R$ such that the problem (0.3), (0.4) has no solution.

1.1. Main Results.

Theorem 1.2. *Assume that* $\lambda \in [0, 1[$*, the operator* ℓ *admits the representation* $\ell = \ell_0 - \ell_1$*, where*

$$\ell_0, \ell_1 \in \mathcal{P}_{ab}, \tag{1.1}$$

and let there exist a function $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ *satisfying the inequalities*

$$\gamma'(t) \geq \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \text{for} \quad t \in [a, b], \tag{1.2}$$

$$\gamma(a) > \lambda\gamma(b), \tag{1.3}$$

$$\gamma(b) - \gamma(a) < 3 + \lambda. \tag{1.4}$$

Then the problem (0.3), (0.4) *has a unique solution.*

Remark 1.2. Theorem 1.2 is nonimprovable in a certain sense. More precisely, the strict inequality (1.4) cannot be replaced by nonstrict one (see On Remark 1.2 and Example 3.1).

Remark 1.3. Let $\lambda \in [1, +\infty[$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Introduce the operator $\psi : L([a,b];R) \to L([a,b];R)$ by

$$\psi(w)(t) \stackrel{def}{=} w(a+b-t) \quad \text{for} \quad t \in [a,b].$$

Let φ be a restriction of ψ to the space $C([a,b];R)$. Put $\vartheta = \frac{1}{\lambda}$, and

$$\widehat{\ell}_0(w)(t) \stackrel{def}{=} \psi(\ell_0(\varphi(w)))(t), \qquad \widehat{\ell}_1(w)(t) \stackrel{def}{=} \psi(\ell_1(\varphi(w)))(t) \quad \text{for} \quad t \in [a,b].$$

It is clear that if u is a solution of the problem $(0,3_0)$, $(0,4_0)$, then the function $v \stackrel{def}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}_1(v)(t) - \widehat{\ell}_0(v)(t), \qquad v(a) - \vartheta v(b) = 0, \tag{1.5}$$

and, conversely, if v is a solution of the problem (1.5), then the function $u \stackrel{def}{=} \varphi(v)$ is a solution of the problem $(0,3_0)$, $(0,4_0)$.

It is also obvious that if a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfies the inequality (1.2), then $\beta \stackrel{def}{=} \varphi(\gamma)$ satisfies the inequality

$$\beta'(t) \le -\widehat{\ell}_0(\beta)(t) - \widehat{\ell}_1(1)(t) \quad \text{for} \quad t \in [a,b], \tag{1.6}$$

and, conversely, if a function $\beta \in \widetilde{C}([a,b];]0,+\infty[)$ satisfies the inequality (1.6), then $\gamma \stackrel{def}{=} \varphi(\beta)$ satisfies the inequality (1.2).

In view of Remark 1.3 the following statement is an immediate consequence of Theorem 1.2.

Theorem 1.3. *Let $\lambda \in]1,+\infty[$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let there exist a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities*

$$\gamma'(t) \le -\ell_1(\gamma)(t) - \ell_0(1)(t) \quad \textit{for} \quad t \in [a,b], \tag{1.7}$$

$$\gamma(a) < \lambda\gamma(b), \tag{1.8}$$

$$\gamma(a) - \gamma(b) < 3 + \frac{1}{\lambda}\,. \tag{1.9}$$

Then the problem (0.3), (0.4) *has a unique solution.*

Remark 1.4. According to Remarks 1.2 and 1.3, Theorem 1.3 is nonimprovable in that sense that the strict inequality (1.9) cannot be replaced by nonstrict one.

Theorem 1.2 implies the following assertions for the problem (0.6), (0.4).

Corollary 1.1. *Let $\lambda \in [0,1[$, $p, g \in L([a,b];R_+)$, and $\tau \in \mathcal{M}_{ab}$ be such that*

$$\lambda \exp\left(\int_a^b p(s)ds\right) < 1, \tag{1.10}$$

$$(t - \tau(t))p(t) \ge 0 \quad \textit{for} \quad t \in [a,b], \tag{1.11}$$

and

$$\frac{1-\lambda}{1-\lambda\exp\Big(\int\limits_a^b p(s)ds\Big)}\int\limits_a^b g(s)\exp\Big(\int\limits_s^b p(\xi)d\xi\Big)ds < 3+\lambda. \tag{1.12}$$

Then the problem (0.6), (0.4) *has a unique solution.*

Corollary 1.2. *Let* $\lambda \in [0,1[$, $p, g \in L([a,b];R_+)$, *and* $\tau \in \mathcal{M}_{ab}$ *be such that*

$$(1-\lambda)\bigg(\int\limits_a^b g(s)ds + \alpha_1\bigg) + (3+\lambda)\beta_1 < 3+\lambda, \tag{1.13}$$

where

$$\alpha_1 = \int\limits_a^b p(s)\bigg(\int\limits_a^{\tau(s)} g(\xi)d\xi\bigg)\exp\Big(\int\limits_s^b p(\xi)d\xi\Big)ds, \tag{1.14}$$

$$\beta_1 = \lambda\exp\bigg(\int\limits_a^b p(s)ds\bigg) + \int\limits_a^b p(s)\sigma(s)\bigg(\int\limits_s^{\tau(s)} p(\xi)d\xi\bigg)\exp\Big(\int\limits_s^b p(\xi)d\xi\Big)ds, \tag{1.15}$$

$$\sigma(t) = \frac{1}{2}\left(1+\operatorname{sgn}(\tau(t)-t)\right) \quad \textit{for} \quad t\in[a,b]. \tag{1.16}$$

Then the problem (0.6), (0.4) *has a unique solution.*

Remark 1.5. Corollaries 1.1 and 1.2 are nonimprovable in a certain sense. More precisely, the strict inequalities (1.12) in Corollary 1.1 and (1.13) in Corollary 1.2 cannot be replaced by nonstrict ones (see On Remark 1.2 and Example 3.1).

Theorem 1.3 implies the following statements.

Corollary 1.3. *Let* $\lambda \in]1,+\infty[$, $p, g \in L([a,b];R_+)$, *and* $\mu \in \mathcal{M}_{ab}$ *be such that*

$$\exp\bigg(\int\limits_a^b g(s)ds\bigg) < \lambda, \tag{1.17}$$

$$(\mu(t)-t)g(t) \geq 0 \quad \textit{for} \quad t\in[a,b], \tag{1.18}$$

and

$$\frac{\lambda-1}{\lambda-\exp\Big(\int_a^b g(\xi)d\xi\Big)}\int\limits_a^b p(s)\exp\Big(\int\limits_a^s g(\xi)d\xi\Big)ds < 3+\frac{1}{\lambda}\,. \tag{1.19}$$

Then the problem (0.6), (0.4) *has a unique solution.*

Corollary 1.4. *Let $\lambda \in]1, +\infty[$, $p, g \in L([a,b]; R_+)$, and $\mu \in \mathcal{M}_{ab}$ be such that*

$$\frac{\lambda - 1}{\lambda}\left(\int_a^b p(s)ds + \alpha_2\right) + \left(3 + \frac{1}{\lambda}\right)\beta_2 < 3 + \frac{1}{\lambda}\,, \tag{1.20}$$

where

$$\alpha_2 = \int_a^b g(s)\Big(\int_{\mu(s)}^b p(\xi)d\xi\Big) \exp\Big(\int_a^s g(\xi)d\xi\Big)ds, \tag{1.21}$$

$$\beta_2 = \frac{1}{\lambda}\exp\Big(\int_a^b g(s)ds\Big) + \int_a^b g(s)\sigma(s)\Big(\int_{\mu(s)}^s g(\xi)d\xi\Big)\exp\Big(\int_a^s g(\xi)d\xi\Big)ds, \tag{1.22}$$

$$\sigma(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}(t - \mu(t))\Big) \quad \textit{for} \quad t \in [a,b]. \tag{1.23}$$

Then the problem (0.6), (0.4) *has a unique solution.*

Remark 1.6. Corollaries 1.3 and 1.4 are nonimprovable in a certain sense. More precisely, the strict inequalities (1.19) in Corollary 1.3 and (1.20) in Corollary 1.4 cannot be replaced by nonstrict ones.

1.2. Proofs of Main Results. To prove Theorem 1.2, we need a result from [14]. Let us first introduce the following definition.

Definition 1.1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $V^+(\lambda)$ (resp. $V^-(\lambda)$) if the homogeneous problem $(0, 3_0)$, $(0, 4_0)$ has only a trivial solution and, for arbitrary $q_0 \in L([a,b]; R_+)$ and $c_0 \in R_+$, the solution of the problem (0.3), (0.4) is nonnegative (resp. nonpositive).

Remark 1.7. It follows immediately from Definition 1.1 that $\ell \in V^+(\lambda)$ (resp. $\ell \in V^-(\lambda)$) iff a certain theorem on differential inequalities holds for the equation (0.3), i.e., if $u, v \in \widetilde{C}([a,b]; R)$ satisfy the inequalities

$$u'(t) \le \ell(u)(t) + q_0(t), \qquad v'(t) \ge \ell(v)(t) + q_0(t) \quad \text{for} \quad t \in [a,b],$$
$$u(a) - \lambda u(b) \le v(a) - \lambda v(b),$$

then $u(t) \le v(t)$ (resp. $u(t) \ge v(t)$) for $t \in [a,b]$.

Lemma 1.1 ([14]). *Let $\lambda \in [0, 1[$ and $\ell \in \mathcal{P}_{ab}$. Then $\ell \in V^+(\lambda)$ iff there exist $\gamma \in \widetilde{C}([a,b];]0, +\infty[)$ satisfying*

$$\gamma'(t) \ge \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b],$$
$$\gamma(a) > \lambda\gamma(b).$$

Proof of Theorem 1.2. According to Theorem 1.1, it is sufficient to show that the homogeneous problem $(0, 3_0)$, $(0, 4_0)$ has only a trivial solution.

Assume that, on the contrary, the problem $(0, 3_0)$, $(0, 4_0)$ has a nontrivial solution u. By virtue of (1.1) and Lemma 1.1, it follows from (1.2) and (1.3) that $\ell_0 \in V^+(\lambda)$. Consequently, according to Definition 1.1, u must change its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = -\min\{u(t) : t \in [a,b]\}, \tag{1.24}$$

and choose $t_M, t_m \in [a,b]$ so that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{1.25}$$

Obviously,

$$M > 0, \qquad m > 0 \tag{1.26}$$

and without loss of generality we can assume that $t_M < t_m$. In view of (1.26), the relations $(0, 3_0)$, $(0, 4_0)$, (1.2) and (1.3) yield

$$\begin{gathered} (M\gamma(t) + u(t))' \geq \ell_0(M\gamma + u)(t) + \ell_1(M - u)(t) \quad \text{for} \quad t \in [a,b], \\ M\gamma(a) + u(a) - \lambda(M\gamma(b) + u(b)) \geq 0, \end{gathered} \tag{1.27}$$

and

$$\begin{gathered} (m\gamma(t) - u(t))' \geq \ell_0(m\gamma - u)(t) + \ell_1(m + u)(t) \quad \text{for} \quad t \in [a,b], \\ m\gamma(a) - u(a) - \lambda(m\gamma(b) - u(b)) \geq 0. \end{gathered} \tag{1.28}$$

Hence, according to (1.1), (1.24), the condition $\ell_0 \in V^+(\lambda)$, and Remark 1.7, it follows that

$$M\gamma(t) + u(t) \geq 0, \qquad m\gamma(t) - u(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

By virtue of the last two inequalities, (1.1), and (1.24), from (1.27) and (1.28) we get

$$(M\gamma(t) + u(t))' \geq 0, \qquad (m\gamma(t) - u(t))' \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{1.29}$$

In view of (1.25) and (1.26), the integration of the first inequality in (1.29) from t_M to t_m results in

$$M\gamma(t_m) - m - M\gamma(t_M) - M \geq 0,$$

i.e.,

$$\gamma(t_m) - \gamma(t_M) \geq 1 + \frac{m}{M}. \tag{1.30}$$

On the other hand, on account of (1.25) and (1.26), the integration of the second inequality in (1.29) from a to t_M and from t_m to b yields

$$\begin{gathered} m\gamma(t_M) - M - m\gamma(a) + u(a) \geq 0, \\ m\gamma(b) - u(b) - m\gamma(t_m) - m \geq 0. \end{gathered}$$

Summing these two inequalities and taking into account that

$$u(b) - u(a) = u(b)(1-\lambda) \geq -m(1-\lambda),$$

we get

$$\gamma(t_M) - \gamma(t_m) + \gamma(b) - \gamma(a) \geq \lambda + \frac{M}{m}. \tag{1.31}$$

Now from (1.30) and (1.31) we have

$$\gamma(b) - \gamma(a) \geq 1 + \lambda + \frac{m}{M} + \frac{M}{m} \geq 3 + \lambda,$$

which contradicts (1.4). $\square$

Proof of Corollary 1.1. According to (1.12), there exists $\varepsilon > 0$ such that

$$\frac{\varepsilon}{1 - \lambda \exp\Big(\int\limits_a^b p(s)ds\Big)} \left(\exp\Big(\int\limits_a^b p(s)ds \Big) - 1 \right) + \frac{1-\lambda}{1 - \lambda \exp\Big(\int\limits_a^b p(s)ds\Big)} \int\limits_a^b g(s) \exp\Big(\int\limits_s^b p(\xi)d\xi \Big) ds < 3 + \lambda.$$

Put

$$\begin{aligned}\gamma(t) = {} & \frac{\varepsilon}{1 - \lambda \exp\Big(\int\limits_a^b p(s)ds\Big)} \exp\Big(\int\limits_a^t p(s)ds \Big) \\ & + \frac{1}{1 - \lambda \exp\Big(\int\limits_a^b p(s)ds\Big)} \int\limits_a^t g(s) \exp\Big(\int\limits_s^t p(\xi)d\xi \Big) ds \\ & + \frac{\lambda \exp\Big(\int\limits_a^b p(s)ds\Big)}{1 - \lambda \exp\Big(\int\limits_a^b p(s)ds\Big)} \int\limits_t^b g(s) \exp\Big(\int\limits_s^t p(\xi)d\xi \Big) ds \quad \text{for} \quad t \in [a,b].\end{aligned}$$

Then γ is a solution of the problem

$$\gamma'(t) = p(t)\gamma(t) + g(t), \qquad \gamma(a) - \lambda\gamma(b) = \varepsilon. \tag{1.32}$$

Since $\varepsilon > 0$, in view of (1.10), we have $\gamma(t) > 0$ for $t \in [a,b]$. Consequently, (1.32) implies $\gamma'(t) \geq 0$ for $t \in [a,b]$, and (1.11) yields

$$p(t)\gamma(t) \geq p(t)\gamma(\tau(t)) \quad \text{for} \quad t \in [a,b]. \tag{1.33}$$

Therefore, by virtue of (1.32) and (1.33), the function γ satisfies the inequalities (1.2), (1.3), and (1.4) with

$$\ell_0(w)(t) \overset{def}{=} p(t)w(\tau(t)), \quad \ell_1(w)(t) \overset{def}{=} g(t)w(\mu(t)) \\ \text{for} \quad t \in [a,b]. \qquad \square \tag{1.34}$$

Proof of Corollary 1.2. Let the operators ℓ_0 and ℓ_1 be defined by (1.34). From (1.13) it follows that $\beta_1 < 1$. Consequently, by [14, Theorem 2.1 c)], we have $\ell_0 \in V^+(\lambda)$. Choose $\delta > 0$ and $\varepsilon > 0$ such that

$$(1-\lambda)(1-\beta_1)^{-1}\left(\alpha_1 + \int_a^b g(s)ds\right) < 3 + \lambda - \delta, \tag{1.35}$$

$$\varepsilon < \frac{\delta(1-\beta_1)}{1-\lambda}\exp\left(-\int_a^b p(s)ds\right). \tag{1.36}$$

According to the condition $\ell_0 \in V^+(\lambda)$ and Theorem 1.1, the problem

$$\gamma'(t) = p(t)\gamma(\tau(t)) + g(t), \tag{1.37}$$

$$\gamma(a) - \lambda\gamma(b) = \varepsilon \tag{1.38}$$

has a unique solution γ. It is clear that the conditions (1.2) and (1.3) are fulfilled. Due to the conditions $\ell_0 \in V^+(\lambda)$, $g \in L([a,b];R_+)$, and $\varepsilon > 0$, we get $\gamma(t) \geq 0$ for $t \in [a,b]$. Hence, by (1.37), we find that $\gamma(t) > 0$ for $t \in [a,b]$. On the other hand, γ is a solution of the equation

$$\gamma'(t) = p(t)\gamma(t) + p(t)\int_t^{\tau(t)} p(s)\gamma(\tau(s))ds + p(t)\int_t^{\tau(t)} g(s)ds + g(t).$$

Hence, the Cauchy formula implies

$$\gamma(b) \leq \beta_1\gamma(b) + \alpha_1 + \int_a^b g(s)ds + \varepsilon\exp\left(\int_a^b p(s)ds\right).$$

The last inequality results in

$$\gamma(b) \leq (1-\beta_1)^{-1}\left(\alpha_1 + \int_a^b g(s)ds\right) + \varepsilon(1-\beta_1)^{-1}\exp\left(\int_a^b p(s)ds\right)$$

and thus, in view of (1.35), (1.36), and (1.38), we have

$$\gamma(b) - \gamma(a) \leq (1-\lambda)\gamma(b) < 3 + \lambda.$$

Therefore the assumptions of Theorem 1.2 are fulfilled. $\square$

Corollaries 1.3 and 1.4 can be proved analogously.

2. Nonlinear Problem

In what follows we will assume that $q \in K([a,b] \times R; R_+)$ is nondecreasing in the second argument and satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0. \tag{2.1}$$

2.1. Main Results.

Theorem 2.1. *Let* $\lambda \in [0,1[$, $c \in R_+$,

$$h(v) \operatorname{sgn} v(a) \leq c \quad \textit{for} \quad v \in C([a,b]; R), \tag{2.2}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that the inequality*

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)] \operatorname{sgn} v(t) \leq q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b] \tag{2.3}$$

holds on the set $B^1_{\lambda c}([a,b]; R)$. *If, moreover, there exists* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying inequalities* (1.2), (1.3), *and*

$$\gamma(b) - \gamma(a) < 2, \tag{2.4}$$

then the problem (0.1), (0.2) *has at least one solution.*

Remark 2.1. Theorem 2.1 is nonimprovable in a certain sense. More precisely, the inequality (2.4) cannot be replaced by the inequality

$$\gamma(b) - \gamma(a) \leq 2 + \varepsilon \tag{2.5}$$

no matter how small $\varepsilon > 0$ is (see On Remark 2.1 and Example 3.2).

Theorem 2.2. *Let* $\lambda \in [0,1[$,

$$[h(v) - h(w)] \operatorname{sgn}(v(a) - w(a)) \leq 0 \quad \textit{for} \quad v, w \in C([a,b]; R), \tag{2.6}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that the inequality*

$$\begin{aligned} \Big[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) \\ + \ell_1(v-w)(t)\Big] \operatorname{sgn}(v(t) - w(t)) \leq 0 \quad \textit{for} \quad t \in [a,b] \end{aligned} \tag{2.7}$$

is fulfilled on the set $B^1_{\lambda c}([a,b]; R)$, *where* $c = |h(0)|$. *If, moreover, there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the inequalities* (1.2), (1.3), *and* (2.4), *then the problem* (0.1), (0.2) *is uniquely solvable.*

Remark 2.2. Theorem 2.2 is nonimprovable in the sense that the inequality (2.4) cannot be replaced by the inequality (2.5) no matter how small $\varepsilon > 0$ is (see On Remark 2.2).

Remark 2.3. Let $\lambda \in [1,+\infty[$, and φ, ψ be the operators defined in Remark 1.3. Put $\vartheta = \frac{1}{\lambda}$, and

$$\widehat{F}(w)(t) \overset{def}{=} -\psi(F(\varphi(w)))(t) \quad \text{for} \quad t \in [a,b], \qquad \widehat{h}(w) \overset{def}{=} -\vartheta h(\varphi(w)).$$

It is clear that if u is a solution of the problem (0.1), (0.2), then the function $v \overset{def}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \qquad v(a) - \vartheta v(b) = \widehat{h}(v), \tag{2.8}$$

and vice versa, if v is a solution of the problem (2.8), then the function $u \overset{def}{=} \varphi(v)$ is a solution of the problem (0.1), (0.2).

Therefore, according to Remarks 1.3 and 2.3, Theorems 2.1 and 2.2 imply

Theorem 2.3. *Let $\lambda \in]1,+\infty[$, $c \in R_+$,*

$$h(v)\operatorname{sgn} v(b) \geq -c \quad \textit{for} \quad v \in C([a,b];R), \tag{2.9}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)]\operatorname{sgn} v(t) \geq -q(t, \|v\|_C) \quad \textit{for} \quad t \in [a,b]$$

is fulfilled on the set $B^2_{\lambda c}([a,b];R)$. If, moreover, there exists $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying inequalities (1.7), (1.8), *and*

$$\gamma(a) - \gamma(b) < 2, \tag{2.10}$$

then the problem (0.1), (0.2) *has at least one solution.*

Theorem 2.4. *Let $\lambda \in]1,+\infty[$,*

$$[h(v) - h(w)]\operatorname{sgn}(v(b) - w(b)) \geq 0 \quad \textit{for} \quad v, w \in C([a,b];R), \tag{2.11}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality

$$\begin{gathered}[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) \\ +\ell_1(v-w)(t)]\operatorname{sgn}(v(t) - w(t)) \geq 0 \quad \textit{for} \quad t \in [a,b]\end{gathered}$$

is fulfilled on the set $B^2_{\lambda c}([a,b];R)$, where $c = |h(0)|$. If, moreover, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying inequalities (1.7), (1.8), *and* (2.10), *then the problem* (0.1), (0.2) *is uniquely solvable.*

Remark 2.4. According to Remarks 1.3 and 2.1–2.3, Theorems 2.3 and 2.4 are nonimprovable in the sense that the strict inequality (2.10) cannot be replaced by the inequality

$$\gamma(a) - \gamma(b) \leq 2 + \varepsilon$$

no matter how small $\varepsilon > 0$ is.

Theorems 2.1 and 2.2 imply the following results for the problem (0.5), (0.2).

Corollary 2.1. *Let* $\lambda \in [0,1[$, $c \in R_+$, $p, g \in L([a,b];R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and let* (2.2) *and the condition*

$$f(t,x,y)\operatorname{sgn} x \le q(t,|x|) \quad \textit{for} \quad t \in [a,b], \ x, y \in R \tag{2.12}$$

be fulfilled. If, moreover, the inequalities (1.10), (1.11), *and*

$$\frac{1-\lambda}{1-\lambda\exp\Big(\int\limits_a^b p(s)ds\Big)} \int\limits_a^b g(s)\exp\Big(\int\limits_s^b p(\xi)d\xi\Big)ds < 2 \tag{2.13}$$

hold, then the problem (0.5), (0.2) *has at least one solution.*

Corollary 2.2. *Let* $\lambda \in [0,1[$, $c \in R_+$, $p, g \in L([a,b];R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and let conditions* (2.2) *and* (2.12) *be fulfilled. If, moreover,*

$$(1-\lambda)\bigg(\int\limits_a^b g(s)ds + \alpha_1\bigg) + 2\beta_1 < 2 \tag{2.14}$$

holds, where α_1 *and* β_1 *are defined by* (1.14) *and* (1.15) *with* σ *given by* (1.16), *then the problem* (0.5), (0.2) *has at least one solution.*

Remark 2.5. Corollaries 2.1 and 2.2 are nonimprovable in a certain sense. More precisely, the strict inequalities (2.13) in Corollary 2.1 and (2.14) in Corollary 2.2 cannot be replaced by the inequalities

$$\frac{1-\lambda}{1-\lambda\exp\Big(\int\limits_a^b p(s)ds\Big)} \int\limits_a^b g(s)\exp\Big(\int\limits_s^b p(\xi)d\xi\Big)ds \le 2+\varepsilon \tag{2.15}$$

and

$$(1-\lambda)\bigg(\int\limits_a^b g(s)ds + \alpha_1\bigg) + 2\beta_1 \le 2+\varepsilon, \tag{2.16}$$

no matter how small $\varepsilon > 0$ is (see On Remark 2.1 and Example 3.2).

Corollary 2.3. *Let* $\lambda \in [0,1[$, $p, g \in L([a,b];R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and let* (2.6) *and the condition*

$$\begin{gathered}[f(t,x_1,y_1) - f(t,x_2,y_2)]\operatorname{sgn}(x_1-x_2) \le 0 \\ \textit{for} \quad t \in [a,b], \quad x_1, x_2, y_1, y_2 \in R\end{gathered} \tag{2.17}$$

be fulfilled. If, moreover, inequalities (1.10), (1.11), *and* (2.13) *hold, then the problem* (0.5), (0.2) *is uniquely solvable.*

Corollary 2.4. *Let* $\lambda \in [0,1[$, $p, g \in L([a,b];R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and let the conditions* (2.6) *and* (2.17) *be fulfilled. If, moreover, the inequality* (2.14) *holds, where* α_1 *and* β_1 *are defined by* (1.14) *and* (1.15) *with* σ *given by* (1.16), *then the problem* (0.5), (0.2) *is uniquely solvable.*

Remark 2.6. Corollaries 2.3 and 2.4 are nonimprovable in that sense that the strict inequalities (2.13) in Corollary 2.3 and (2.14) in Corollary 2.4 cannot be replaced by inequalities (2.15) and (2.16), no matter how small $\varepsilon > 0$ is (see On Remark 2.2).

The following corollaries of Theorems 2.3 and 2.4 hold true.

Corollary 2.5. *Let* $\lambda \in]1, +\infty[$, $c \in R_+$, $p, g \in L([a,b]; R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and let* (2.9) *and the condition*

$$f(t,x,y)\operatorname{sgn} x \geq -q(t,|x|) \quad for \quad t \in [a,b],\ x,y \in R \tag{2.18}$$

be fulfilled. If, moreover, inequalities (1.17), (1.18), *and*

$$\frac{\lambda - 1}{\lambda - \exp\Big(\int\limits_a^b g(\xi)d\xi\Big)} \int\limits_a^b p(s)\exp\Big(\int\limits_a^s g(\xi)d\xi\Big)ds < 2 \tag{2.19}$$

hold, then the problem (0.5), (0.2) *has at least one solution.*

Corollary 2.6. *Let* $\lambda \in]1, +\infty[$, $c \in R_+$, $p, g \in L([a,b]; R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and the conditions* (2.9) *and* (2.18) *be fulfilled. If, moreover,*

$$\frac{\lambda - 1}{\lambda}\left(\int\limits_a^b p(s)ds + \alpha_2\right) + 2\beta_2 < 2, \tag{2.20}$$

where α_2 *and* β_2 *are defined by* (1.21) *and* (1.22) *with* σ *given by* (1.23),*then the problem* (0.5), (0.2) *has at least one solution.*

Corollary 2.7. *Let* $\lambda \in]1, +\infty[$, $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$ *and let* (2.11) *and the condition*

$$\begin{gathered}[f(t,x_1,y_1) - f(t,x_2,y_2)]\operatorname{sgn}(x_1 - x_2) \geq 0 \\ for \quad t \in [a,b], \quad x_1, x_2, y_1, y_2 \in R,\end{gathered} \tag{2.21}$$

be fulfilled. If, moreover, inequalities (1.17), (1.18), *and* (2.19) *hold, then the problem* (0.5), (0.2) *is uniquely solvable.*

Corollary 2.8. *Let* $\lambda \in]1, +\infty[$, $p, g \in L([a,b]; R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, *and let the conditions* (2.11) *and* (2.21) *be fulfilled. If, moreover, the inequality* (2.20) *holds, where* α_2 *and* β_2 *are defined by* (1.21) *and* (1.22) *with* σ *given by* (1.23), *then the problem* (0.5), (0.2) *is uniquely solvable.*

Remark 2.7. Corollaries 2.5–2.8 are nonimprovable in the sense that the strict inequality (2.19) in Corollaries 2.5 and (2.7), and the strict inequality (2.20) in Corollaries 2.6 and (2.8) cannot be replaced by the inequalities

$$\frac{\lambda - 1}{\lambda - \exp\Big(\int\limits_a^b g(\xi)d\xi\Big)} \int\limits_a^b p(s)\exp\Big(\int\limits_a^s g(\xi)d\xi\Big)ds \leq 2 + \varepsilon$$

and

$$\frac{\lambda-1}{\lambda}\left(\int_a^b p(s)ds+\alpha_2\right)+2\beta_2\le 2+\varepsilon,$$

no matter how small $\varepsilon>0$ is.

2.2. Auxiliary Propositions. First we formulate the result from [24, Theorem 1] in the form suitable for us.

Lemma 2.1. *Let there exist a positive number ρ and an operator $\ell\in\mathcal{L}_{ab}$ such that the homogeneous problem $(0,3_0)$, $(0,4_0)$ has only a trivial solution, and let for every $\delta\in\,]0,1[$, an arbitrary function $u\in\widetilde{C}([a,b];R)$ satisfying the relations*

$$u'(t)=\ell(u)(t)+\delta[F(u)(t)-\ell(u)(t)],\qquad u(a)-\lambda u(b)=\delta h(u),\quad (2.22)$$

admits the estimate

$$\|u\|_C\le\rho. \quad (2.23)$$

Then the problem (0.1), (0.2) *has at least one solution.*

Definition 2.1. We say that an operator $\ell\in\mathcal{L}_{ab}$ belongs to the set $U(\lambda)$ if there exists a positive number r such that, for arbitrary $q^*\in L([a,b];R_+)$ and $c\in R_+$, every function $u\in\widetilde{C}([a,b];R)$ satisfying the inequalities

$$[u(a)-\lambda u(b)]\operatorname{sgn}u(a)\le c, \quad (2.24)$$

$$[u'(t)-\ell(u)(t)]\operatorname{sgn}u(t)\le q^*(t)\quad\text{for}\quad t\in[a,b], \quad (2.25)$$

admits the estimate

$$\|u\|_C\le r\left(c+\|q^*\|_L\right). \quad (2.26)$$

Lemma 2.2. *Let $c\in R_+$,*

$$h(v)\operatorname{sgn}v(a)\le c\quad\textit{for}\quad v\in C([a,b];R), \quad (2.27)$$

and let there exist $\ell\in U(\lambda)$ such that the inequality

$$[F(v)(t)-\ell(v)(t)]\operatorname{sgn}v(t)\le q(t,\|v\|_C)\quad\textit{for}\quad t\in[a,b] \quad (2.28)$$

is fulfilled on the set $B^1_{\lambda c}([a,b];R)$. Then the problem (0.1), (0.2) *has at least one solution.*

Proof. First of all we note that, due to the condition $\ell\in U(\lambda)$, the homogeneous problem $(0,3_0)$, $(0,4_0)$ has only a trivial solution.

Let r be the number appearing in Definition 2.1. According to (2.1), there exists $\rho>2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)ds<\frac{1}{2r}\quad\text{for}\qquad x>\rho.$$

Now assume that $u \in \widetilde{C}([a,b];R)$ satisfies (2.22) for some $\delta \in]0,1[$. Then, according to (2.27), u satisfies the inequality (2.24), i.e., $u \in B^1_{\lambda c}([a,b];R)$. By (2.28) we obtain that the inequality (2.25) is fulfilled with $q^*(t) = q(t, \|u\|_C)$ for $t \in [a,b]$. Hence, according to the condition $\ell \in U(\lambda)$ and the definition of the number ρ, we arrive at the estimate (2.23).

Since ρ depends neither on u nor on δ, it follows from Lemma 2.1 that the problem (0.1), (0.2) has at least one solution. □

Lemma 2.3. *Let*

$$[h(v) - h(w)] \operatorname{sgn}(v(a) - w(a)) \le 0 \quad \textit{for} \quad v, w \in C([a,b];R) \tag{2.29}$$

and let there exist $\ell \in U(\lambda)$ such that the inequality

$$[F(v)(t) - F(w)(t) - \ell(v-w)(t)] \operatorname{sgn}(v(t) - w(t)) \le 0 \quad \textit{for} \quad t \in [a,b] \tag{2.30}$$

holds on the set $B^1_{\lambda c}([a,b];R)$, where $c = |h(0)|$. Then the problem (0.1), (0.2) *is uniquely solvable.*

Proof. It follows from (2.29) that the condition (2.27) is fulfilled with $c = |h(0)|$. By (2.30) we see that on the set $B^1_{\lambda c}([a,b];R)$ the inequality (2.28) holds with $q \equiv |F(0)|$. Consequently, all the assumptions of Lemma 2.2 are satisfied and therefore the problem (0.1), (0.2) has at least one solution. It remains to show that the problem (0.1), (0.2) has at most one solution.

Let u_1, u_2 be arbitrary solutions of the problem (0.1), (0.2). Put $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. Then (2.29) and (2.30) yield

$$[u(a) - \lambda u(b)] \operatorname{sgn} u(a) \le 0,$$
$$[u'(t) - \ell(u)(t)] \operatorname{sgn} u(t) \le 0 \quad \text{for} \quad t \in [a,b].$$

These two inequalities, together with the assumption $\ell \in U(\lambda)$, result in $u \equiv 0$. Consequently, $u_1 \equiv u_2$. □

Lemma 2.4. *Let $\ell_0 \in \mathcal{L}_{ab}$ and the homogeneous problem*

$$v'(t) = \ell_0(v)(t), \qquad v(a) - \lambda v(b) = 0$$

have only the trivial solution. Then there exists a positive number r_0 such that for any $q^* \in L([a,b];R)$ and $c \in R$, the solution v of the problem

$$v'(t) = \ell_0(v)(t) + q^*(t), \qquad v(a) - \lambda v(b) = c \tag{2.31}$$

admits the estimate

$$\|v\|_C \le r_0 \left(|c| + \|q^*\|_L\right). \tag{2.32}$$

Proof. Let

$$R \times L([a,b];R) = \{(c, q^*) : c \in R, q^* \in L([a,b];R)\}$$

denote the Banach space with the norm

$$\|(c, q^*)\|_{R \times L} = |c| + \|q^*\|_L,$$

and Ω be the operator which to every $(c, q^*) \in R \times L([a,b]; R)$ assigns the solution v of the problem (2.31). According to Theorem 1.4 from [21], $\Omega : R \times L([a,b]; R) \to C([a,b]; R)$ is a linear bounded operator. Let r_0 be the norm of Ω. Then, clearly, the inequality

$$\|\Omega(c, q^*)\|_C \leq r_0\Big(|c| + \|q^*\|_L\Big)$$

holds for arbitrary $(c, q^*) \in R \times L([a,b]; R)$. Consequently, the solution $v = \Omega(c, q^*)$ of the problem (2.31) admits the estimate (2.32). □

Lemma 2.5. *Let $\lambda \in [0,1[$ and the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the condition* (1.1). *Let, moreover, there exist a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities* (1.2), (1.3) *and* (2.4). *Then $\ell \in U(\lambda)$.*

Proof. Let $q^* \in L([a,b]; R_+)$, $c \in R_+$ and $u \in \widetilde{C}([a,b]; R)$ satisfy the inequalities (2.24) and (2.25). It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t), \tag{2.33}$$

where

$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \quad \text{for} \quad t \in [a,b].$$

Obviously,

$$\widetilde{q}(t) \operatorname{sgn} u(t) \leq q^*(t) \quad \text{for} \quad t \in [a,b]. \tag{2.34}$$

According to (1.1), (1.2), (1.3), and Lemma 1.1, we see that $\ell_0 \in V^+(\lambda)$. Therefore, the assumptions of Lemma 2.4 are fulfilled. Let r_0 be the number appearing in Lemma 2.4 and put

$$r = r_0\Big(1 + 4\Big(1 + \gamma(b) - \gamma(a)\Big)\Big(4 - (\gamma(b) - \gamma(a))^2\Big)^{-1}\Big). \tag{2.35}$$

We will show that (2.26) holds with r defined by (2.35).

Let us first suppose that u does not change its sign. Then, in view of (1.1) and (2.34), the equality (2.33) yields

$$|u(t)|' \leq \ell_0(|u|)(t) + q^*(t) \quad \text{for} \quad t \in [a,b],$$

and from (2.24) we get

$$|u(a)| - \lambda|u(b)| \leq c.$$

Therefore, by Remark 1.7, the condition $\ell_0 \in V^+(\lambda)$ implies

$$|u(t)| \leq v(t) \quad \text{for} \quad t \in [a,b],$$

where v is a solution of the problem (2.31). Due to Lemma 2.4, the function v admits the estimate (2.32), and so the estimate (2.26) holds.

Now assume that u changes its sign. Define numbers M and m by (1.24) and choose $t_M, t_m \in [a,b]$ such that (1.25) holds. It is clear that (1.26) is fulfilled and either

$$t_M < t_m \tag{2.36}$$

or

$$t_m < t_M. \tag{2.37}$$

According to (1.1), (1.2), (1.3), (1.24), (1.26), and (2.31), we have

$$\begin{gathered}(M\gamma(t)+v(t))' \geq \ell_0(M\gamma+v)(t)+M\ell_1(1)(t)+q^*(t)\\ \geq \ell_0(M\gamma+v)(t)+\ell_1([u]_+)(t)+q^*(t) \quad \text{for} \quad t\in[a,b],\\ M\gamma(a)+v(a)-\lambda(M\gamma(b)+v(b)) \geq c,\end{gathered} \tag{2.38}$$

and

$$\begin{gathered}(m\gamma(t)+v(t))' \geq \ell_0(m\gamma+v)(t)+m\ell_1(1)(t)+q^*(t)\\ \geq \ell_0(m\gamma+v)(t)+\ell_1([u]_-)(t)+q^*(t) \quad \text{for} \quad t\in[a,b],\\ m\gamma(a)+v(a)-\lambda(m\gamma(b)+v(b)) \geq c.\end{gathered} \tag{2.39}$$

On the other hand, in view of (2.34) and (2.24), the equality (2.33) yields

$$\begin{gathered}[u(t)]_+' \leq \ell_0([u]_+)(t)+\ell_1([u]_-)(t)+q^*(t) \quad \text{for} \quad t\in[a,b],\\ [u(a)]_+-\lambda[u(b)]_+ \leq c,\end{gathered} \tag{2.40}$$

and

$$\begin{gathered}[u(t)]_-' \leq \ell_0([u]_-)(t)+\ell_1([u]_+)(t)+q^*(t) \quad \text{for} \quad t\in[a,b],\\ [u(a)]_- -\lambda[u(b)]_- \leq c.\end{gathered} \tag{2.41}$$

In view of the condition $\ell_0 \in V^+(\lambda)$ and Remark 1.7, it follows from (2.38) and (2.41), and from (2.39) and (2.40), that

$$M\gamma(t)+v(t) \geq [u(t)]_- \quad \text{and} \quad m\gamma(t)+v(t) \geq [u(t)]_+ \quad \text{for} \quad t\in[a,b]. \tag{2.42}$$

Inequalities (2.38)–(2.41), by virtue of (2.42) and the assumption $\ell_0 \in \mathcal{P}_{ab}$, imply

$$[u(t)]_-' \leq (M\gamma(t)+v(t))', \quad [u(t)]_+' \leq (m\gamma(t)+v(t))' \quad \text{for} \quad t\in[a,b]. \tag{2.43}$$

Note also that in view of the condition $\ell_0 \in V^+(\lambda)$, we have

$$v(t) \geq 0 \quad \text{for} \quad t\in[a,b]. \tag{2.44}$$

Let us first suppose that (2.36) is fulfilled. On account of (1.25), (1.26), and (2.44), the integration of the first inequality in (2.43) from t_M to t_m results in

$$m \leq M\gamma(t_m)+v(t_m)-M\gamma(t_M)-v(t_M) \leq M(\gamma(t_m)-\gamma(t_M))+\|v\|_C. \tag{2.45}$$

On the other hand, in view of (1.25), (1.26), and (2.44), the integration of the second inequality in (2.43) from a to t_M and from t_m to b yields

$$\begin{gathered}M-[u(a)]_+ \leq m\gamma(t_M)+v(t_M)-m\gamma(a)-v(a)\\ \leq m(\gamma(t_M)-\gamma(a))-v(a)+\|v\|_C,\end{gathered} \tag{2.46}$$

$$[u(b)]_+ \leq m\gamma(b)+v(b)-m\gamma(t_m)-v(t_m) \leq m(\gamma(b)-\gamma(t_m))+v(b). \tag{2.47}$$

Multiplying both sides of (2.47) by λ and taking into account that $m > 0$, $\lambda < 1$ and γ is a nondecreasing function, we obtain

$$\lambda[u(b)]_+ \leq m(\gamma(b) - \gamma(t_m)) + \lambda v(b).$$

Summing the last inequality and (2.46) and taking into account (2.31) and (2.40), we get

$$M \leq m(\gamma(t_M) - \gamma(t_m) + \gamma(b) - \gamma(a)) + \|v\|_C. \tag{2.48}$$

In view of (1.24), (2.36), and the condition $\gamma'(t) \geq 0$ for $t \in [a,b]$, it follows from (2.45) and (2.48) that

$$\begin{aligned}\|u\|_C \leq \|u\|_C(\gamma(t_m) - \gamma(t_M))(\gamma(t_M) - \gamma(t_m) + \gamma(b) - \gamma(a))\\ + (1 + \gamma(b) - \gamma(a))\|v\|_C.\end{aligned}$$

Consequently, by virtue of the inequality

$$AB \leq \frac{1}{4}(A+B)^2, \tag{2.49}$$

$$\|u\|_C \leq \frac{\|u\|_C}{4}\Big(\gamma(b) - \gamma(a)\Big)^2 + \Big(1 + \gamma(b) - \gamma(a)\Big)\|v\|_C.$$

Hence, by(2.4),

$$\|u\|_C \leq 4\Big(1 + \gamma(b) - \gamma(a)\Big)\Big(4 - (\gamma(b) - \gamma(a))^2\Big)^{-1}\|v\|_C. \tag{2.50}$$

Therefore, according to (2.32) and (2.35), the estimate (2.26) holds.

Now suppose that (2.37) is fulfilled. On account of (1.25), (1.26), and (2.44), the integration of the second inequality in (2.43) from t_m to t_M results in

$$\begin{aligned}M &\leq m\gamma(t_M) + v(t_M) - m\gamma(t_m) - v(t_m)\\ &\leq m(\gamma(t_M) - \gamma(t_m)) + \|v\|_C.\end{aligned} \tag{2.51}$$

On the other hand, in view of (1.25), (1.26), and (2.44), the integration of the first inequality in (2.43) from a to t_m and from t_M to b yields

$$\begin{aligned}m - [u(a)]_- &\leq M\gamma(t_m) + v(t_m) - M\gamma(a) - v(a)\\ &\leq M(\gamma(t_m) - \gamma(a)) - v(a) + \|v\|_C,\end{aligned} \tag{2.52}$$

$$[u(b)]_- \leq M\gamma(b) + v(b) - M\gamma(t_M) - v(t_M) \leq M(\gamma(b) - \gamma(t_M)) + v(b). \tag{2.53}$$

Multiplying both sides of (2.53) by λ and taking into account that $M > 0$, $\lambda < 1$ and γ is a nondecreasing function, we get

$$\lambda[u(b)]_- \leq M(\gamma(b) - \gamma(t_M)) + \lambda v(b).$$

Summing the last inequality and (2.52) and taking into account (2.31) and (2.41), we show that

$$m \leq M(\gamma(t_m) - \gamma(t_M) + \gamma(b) - \gamma(a)) + \|v\|_C. \tag{2.54}$$

Accordingg to (1.24), (2.37), and the condition $\gamma'(t) \geq 0$ for $t \in [a,b]$, it follows from (2.51) and (2.54) that

$$\begin{aligned}\|u\|_C leq \|u\|_C(\gamma(t_M)-\gamma(t_m))(\gamma(t_m)-\gamma(t_M)+\gamma(b)-\gamma(a))\\ + (1+\gamma(b)-\gamma(a))\|v\|_C.\end{aligned}$$

Consequently, by virtue of (2.4) and (2.49), the inequality (2.50) is fulfilled. Therefore, according to (2.32) and (2.35), the estimate (2.26) holds. □

2.3. Proofs of Main Results. Theorem 2.1 follows from Lemmas 2.2 and 2.5, whereas Theorem 2.2 is a consequence Lemmas 2.3 and 2.5.

Proof of Corollary 2.1. Obviously, the condition (2.12) yields (2.3), where

$$\begin{gathered}F(v)(t) \stackrel{def}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) + f(t, v(t), v(\nu(t))),\\ \ell_0(v)(t) \stackrel{def}{=} p(t)v(\tau(t)), \qquad \ell_1(v)(t) \stackrel{def}{=} g(t)v(\mu(t)) \quad \text{for} \quad t \in [a,b].\end{gathered} \tag{2.55}$$

Moreover, similarly to the proof of Corollary 1.1 one can show that, according to the conditions (1.10), (1.11), and (2.13), there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities (1.2), (1.3), and (2.4). Therefore, the assumptions of Theorem 2.1 are satisfied. □

Proof of Corollary 2.4. Obviously, the condition (2.17) yields the condition (2.7) with F, ℓ_0 and ℓ_1 defined by (2.55). Moreover, analogously to the proof of Corollary 1.2, according to the condition (2.14), where α_1 and β_1 are defined by (1.14) and (1.15) with σ given by (1.16), one can show that there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities (1.2), (1.3), and (2.4). Therefore, the assumptions of Theorem 2.2 are fulfilled. □

Corollaries 2.2, 2.3 and 2.5–2.8 can be proved in a similar manner.

3. On Remarks 1.2, 2.1 and 2.2

On Remark 1.2. In Example 3.1, we have constructed an operator $\ell \in \mathcal{L}_{ab}$ such that the homogeneous problem $(0,3_0)$, $(0,4_0)$ has a nontrivial solution. Then, according to Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.3), (0.4) has no solution.

Example 3.1. Let $\lambda \in [0,1[$, $a=0$, $b=4$, $\varepsilon \geq 0$, and let

$$\ell_0 \equiv 0, \qquad \ell_1(v)(t) \stackrel{def}{=} g(t)v(\mu(t)) \quad \text{for} \quad t \in [a,b], \tag{3.1}$$

where

$$g(t) = \begin{cases} 1+\lambda & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ \varepsilon & \text{for } t \in [3,4] \end{cases}, \qquad \mu(t) = \begin{cases} 3 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ 2 & \text{for } t \in [3,4] \end{cases}.$$

Obviously, $\|g\|_L = 3 + \lambda + \varepsilon$. Choose $\delta > 0$ such that $\delta > \frac{\lambda\|g\|_L}{1-\lambda}$ and define the function γ by

$$\gamma(t) = \delta + \int_a^t g(s)ds \quad \text{for} \quad t \in [a,b]. \tag{3.2}$$

It is clear that $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfies the conditions (1.2), (1.3), and

$$\gamma(b) - \gamma(a) = 3 + \lambda + \varepsilon.$$

On the other hand, the problem $(0,3_0)$, $(0,4_0)$ has a nontrivial solution

$$u(t) = \begin{cases} \lambda - (1+\lambda)t & \text{for } t \in [0,1[\\ t-2 & \text{for } t \in [1,3[\\ 1 & \text{for } t \in [3,4] \end{cases}.$$

Example 3.1 shows that the strict inequality (1.4) in Theorem 1.2 cannot be replaced by nonstrict one. This example also shows that the strict inequalities (1.12) in Corollary 1.1 and (1.13) in Corollary 1.2 cannot be replaced by nonstrict ones.

On Remark 2.1. In Example 3.2 functions $g, z \in L([a,b];R_+)$ and $\mu \in \mathcal{M}_{ab}$ are constructed such that the problem

$$u'(t) = -g(t)u(\mu(t)) - z(t)u(t), \qquad u(a) - \lambda u(b) = 0 \tag{3.3}$$

has a nontrivial solution. Then, by Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.1), (0.2), with

$$F(v)(t) \stackrel{def}{=} -g(t)v(\mu(t)) - z(t)v(t) + q_0(t) \quad \text{for} \quad t \in [a,b], \qquad h(v) \stackrel{def}{=} c_0,$$

has no solution, while the conditions (2.2) and (2.3) are fulfilled, where ℓ_0, ℓ_1 are defined by (3.1), $q \equiv |q_0|$, and $c = |c_0|$.

Example 3.2. Let $\lambda \in [0,1[$, $\varepsilon > 0$, and choose $\eta_1 \in [0,\lambda]$ such that $0 < \eta_1 < \lambda$ if $\lambda \neq 0$, and $\eta_2 \in]0,1[$ such that $\eta_1 + \eta_2 \leq \varepsilon$. Put $a = 0$, $b = 5$, $t_0 = \frac{\eta_1}{1+\eta_1} + 1$,

$$g(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup [2,3[\\ 1+\eta_1 & \text{for } t \in [1,2[\\ 1+\eta_2 & \text{for } t \in [3,4[\\ \varepsilon - \eta_1 - \eta_2 & \text{for } t \in [4,5] \end{cases}, \quad \mu(t) = \begin{cases} 5 & \text{for } t \in [0,2[\\ 2 & \text{for } t \in [2,4[\\ t_0 & \text{for } t \in [4,5] \end{cases},$$

$$z(t) = \begin{cases} z_0(t) & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\cup [3,5] \\ \dfrac{(1-\eta_2)}{(1-\eta_2)(2-t)+1} & \text{for } t \in [2,3[\end{cases},$$

where

$$z_0(t) = \begin{cases} 0 & \text{if } \lambda = 0 \\ \dfrac{\lambda - \eta_1}{\lambda - (\lambda - \eta_1)t} & \text{if } \lambda \neq 0 \end{cases} \quad \text{for} \quad t \in [0, 1[\,.$$

Obviously, $\|g\|_L = 2 + \varepsilon$. Choose $\delta > 0$ such that $\delta > \frac{\lambda \|g\|_L}{1-\lambda}$ and define the function γ by (3.2). It is clear that $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ satisfies the conditions (1.2), (1.3), and

$$\gamma(b) - \gamma(a) = 2 + \varepsilon.$$

On the other hand, the problem (3.3) has a nontrivial solution

$$u(t) = \begin{cases} (\lambda - \eta_1)t - \lambda & \text{for } t \in [0, 1[\\ (\eta_1 + 1)(t - 1) - \eta_1 & \text{for } t \in [1, 2[\\ (1 - \eta_2)(2 - t) + 1 & \text{for } t \in [2, 3[\\ (1 + \eta_2)(3 - t) + \eta_2 & \text{for } t \in [3, 4[\\ -1 & \text{for } t \in [4, 5] \end{cases}.$$

Example 3.2 shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the inequality (2.5), no matter how small $\varepsilon > 0$ is. Example 3.2 also shows that the strict inequalities (2.13) in Corollary 2.1 and (2.14) in Corollary 2.2 cannot be replaced by the inequalities (2.15) and (2.16), no matter how small $\varepsilon > 0$ is.

On Remark 2.2. Example 3.2 shows that the strict inequality (2.4) in Theorem 2.2, resp.(2.13) in Corollary 2.3, resp. (2.14) Corollary 2.4, cannot be replaced by the inequaity (2.5), resp. (2.15), resp. (2.16), no matter how small $\varepsilon > 0$ is.

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