Georgian Mathematical Journal
Volume 9 (2002), Number 2, 309–324

A PROBLEM OF LINEAR CONJUGATION FOR ANALYTIC FUNCTIONS WITH BOUNDARY VALUES FROM THE ZYGMUND CLASS

V. KOKILASHVILI AND V. PAATASHVILI

Abstract. The solvability conditions are established for a problem of linear conjugation for analytic functions with boundary values from the Zygmund class $L(\ln^+ L)^\alpha$ when the conjugation coefficient is piecewise-continuous in the Hölder sense. Solutions of the problem are constructed in explicit form.

2000 Mathematics Subject Classification: 30E20, 30E25, 47B38.
Key words and phrases: Analytic functions, discontinuous boundary value problem of linear conjugation, Cauchy type integral, singular integral.

Depending on assumptions made for the boundary values of the sought functions, boundary value problems of the analytic function theory are traditionally divided into three groups: continuous, piecewise-continuous and discontinuous (see [1], p. 6).

Continuous and piecewise-continuous problems have been studied with sufficient completeness. The main results obtained for these problems are presented in the monographs [2] and [3].

As for the group of discontinuous problems, only those have been studied, in which the sought functions are required to be representable by a Cauchy type integral with a density summable to the p-th degree with $p > 1$ in the Lebesgue sense (see, e.g., [1], [4]–[9]). Concerning the development of this subject-matter we refer the reader to [10]. The study of a discontinuous problem of linear conjugation in the class of functions representable by a Cauchy type integral with a summable density is fraught with certain difficulties (see, e.g., [1], pp. 132–133]). Hence we find it interesting to further widen the class of sought functions.

In this paper we investigate a discontinuous problem of linear conjugation in the class of functions representable by a Cauchy type integral whose boundary values belong to some Zygmund class. True, this class is more narrow than the set of all functions representable by a Cauchy type integral, but it is much wider than the class of Cauchy type integrals with densities summable to the p-th degree with $p > 1$. Moreover, the boundary curve is taken from a wide set of curves, while the conjugation coefficient is assumed to be piecewise-continuous in the Hölder sense.

ISSN 1072-947X / \$8.00 /

In Subsection 1 we introduce the required classes of functions and operators, formulate the problem and discuss some known results. In Subsection 2 some properties of Cauchy singular integrals are established when their density belongs to the Zygmund class or to the weight Zygmund class. In Subsection 3 the problem of linear conjugation is studied in the case, where the coefficient G of the boundary function Φ^- belongs to the Hölder class. Subsection 4 deals with the case, where G is piecewise-continuous in the Hölder sense. A free term in the boundary condition is taken from a weight Zygmund class chosen depending on discontinuity values of the function G. In that case we obtain the branch of the function $\arg G$ corresponding to the chosen class of solutions. An increment of this functions plays the role of a problem index. In both considered cases, solutions are constructed in explicit form. Various particular cases are considered.

1^0. On the discontinuous boundary value problems. Let Γ be a rectifiable, oriented, closed curve bounding the finite domain D^+ and the infinite domain D^-.

Denote by $K^p(\Gamma)$, $p \geq 1$, the set of those analytic functions in the plane cut along Γ, which are representable in the form

$$\Phi(z) = (K_\Gamma\, \varphi)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}, \quad z\overline{\in}\Gamma, \tag{1.1}$$

where $\varphi \in L^p(\Gamma)$.

Almost at all points $t \in \Gamma$, functions from $K^p(\Gamma)$ have angular boundary values $\Phi^+(t)$ and $\Phi^-(t)$ from D^+ and D^-, respectively (see. e.g., [9], p. 29).

The following problem of linear conjugation is thoroughly investigated under various assumptions for Γ: Define a function $\Phi \in K^p(\Gamma)$, $p > 1$, whose angular boundary values $\Phi^\pm(t)$ satisfy, almost everywhere on Γ, the condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{1.2}$$

where G, g are given functions on Γ, $g \in L^p(\Gamma)$ and G satisfies a number of additional conditions ([1], [4–9] and others).

If Γ is a curve such that the Cauchy singular operator

$$S = S_\Gamma : f \to S_\Gamma f, \;\; (S_\Gamma f)(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}, \;\; t \in \Gamma, \tag{1.3}$$

is continuous in spaces $L^p(\Gamma)$, $p > 1$, then $K^p(\Gamma)$ coincides with the set of those analytic functions in the plane cut along Γ, whose restrictions on D^+ and D^- belong to the Smirnov classes $E^p(D^+)$ and $E^p(D^-)$, respectively (see [11]–[12] and also [9], p. 30). Here $E^p(D)$ is the class of those analytic functions Φ in D, for which

$$\sup_r \int\limits_{\Gamma_r} |\Phi(z)|^p |dz| < \infty,$$

where Γ_r is the image of a circumference of radius r, for a conformal mapping of the unit circle on D (see, e.g., [13], p. 205; [14], p. 422).

As is known, we have $\Phi \in E^p(D)$ if there exists a sequence of curves Γ_n which converge to the boundary of the domain D and along which the integral means are uniformly bounded ([15], see also [14], p. 422).

We immediately note that the operator S_Γ is continuous in spaces $L^p(\Gamma)$, $p > 1$, if and only if Γ is a regular curve, i.e.,

$$\sup_{\rho>0,\zeta\in\Gamma} l_\zeta(\rho)\rho^{-1} < \infty, \tag{1.4}$$

where $l_\zeta(\rho)$ is an arc-length measure of that part of Γ which is contained in the circle with center at ζ radius ρ [16].

In what follows the set of regular curves will be denoted by R.

Denote by $E^p(\Gamma)$ the set of those analytic functions in the plane cut along Γ, whose restrictions on the domains D^+ and D^- belong to the classes $E^p(D^+)$ and $E^p(D^-)$, respectively.

From the above result it follows that if $\Gamma \in R$, then

$$K^p(\Gamma) = E^p(\Gamma), \quad p > 1. \tag{1.5}$$

If however $p = 1$, then it is easy to show that

$$E^1(\Gamma) \subset K^1(\Gamma), \quad E^1(\Gamma) \neq K^1(\Gamma). \tag{1.6}$$

Even under strict requirements for the smoothness of G, the investigation of the problem of linear conjugation in the class $K^1(\Gamma)$ is a matter of some difficulty. If however for the sought functions in representation (1.1) we replace the Lebesgue integral by an integral generalized in a certain sense, then we are able to investigate the conjugation problem under the assumptions that $g \in L^1(\Gamma)$ and G belongs to the Hölder class $H(\Gamma)$ and $G \neq 0$ ([17], see also [1], pp. 132–134; [9], pp. 134–135).

When Γ is the Lyapunov curve, the boundary value problem (1.2) was investigated in the class $E^1(\Gamma)$, too, under the assumptions [18]

$$g,\, S_\Gamma\, g \in L^1(\Gamma), \quad G \in H(\Gamma), \quad G \neq 0. \tag{1.7}$$

This result was further extended to the curves Γ satisfying the chord condition, i.e., to the curves for which the relation of the length of the shortest arc connecting its two points to the length of the chord contracting them is expressed as a bounded function ([19], see also [9], pp. 117–119).

Assumptions (1.7) for g are optimal and must be fulfilled when we consider the problem of linear conjugation in the class $E^1(\Gamma)$. Besides, we have to require that a rather strict requirement that $G \in H(\Gamma)$ be fulfilled.

Therefore it is natural to pose the problem of narrowing the set of functions g in the boundary condition (1.2) so that in the case of discontinuous coefficients G, too, we could study the conjugation problem in the class $E^1(\Gamma)$ or, which is better, in its wide subclasses different from $E^p(\Gamma)$.

We show below that as such sets we can consider the well-known Zygmund classes.

2^0. On the mapping properties of a singular operator.

Let $\Gamma \in R$ and ν be an arc-length measure on Γ. For $z \in \Gamma$ and $r > 0$ we denote by $B(z,r)$ a circle with center at z and radius r.

Definition 1. A measurable (in the sense of the measure ν) a.e. positive function w on Γ belongs to the class $A_1(\Gamma)$ if the condition

$$\frac{1}{\nu(B(z,r)\cap\Gamma)} \int\limits_{B(z,r)\cap\Gamma} w(\tau)d\nu \leq c \operatorname*{ess\,inf}_{t\in B(z,r)\cap\Gamma} w(t) \tag{2.1}$$

is fulfilled, where the constant c does not depend on $z \in \Gamma$ and $r > 0$.

It is well known that a curve $\Gamma \in R$ with measure ν and Euclidean metric becomes a space of homogeneous type (see, e.g., [20]).

Proposition 1. *Let $t_0 \in \Gamma$, $\Gamma \in R$. Then a function $w_0(t) = |t-t_0|^\alpha$, $-1 < \alpha \leq 0$, belongs to the class $A_1(\Gamma)$.*

For the proof of this fact in a more general case of a space of homogeneous type see [21] (Proposition 1.5.9, p. 43).

For a given weight function w on Borel subsets of the curve Γ we define the measure

$$\mu_w e = \int\limits_e w(t)d\nu$$

Proposition 2. *Let $\Gamma \in R$ and $w \in A_1(\Gamma)$. Then there exists a positive constant c such that the inequality*

$$\mu_w\Big\{t : |S_\Gamma\, f(t)| > \lambda\Big\} \leq c\lambda^{-1} \int\limits_\Gamma |f(t)|w(t)d\nu \tag{2.2}$$

holds for any $\lambda > 0$.

For the case of a space of homogeneous type this assertion is established in [22].

Theorem 1. *Let $\alpha \geq 1$, $\Gamma \in R$ and $w \in A_1(\Gamma)$. then there exist positive constants c_1 and c_2 such that the inequalities*

$$\int\limits_\Gamma w(t)\Big|S\Big(\frac{f}{w}\Big)(t)\Big| \log^{\alpha-1}\Big(2+\Big|S\Big(\frac{f}{w}\Big)(t)\Big|\Big)d\nu$$
$$\leq c_1 \int\limits_\Gamma |f(t)| \log^\alpha\Big(2+\frac{|f(t)|}{w(t)}\Big)d\nu + c_2 \tag{2.3}$$

take place.

Theorem 1 is a corollary of the following assertion.

Proposition 3. *Let $\alpha \geq 1$, $\Gamma \in R$ and $w \in A_1(\Gamma)$. Then the weight Zygmund inequality*

$$\int_\Gamma |Sf(t)| \log^{\alpha-1}\Big(2+|Sf(t)|\Big)w(t)d\nu \\ \leq c_1 \int_\Gamma |f(t)| \log^{\alpha}\Big(2+|f(t)|\Big)w(t)d\nu + c_2 \tag{2.4}$$

is valid, where the constants c_1 and c_2 do not depend on the function f.

The proof is carried out by the known technique using the function truncation method. We give the proof for the completeness of the exposition.

Assume that $\Phi(\lambda) = \lambda \log^{\alpha-1}(2+\lambda)$. We have

$$\int_\Gamma \Phi(|Sf(t)|)w(t)d\nu = \int_0^\infty \mu_w\Big\{t : |Sf(t)| > \lambda\Big\}d\Phi(\lambda).$$

Let

$$^{\lambda}f(t) = \begin{cases} |f(t)| & \text{for} \quad |f(t)| > \lambda, \\ 0 & \text{for} \quad |f(t)| \leq \lambda, \end{cases}$$

and

$$_{\lambda}f(t) = \begin{cases} |f(t)| & \text{for} \quad |f(t)| \leq \lambda, \\ 0 & \text{for} \quad |f(t)| > \lambda. \end{cases}$$

Then

$$\int_\Gamma \Phi\Big(|Sf(t)|\Big)w(t)d\nu \leq c_1\bigg(\int_\Gamma w(t)d\nu + \int_1^\infty \mu_w\Big\{t : |S^\lambda f(t)| > \frac{\lambda}{2}\Big\}d\Phi(\lambda) \\ + \int_1^\infty \mu_w\Big\{t : |S_\lambda f(t)| > \frac{\lambda}{2}\Big\}d\Phi(\lambda). \tag{2.5}$$

Now use the fact that inequality (2.2) is fulfilled for $w \in A_1(\Gamma)$. Estimating the second term in the right-hand side of inequality (2.5), we obtain

$$\int_1^\infty \mu_w\Big\{t : |S^\lambda f(t)| > \frac{\lambda}{2}\Big\}d\Phi(\lambda) \leq c\int_1^\infty \frac{1}{\lambda}\bigg(\int_\Gamma |S^\lambda f(t)|w(t)d\nu\bigg)d\Phi(\lambda) \\ = c\int_1^\infty \frac{1}{\lambda}\bigg(\int_\Gamma \chi\Big\{t : |f(t)| > \lambda\Big\}|f(t)|w(t)d\nu\bigg)d\Phi(\lambda).$$

Applying the Fubini theorem, we get an estimate

$$\int\limits_1^\infty \mu_w\Big\{t : |S^\lambda f(t)| > \frac{\lambda}{2}\Big\} d\Phi(\lambda) \le c \int\limits_\Gamma |f(t)| \Big(\int\limits_1^{|f(t)|} \frac{d\Phi(\lambda)}{\lambda} \Big) w(t) d\nu$$
$$\le c \int\limits_\Gamma |f(t)| \log^\alpha \Big(2 + |f(t)|\Big) w(t) d\nu.$$

Analogously, taking into account that for $\Gamma \in R$ the operators S is of weak type (2.2), we obtain

$$\int\limits_1^\infty \mu_w\Big\{t : |S^\lambda f(t)| > \frac{\lambda}{2}\Big\} d\Phi(\lambda) \le c \int\limits_\Gamma |f(t)|^2 \Big(\int\limits_{|f(t)|}^\infty \frac{d\Phi(\lambda)}{\lambda^2} \Big) w(t) d\nu$$
$$\le c \int\limits_\Gamma |f(t)| \log^\alpha \Big(2 + |f(t)|\Big) w(t) d\nu.$$

Finally, from (2.5) we conclude that

$$\int\limits_\Gamma |Sf(t)| \log^{\alpha-1} \Big(2 + |Sf(t)|\Big) w(t) d\nu$$
$$\le c_1 \int\limits_\Gamma |f(t)| \log^\alpha \Big(2 + |f(t)|\Big) w(t) d\nu + c_2. \quad \Box \tag{2.6}$$

Corollary. *If* $\alpha \ge 1$, $w \in A_1(\Gamma)$ *and* $\sup \frac{1}{w} = M < \infty$, *then there exist constants* c_1 *and* c_2 *such that*

$$\int\limits_\Gamma \Big| wS\Big(\frac{f}{w}\Big)\Big| \log^{\alpha-1} \Big(2 + \Big|S\Big(\frac{f}{w}\Big)(t)\Big|\Big) d\nu \le c_1 \int\limits_\Gamma |f| \log^\alpha \Big(2 + |f|\Big) d\nu + c_2. \tag{2.7}$$

Proof. From (2.3) it follows that

$$\int\limits_\Gamma \Big| wS\Big(\frac{f}{w}\Big)\Big| \log^{\alpha-1} \Big(2 + \Big|S\Big(\frac{f}{w}\Big)\Big|\Big) d\nu \le c_1 \int\limits_\Gamma |f| \log^\alpha \Big(2 + \frac{|f|}{w}\Big) d\nu + c_2$$
$$\le c_1 \int\limits_\Gamma |f| \log^\alpha \Big(2 + M|f|\Big) d\nu + c_2.$$

If $M \le 1$, then (2.7) is obvious; if however $M > 1$, then

$$\int\limits_\Gamma \Big| wS\Big(\frac{f}{w}\Big)\Big| \log^{\alpha-1} \Big(2 + \Big|S\Big(\frac{f}{w}\Big)\Big|\Big) d\nu \le c_1 \int\limits_\Gamma |f| \log^\alpha \Big(2M + 2M|f|\Big) d\nu + c_2$$
$$\le c_1 2^\alpha \Big(\int\limits_\Gamma |f| \log^\alpha M d\nu + \int\limits_\Gamma |f| \log^\alpha \Big(2 + |f|\Big) d\nu \Big) + c_2$$
$$\le c_1 2^\alpha \Big(\log^\alpha M + 1 \Big) \int\limits_\Gamma |f| \log^\alpha (2 + |f|) d\nu + c_2$$

$$= \widetilde{c}_1 \int_\Gamma |f| \log^\alpha \Big(2 + |f|\Big) d\nu + c_2.$$

In particular, for $\alpha = 1$ we have

$$\int_\Gamma \Big| S\Big(\frac{f}{w}\Big) \Big| d\nu \le \widetilde{c}_1 \int_\Gamma |f| \log \Big(2 + |f|\Big) d\nu + c_2. \qquad \square$$

Definition 2. Let $\alpha > 0$ and w be a weight function. We say that the function f belongs to the Zygmund class $Z_{\alpha,w}(\Gamma)$ if

$$\int_\Gamma |f(t)| \log^\alpha \Big(2 + \frac{|f(t)|}{w}\Big) d\nu < \infty. \tag{2.8}$$

If $w(t) \equiv 1$, then we write $Z_{\alpha,1}(\Gamma) = Z_\alpha(\Gamma)$, $Z_1(\Gamma) = Z(\Gamma)$.

Set $Z_0(\Gamma) = L(\Gamma)$.

Now the assertion of Theorem 1 can be reformulated as follows.

Theorem 1$'$. *If $\Gamma \in R$, $w \in A_1(\Gamma)$, $\alpha \ge 1$, then the operator*

$$T : f \to Tf, \quad (Tf)(t) = \frac{w(\tau)}{\pi i} \int_\Gamma \frac{f(\tau)}{w(\tau)} \frac{d\tau}{\tau - t} \tag{2.9}$$

maps the class $Z_{\alpha,w}(\Gamma)$ into the class $Z_{\alpha-1,w}(\Gamma)$.

In what follows we will need

Proposition 4. *Let the finite curve Γ belong to R and $0 < \delta < 1$. Then the function*

$$F_\delta(t) = \int_\Gamma \frac{d\nu(\tau)}{|\tau - t|^\delta}$$

is bounded on Γ.

Proof. Let $D(z,r) = B(z,r) \cap \Gamma$, $z \in \Gamma$. We have

$$\int_\Gamma \frac{d\nu(\tau)}{|\tau - t|^\delta} \le \sum_{k=0}^\infty \int_{D(t,2^{-k}) \setminus D(t,2^{-k-1})} \frac{d\nu(\tau)}{|\tau - t|^\delta} + \int_{|\tau - t| \ge 1} \frac{d\nu(\tau)}{|\tau - t|^\delta}$$

$$\le c_1 + c_2 \sum_{k=0}^\infty \frac{1}{2^{-k\delta}} \nu(D(t, 2^{-k})) \le c_1 + c_2 \sum_{k=0}^\infty \frac{1}{2^{k(1-\delta)}} \le c_3. \qquad \square$$

Corollary. *If $\Gamma \in R$ and $\varphi \in H(\Gamma)$, then the function*

$$Mf(t) = \int_\Gamma \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau$$

is bounded on Γ.

3^0. Problem of linear conjugation (1.2) when $\Gamma \in R$, $G \in H(\Gamma)$, $G \ne 0$, and $g \in Z_{\alpha+1}(\Gamma)$.

Let Γ be the closed Jordan rectifiable curve which bounds the domains D^+ and D^- ($\infty \in D^-$). Consider the following boundary value problem: Define a function $\Phi \in E^1(\Gamma)$ whose boundary values $\Phi^\pm(t)$ belong to the class $Z_\alpha(\Gamma)$, $\alpha \geq 0$, and for almost all $t \in \Gamma$ satisfy condition (1.2).

If we denote by $E^{1,\alpha}(\Gamma)$ the subset of all those analytic functions from $E^1(\Gamma)$, for which $\Phi^\pm \in Z_\alpha(\Gamma)$, then the problem posed is reformulated as follows: Define solutions of problem (1.2) which belong to the class $E^{1,\alpha}(\Gamma)$, ($E^{1,0}(\Gamma) = E^1(\Gamma)$).

Let

$$\Gamma \in R, \ \ G(t) \in H(\Gamma), \ \ G(t) \neq 0, \ \ g(t) \in Z_{\alpha+1}(\Gamma). \tag{3.1}$$

It is assumed that $\arg_c G(t)$ is that one-valued branch of the multi-valued function $\arg G(t)$ which changes continuously on the set $\Gamma \backslash \{c\}$, $c \in \Gamma$. Let

$$\varkappa = \frac{\arg G(c-) - \arg G(c+)}{2\pi}. \tag{3.2}$$

It is obvious that $\varkappa$ does not depend on a choice of the point c. Let further $a \in D^+$ and $G_\varkappa(t) = G(t)(t-a)^{-\varkappa}$. Then the function $\arg G_\varkappa(t)$ is continuous on Γ.

Let

$$\gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln G_\varkappa(t) dt}{t-z}. \tag{3.3}$$

Consider the function

$$X(z) = \begin{cases} \exp \gamma(z), & z \in D^+, \\ (z-a)^{-\varkappa} \exp \gamma(z), & z \in D^-. \end{cases} \tag{3.4}$$

Since $G \in H(\Gamma)$, $G \neq 0$, the function $\ln G_\varkappa(t) = \ln G(t)(t-a)^{-\varkappa}$ belongs to $H(\Gamma)$. Therefore, applying Proposition 4, we easily establish that the function $\gamma(z)$ is bounded in $D^+ \cup D^-$. Hence it follows that the functions $X^{\pm 1}(z)$ are bounded in D^+. In the domain D^-, $X(z)$ has at the point $z = \infty$, a pole of order $(-\varkappa)$ for $\varkappa < 0$, while $X^{-1}(z)$ is a bounded function. If however $\varkappa > 0$, then $X(t)$ is bounded and $X^{-1}(z)$ has a pole of order $\varkappa$. Hence, as usual (see, e.g., [2], [4]), we establish that all possible solutions of problem (1.2) of the class $E^1(\Gamma)$ are contained in the set of functions

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z} + P_{\varkappa-1}(z) X(z), \tag{3.5}$$

where $P_{\varkappa-1}(z)$ is a arbitrary polynomial of order $\varkappa - 1$ for $\varkappa - 1 \geq 0$, and $P_{\varkappa-1}(z) \equiv 0$ for $\varkappa - 1 < 0$.

The function $\Phi(z)$ given by (3.5) is a solution of the problem posed if $\Phi(z)$ is analytic in D^- and $\Phi^\pm \in Z_\alpha(\Gamma)$.

If $\varkappa \geq 0$, then the first of these conditions is fulfilled for any g and $P_{\varkappa-1}$. When $\varkappa < 0$, the function $\Phi(z)$ analytic in D^- if and only if the conditions

$$\int_\Gamma g(t)[X^+(t)]^{-1}t^k dt = 0, \quad k = 0, 1, \dots, -\varkappa - 1, \tag{3.6}$$

are fulfilled.

Further,

$$\Phi^\pm(t) = \frac{1}{2}\, g(t) + \frac{X^+(t)}{2\pi i} \int_\Gamma \frac{g(\tau)}{X^+(\tau)}\, \frac{d\tau}{\tau - z} + X^+(t)P_{\varkappa-1}(t).$$

whence it follows that $\Phi^\pm \in Z_\alpha(\Gamma)$ if and only if $T_g = X^+ \int_\Gamma \frac{g}{X^+}$ belongs to $Z_\alpha(\Gamma)$. Since $[X^+(t)]^{\pm 1}$ are bounded functions, then we can apply the corollary of Proposition 3. Replacing α in (2.7) by the number $\alpha + 1$, we obtain $T_g \in Z_\alpha(\Gamma)$. Hence $\Phi^\pm \in Z_\alpha(\Gamma)$ and therefore (3.5) gives a general solution of the problem. Thus the following theorem is true.

Theorem 2. *Let a solution Φ of the boundary value problem* (1.2) *be sought for in the class $E^{1,\alpha}(\Gamma)$, $\alpha \geq 0$, and let conditions* (3.1) *be fulfilled, and the number $\varkappa$ be defined by equality* (3.2)*. Then:*

(i) *for $\varkappa \geq 0$ the problem is undoubtedly solvable and all its solutions are given by equality* (3.5)*, where $P_{\varkappa-1}(z)$ is an arbitrary polynomial of order $\varkappa - 1$* ($P_{-1}(z) \equiv 0$) *and $X(z)$ is defined by equalities* (3.4), (3.3)*;*

(ii) *for $\varkappa < 0$ the problem is solvable if and only if conditions* (3.6) *are fulfilled so that there exists a unique solution given by equality* (3.5) *with $P_{-1}(z) \equiv 0$.*

Remark. Since any function $\Phi \in E^1(\Gamma)$ is representable in the form $\Phi(z) = K_\Gamma(\Phi^+ - \Phi^-)$, the considered class of sought functions is contained in the set of functions

$$K^{1,\alpha}(\Gamma) = \Big\{\Phi : \Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)dt}{t - z}, \quad \varphi \in Z_\alpha(\Gamma)\Big\}. \tag{3.7}$$

This class is wider than $E^{1,\alpha}(\Gamma)$.

It is not difficult to show that any solution of problem (1.2) belonging to the class $K^{1,\alpha}(\Gamma)$ must be contained in the set of functions given by equality (3.5). The following assertion is true by virtue of the proven theorem.

Theorem 3. *If conditions* (3.1) *are fulfilled, then any solution of problem* (1.2) *from the class $K^{1,\alpha}(\Gamma)$ is sure to belong to the class $E^{1,\alpha}(\Gamma)$.*

4^0. Problem of linear conjugation when G is piecewise-continuous in the Hölder sense.

Let $A_1, A_2, \dots, A_n$ be the points lying on Γ. If the function G belongs to the Hölder class on the closed arcs $\Gamma_{A_k A_{k+1}}$, $k = 1, 2, \dots, n$, $A_{n+1} = A_1$, then $G(t)$ is piecewise-continuous in the Hölder sense on Γ. We write $G \in H(\Gamma, A_1, A_2, \dots, A_n)$ (in terms of [1], this means that $G \in H_0(\Gamma)$).

To formulate our requirements for the solution of problem (1.2), we have to define preliminarily one weight function. It is constructed with the aid of the function G. For this, we must first define a branch of function's $\arg G(t)$.

Assume that the points A_k, $k = 1, 2, \dots, n$, are numbered according to the growth of the arc abscissa. Fix arbitrarily a value of $\arg G(A_1+)$ and, moving along Γ in the positive direction, change $\arg G(t)$ continuously on the arc A_1A_2. We thus define the value $G(A_2-)$. Now let us define $\arg G(A_2+)$ so that

$$-1 < \beta_1 = \frac{\arg G(A_2-) - \arg G(A_2+)}{2\pi} \le 0.$$

Continuing to define $\arg G(t)$ on the arcs A_kA_{k+1} so that the relations

$$-1 < \beta_k = \frac{\arg G(A_k-) - \arg G(A_k+)}{2\pi} \le 0, \quad k = 2, \dots, n, \tag{4.1}$$

are fulfilled at the points A_k, we come the arc A_nA_1 and the definition of $\arg G(A_1-)$.

Let

$$h = \frac{\arg G(A_1-) - \arg G(A_1+)}{2\pi}. \tag{4.2}$$

Put

$$\varkappa = \begin{cases} h, & \text{if } \ h \in \mathbb{Z}, \\ [h] - 1, & \text{if } \ h \overline{\in} \mathbb{Z}, \end{cases} \tag{4.3}$$

where $[x]$ is the integer part of the number x, $\mathbb{Z}$ is the set of integer numbers.

Consider the function $G_\varkappa(t) = G(t)(t-a)^\varkappa$, $a \in D^+$. Then by virtue of (4.1)–(4.3) we have

$$-1 < \beta_k = (2\pi)^{-1}\Big(\arg G_\varkappa(A_k-) - \arg G_\varkappa(A_k+)\Big)_\Gamma \ \ k = 1, 2, \dots, n. \tag{4.4}$$

Put

$$w(t) = \prod_{k=1}^{n} |t - A_k|^{\beta_k}. \tag{4.5}$$

Further, denote by $R(A_1, A_2, \dots, A_n)$ the set of curves which pass through the points $A_1, A_2, \dots, A_n$, belong to the class R and are smooth in a small neighborhood of $A_1, A_2, \dots, A_n$. Note that the class $R(A_1, \dots, A_n)$ contains all piecewise-smooth curves with the given corner points $A_1, A_2, \dots, A_n$.

We will consider the problem: Define a function $\Phi \in E^1(\Gamma)$ whose angular values $\Phi^\pm(t)$ belong to $Z_{\alpha,w}(\Gamma)$ and which satisfy the boundary condition (1.2).

In what follows the class of sought functions is denote by $E^{1,\alpha}_w(\Gamma)$. It is assumed that

$$\Gamma \in R(A_1, A_2, \dots, A_n), \quad G \in H(\Gamma, A_1, \dots, A_n), \quad G \neq 0, \tag{4.6}$$

$$w(t) = \prod_{k=1}^{n} |t - A_k|^{\beta_k}, \tag{4.7}$$

where the numbers β_k are defined by equalities (4.1)–(4.4) and

$$g \in Z_{\alpha+1,w}(\Gamma), \quad \alpha \geq 0. \tag{4.8}$$

Since $\beta_k \in (-1, 0]$, the weight belongs to the class $A_1(\Gamma)$ (see Proposition 1).

Let $X(z)$ be the function defined by equality (3.4), where

$$\ln G_\varkappa(t) = -\varkappa \ln(t-a) + \ln |G(t)| + i \arg G(t). \tag{4.9}$$

Here $\arg G(t)$ is the above-defined function.

Since this time $G_\varkappa$ is not a function of the class $H(\Gamma)$, it no longer possesses the good properties indicated in Subsection 3^0. However it has some other properties which are sufficient to study problem (1.2) in the class $E_w^{1,\alpha}(\Gamma)$.

Let us discuss the latter properties in greater detail. For simplicity, from this moment on it is assumed that $n = 1$, $A_1 = A$ and $\beta_1 = \beta$.

In the domain D^+ the function X is analytic, and since $G_\varkappa$ is a bounded function, we have $X \in E^\delta(D^+)$ for some $\delta > 0$ (see, e.g., [9], p. 96).

In the domain D^- the function X is analytic for $\varkappa \geq 0$ and has a pole of order $(-\varkappa)$ at infinity if $\varkappa < 0$.

Moreover, there exists a natural number n_0 such that the function $\Big[X(z)(z-A)^\varkappa \cdot (z-z_0)^{-n_0}\Big]$, $z_0 \in D^+$, belongs to $E^\delta(D^-)$ for some $\delta > 0$ (see, e.g., [9], p. 90).

Keeping in mind that $\Gamma \in R(A)$ and using the arguments from [2] (§26), we establish by virtue of Proposition 4 that the following relations holds for the points z lying near Γ:

$$X(z) = (z-A)^{\beta+\alpha i} \exp\left\{\frac{1}{2\pi}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}\right\}, \quad \varphi \in H(\Gamma), \tag{4.10}$$

where

$$\beta + \alpha i = \frac{\ln G_0(A-) - \ln G(A+)}{2\pi i}, \quad (\operatorname{Im}\beta = \operatorname{Im}\alpha = 0),$$

and by virtue of (4.4) $\beta \in (-1, 0]$. Since the curve Γ is piecewise-smooth, the multiplier $\psi(z) = (z-A)^{\alpha i}$ is a bounded function and, moreover, $|\psi| \geq m > 0$. The same refers to the function $\exp\left\{\frac{1}{2\pi}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau-z}\right\}$ by virtue of Proposition 4. Therefore near Γ we have

$$X(z) = \begin{cases} (z-A)^\beta X_0(z), & z \in D^+, \\ (z-A)^\beta (z-a)^{-\varkappa} X_0(z), & z \in D^-, \quad a \in D^+, \end{cases} \tag{4.11}$$

where

$$0 < \inf_{z \in \overline{\Gamma}} |X_0(z)| \leq \sup_{z \in \overline{\Gamma}} |X_0(z)| < \infty. \tag{4.12}$$

By these properties of $X(z)$ we readily conclude that the restriction of the function

$$\Psi(z)=\frac{\Phi(z)}{X(z)} \tag{4.13}$$

on D^+ belongs to $E^1(D^+)$, while in the domain D^- it belongs to $E^1(D^-)$ for $\varkappa\le 0$, and for $\varkappa>0$ there exists a polynomial P of order $\varkappa$ such that $(\Psi-P)\in E^1(D^-)$. This implies that all possible solutions of problem (1.2) from the class $E^1(\Gamma)$ are contained in the set of functions given by equality (3.5).

From (4.11), (4.12) it follows that $P_{x-1}(z)X(z)\in E^1(\Gamma)$. Now from (3.5) we see that $\Phi\in E^1(\Gamma)$ if the function

$$\Phi_g(z)=\frac{X(z)}{2\pi i}\int\limits_{\Gamma}\frac{g(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau-z} \tag{4.14}$$

belongs to $E^1(\Gamma)$.

We will show that this is really so under our assumptions for g and G.

Let $z\in D^+$. Since $\Gamma\in R$ and $\frac{g}{X^+}\in L(\Gamma)$, we have $K\Big(\frac{g}{X^+}\Big)\in\bigcap\limits_{q<1}E^q(D^+)$ (see, e.g., [9], p. 29); besides, $X\in E^\delta(D^+)$ and therefore $\Phi_g\in E^{\delta_1}(D^+)$, $\delta_1>0$. Since $\frac{1}{X^+(t)}$ is a bounded function (this follows from (4.11) and the condition $\beta<0$), we have $\frac{g}{X^+}\in Z_{\alpha,w}(\Gamma)$. Taking into account the Sokhotskii–Plemelj formulas and applying Theorem 1$'$, we conclude that $\Phi^+\in L(\Gamma)$. Since $\Gamma\in R$ is the Smirnov curve [11], we can apply Smirnov's theorem (see, e.g., [13]), by which in this case we have $\Phi_g\in E^1(D^+)$.

Let now $z\in D^-$.

For $\varkappa<0$ the function Φ_g has a pole at infinity. For this function to belong to $E^1(D^-)$, it is necessary that the conditions (3.6) be fulfilled.

Thus assume that these conditions are fulfilled and therefore $\Phi_g(z)$ is analytic in D^-. Let us show that $\Phi_g(z)$ belongs to $E^1(D^-)$.

By the transformation $\zeta=\frac{1}{z-b}$, $b\in D^+$, the domain D^- is mapped into the finite domain D_1^+, while the curve Γ is mapped into the curve Γ_1. The inclusion $\Gamma_1\in R$ is easily proved. Furthermore, the functions $X\Big(b+\frac{1}{\zeta}\Big)$ and

$$\Psi_g(\zeta)=X\Big(b+\frac{1}{\zeta}\Big)\int\limits_{\Gamma}\frac{g(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau-(b+\frac{1}{\zeta})} \tag{4.15}$$

belong to $E^1(D_1^+)$. Indeed, let us write them in the form

$$\begin{aligned}X\Big(b+\frac{1}{\zeta}\Big)&=\zeta^{\varkappa}\exp\Big\{\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\ln G_\varkappa(b+\frac{1}{\sigma})}{\frac{1}{\sigma}-\frac{1}{\zeta}}\,\frac{-d_\sigma}{\sigma^2}\Big\}\\&=\zeta^{\varkappa}\exp\Big\{\frac{\zeta}{2\pi i}\int\limits_{\Gamma_1}\frac{\ln\widetilde{G}(\sigma)}{\sigma}\,\frac{d\sigma}{\sigma-\zeta}\Big\},\ \ \widetilde{G}(\sigma)=G_\varkappa\Big(b+\frac{1}{\sigma}\Big),\end{aligned} \tag{4.16}$$

$$\Psi_g(\zeta) = \frac{X(b+\frac{1}{\zeta})}{2\pi i}\int\limits_{\Gamma_1}\frac{g(b+\frac{1}{\sigma})}{X^+(b+\frac{1}{\sigma})}\,\frac{-d\sigma}{\sigma^2(\frac{1}{\sigma}-\frac{1}{\zeta})}$$
$$= \frac{X(b+\frac{1}{\zeta})\zeta}{2\pi i}\int\limits_{\Gamma_1}\frac{g(b+\frac{1}{\sigma})}{\sigma X^+(b+\frac{1}{\sigma})}\,\frac{d\sigma}{\sigma-\zeta}. \tag{4.17}$$

Since $\frac{\ln\widetilde{G}(\sigma)}{\sigma}$ is a bounded function and $\Gamma_1 \in R$, from (4.16) we conclude that $X(b+\frac{1}{\zeta}) \in E^\eta(D_1^+)$ for some $\eta > 0$ (see [9], p. 96). The Cauchy type integral $\int_{\Gamma_1}\frac{g(b+\frac{1}{\sigma})}{\sigma X^+(b+\frac{1}{\sigma})}\,\frac{d\sigma}{\sigma-\zeta}$ is a function of the class $\bigcap\limits_{q<1} E_q(D_1^-)$ (see [9], p. 29) and (4.17) implies the existence of $\delta > 0$ such that $\Psi_g \in E^\delta(D_1^+)$. Moreover, in view of (4.11) we obtain

$$\Psi_g^+(\zeta_0) = \frac{X_0(b+\frac{1}{\zeta_0})(\frac{1}{\zeta_0}-\frac{1}{A})^\beta}{2\pi i}\int\limits_{\Gamma_1}\frac{g(b+\frac{1}{\sigma})}{X_0(b+\frac{1}{\sigma})(\frac{1}{\sigma}-\frac{1}{A})^\beta(\frac{1}{\sigma}-\frac{1}{\zeta_0})}\,\frac{-d\sigma}{\sigma^2}+g\Big(b+\frac{1}{\zeta_0}\Big)$$
$$= \frac{X_1(\zeta_0)(A-\zeta_0)^\beta}{2\pi i}\int\limits_{\Gamma_1}\frac{g_1(\sigma)}{X_1(\sigma)(A-\sigma)^\beta}\,\frac{d\sigma}{\sigma-\zeta_0}+g_1(\zeta_0),$$

where

$$X_1(\zeta_0) = X_0\Big(b+\frac{1}{\zeta_0}\Big)(A\zeta_0)^{-\beta},\ \ g_1(\sigma) = g\Big(b+\frac{1}{\sigma}\Big)[X_1(\sigma)\sigma]^{-1}.$$

By virtue of (4.11) we have $\inf_{\zeta\in D_1^+}|X_1(\zeta)| > 0$, $\sup_{\zeta\in D_1^+}|X_1(\zeta)| < \infty$.

It is easy to verify that $g(b+\frac{1}{\sigma}) \in Z_{\alpha+1,w}(\Gamma_1)$, $w_1 = \prod\limits_{k=1}^n |\sigma - a_k|^{\beta_k}$. Since $[\sigma X(\sigma)]^{-1}$ is a bounded function on Γ_1, we see that $g_1 \in Z_{\alpha+1,w}(\Gamma_1)$ too. By Theorem 1′ we have $T_{g_1} = (X_1 S\frac{g_1}{X_1}) \in Z_{\alpha,w_1}(\Gamma_1)$ and it is obvious that $Z_{\alpha,w_1}(\Gamma_1) \subseteq L(\Gamma_1)$. Therefore $\Psi^+ \in L(\Gamma_1)$ and thus $\Psi \in E^1(D_1^+)$. This means that there exists a sequence of curves γ_n converging to Γ_1 such that along these curves the integral means of the function Ψ are uniformly bounded. But in that case the curves Γ_n obtained from γ_n by means of the transformation $z = b+\frac{1}{\zeta}$ are such that along them the integral means of the function $\Phi_g(z)$ are bounded and therefore $\Phi_g \in E^1(D^1)$ [15]. Moreover, $\Phi^\pm(t) = \frac{X^\pm(t)}{2\pi i}\int_\Gamma \frac{g(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau-t}$. But $|X^+| \sim w$ and since $X^- = X^+G^{-1}$ and $|G| \geq m > 0$, by Theorem 1′ we have $\Phi^\pm \in Z_{\alpha,w}(\Gamma)$. Therefore $\Phi \in E_w^{1,\alpha}(\Gamma)$.

The above reasoning refers to the case $n = 1$. When n is arbitrary, for z near Γ we have

$$X(z) = \begin{cases} \prod\limits_{k=1}^n (z-A_k)^\beta X_0(z_0), \\ (z-a)^{-\varkappa}\prod\limits_{k=1}^n (z-A_k)^\beta X_0(z), \quad a \in D^+, \end{cases} \tag{4.18}$$

with condition (4.12) on X_0. Applying Theorem $1'$, we find that the solution of problem (1.2) is again given by equality (3.5).

Based on the results obtained, we conclude that the following theorem is valid.

Theorem 4. *Let* $\Gamma \in R(A_1, A_2, \dots, A_n)$, $G \in H(\Gamma, A_1, A_2, \dots, A_n)$, $G \neq 0$, $\varkappa$ *is defined by equality* (4.3) *and*

$$w(t) = \prod_{k=1}^{n} |t - A_k|^{\beta_k}$$

where β_k *is defined by equalities* (4.1)–(4.4).

Then the assertions of Theorem 2 *are valid for the problem of linear conjugation* (1.2) *in the class* $E_w^{1,\alpha}(\Gamma)$.

Corollary. *If* $g \in Z(\Gamma)$, $G \in H(\Gamma, A_1, A_2, \dots, A_n)$, $\Gamma \in R(A_1, A_2, \dots, A_n)$, *then the assertions of Theorem* 2 *are valid for problem* (1.2) *in the class* $E^1(\Gamma)$.

This assertion is easily verified if we apply the corollary of Proposition 3 to the function T_g for $\alpha = 1$.

Remark 4.1. Analogously to the above, one can investigate the linear conjugation problem in the class

$$\widetilde{E}_w^{1,\alpha}(\Gamma) = \Big\{\Phi : \Phi = \Phi_1 + \text{const}, \;\; \Phi_1 \in E_w^{1,\alpha}(\Gamma)\Big\}.$$

In the case the number $\varkappa$ is replaced by the number $\widetilde{\varkappa} = \varkappa + 1$.

Remark 4.2. It is not difficult to verify that Theorem 4 remain in force if Γ is a curve of the class R such that

$$\begin{aligned} 0 < m_{\pm} &= \inf_{Z \in D^{\pm}} \prod_{k=1}^{n} \frac{|(z - A_k)^{\beta_k}|}{|z - A_k|^{\beta}} \\ &\leq \sup_{Z \in D^{\pm}} \prod \frac{|(z - A_k)^{\beta_k}|}{|z - A_k|^{\beta_k}} = M_{\pm} < \infty, \end{aligned} \tag{4.19}$$

where β_k are defined by (4.4).

Remark 4.3. It is easy to show that the index $\varkappa$ of a function G in the class $E_w^{1,\alpha}(\Gamma)$ is not smaller than the index $\varkappa_p(G)$ of the same function in the classes $E^p(\Gamma)$, $p > 1$ (for the definition of $\varkappa_p(G)$ see [1] or [7]).

References

1. B. V. Khvedelidze, The method of Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one complex variable. (Russian) *Itogi Nauki i Tekhniki, Sovrem. Probl. Mat.* **7**(1975), 5–162; English translation in *J. Soviet Math.* **7**(1977), 309–414.

2. N. I. Muskhelishvili, Singular integral equations. (Transl. from Russian) *P. Noordhoff, Groningen*, 1953.

3. F. D. Gakhov, Boundary value problems. *Pergamon Press, London*, 1966; Extended Russian edition: *Nauka, Moscow*, 1977.

4. B. V. Khvedelidze, Linear discontinuous boundary value problems of function theory, singular integral equations, and some of their applications. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **23**(1956), 3–158.

5. I. B. Simonenko, The Riemann boundary value problem for L^p pairs of functions with measurable coefficients and its application to the investigation of singular integrals in the spaces L^p with weight. (Russian) *Izv. Akad. Nauk SSSR. Ser. Mat.* **28**(1964), 277–306.

6. I. Ts. Gokhberg and N. Ya. Krupnik, Introduction to the theory of one-dimensional singular integral operators. (Russian) *Shtiintsa, Kishinev*, 1973.

7. I. I. Danilyuk, Irregular boundary value problems in the plane. (Russian) *Nauka, Moscow*, 1975.

8. A. Böttcher and Y. I. Karlovich, Carleson curves, Muckenhoupt weights and Toeplitz operators. *Birkhäuser Verlag, Basel–Boston–Berlin*, 1997.

9. G. Khuskivadze, V. Kokilashvili, and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14**(1998), 1–195.

10. V. Kokilashvili, A survey of recent results of Georgian mathematicians on boundary value problems for holomorphic functions. *Mem. Differential Equations Math. Phys.* **23**(2001), 85–138.

11. V. P. Havin, Boundary properties of Cauchy type integrals and harmonically conjugated functions in domains with rectifiable boundary. (Russian) *Mat. Sb.* (*N.S.*) **68**(1965), No. 4, 499–517.

12. V. A. Paatashvili, The membership of the class E^p of analytic functions representable by a Cauchy type integral. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **42**(1972), 87–94.

13. I. I. Privalov, Boundary properties of one-valued analytic functions. (Russian) *Nauka, Moscow*, 1950.

14. G. M. Goluzin, Geometric theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1966.

15. M. V. Keldysh and M. A. Lavrentiev, Sur la representation conformé des domines limites par des courbos rectifiables. *Ann. Sci. École Norm. Sup* (3) **54**(1937), (3), 1–38.

16. G. David, Opérateurs intégraux singuliers sur certaines courbes du plan complex. *Ann. Sci. École Norm. Sup.* (4) **17**(1984), 157–189.

17. G. A. Khuskivadze, On conjugate functions and Cauchy type integrals. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **31**(1966), 5–54.

18. A. E. Drachinskii, On the Riemann-Privalov boundary value problem in the class of summable functions. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **32**(1963), No. 2, 271–276.

19. V. A. Paatashvili, Some properties of singular integrals and Cauchy type integrals. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **47**(1975), 34–52.

20. M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.* **61**(1990), 601–628.

21. I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, Weight theory for integral transforms on spaces of homogeneous type. *Pittman Monographs and Surveys in Pure and Applied Mathematics* 92, *Addison Wesley Longman, Harlow*, 1998.

22. S. Hofmann, Weighted norm inequalities and vector valued inequalities for certain rough operators. *Indiana Univ. Math. J.* **42**(1993), 1–14.

(Received 30.11.2001)

Author's addresses:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia