

ORDINARY DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. The method of lower and upper solutions combined with the monotone iterative technique is used for ordinary differential equations with nonlinear boundary conditions. Some existence results are formulated for such problems.

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1. INTRODUCTION

In this paper, we shall consider the following differential problem

$$\begin{cases} x'(t) = f(t, x(t)), & t \in J = [0, T], \quad T > 0, \\ x(0) = g(x(T)), \end{cases} \quad (1)$$

where $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$.

It is well known that the monotone iterative technique is a powerful method used to approximate solutions of several problems (see, for example [5]). The purpose of this paper is to show that it can be applied successfully to problems of type (1). Assuming one-sided Lipschitz condition on f (with respect to the second variable) combined with the corresponding monotonicity conditions on g , it is shown that linear iterations converge to a solution of problem (1). Some comparison results are also formulated.

In many papers, the monotone iterative technique was applied to problem (1) when $g(u) = \lambda u + k$. If $\lambda = 0$, then (1) reduces to the initial value problem for differential equations and this case is considered, for example, in [6], [7], [9], [10]. If $\lambda = 1$ and $k = 0$, then we have a periodic boundary problem considered, for example, in [8], [10], while if $\lambda = -1$ and $k = 0$, then we have an anti-periodic boundary problem, see [10], [11], [12]. A general case where $\lambda, k \in \mathbb{R}$ is discussed in [1]. Nonlinear problems, more general than (1), are studied, for example, in [2], [3], [4].

2. SOME GENERAL FACTS

From Theorem 1.1 proven in [2] follows

Theorem 1. Let $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$. Moreover, we assume that there exist functions $y_0, z_0 \in C^1(J, \mathbb{R})$ such that

$$\begin{aligned} y_0(t) \leq z_0(t), \quad y_0'(t) \leq f(t, y_0(t)), \quad z_0'(t) \geq f(t, z_0(t)), \quad t \in J, \\ y_0(0) \leq g(s) \leq z_0(0) \quad \text{for } y_0(T) \leq s \leq z_0(T). \end{aligned}$$

Then problem (1) has at least one solution in $\Delta = \{w \in C^1(J, \mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), t \in J\}$.

On the basis of Theorem 1 one can easily prove

Theorem 2. Let the conditions of Theorem 1 be satisfied and, moreover, let the function g be nondecreasing. Then problem (1) has, in the set Δ , the minimal and the maximal solution.

As for the uniqueness of the solution of problem (1), the following result holds.

Theorem 3 (Kiguradze). Let the conditions of Theorem 1 be fulfilled and, moreover,

$$f(t, v) - f(t, u) \leq h(t)(v - u) \quad \text{for } t \in J, \quad y_0(t) \leq u \leq v \leq z_0(t), \quad (2)$$

$$g(v) - g(u) \leq L(v - u) \quad \text{for } y_0(T) \leq u \leq v \leq z_0(T), \quad (3)$$

where $h : J \rightarrow \mathbb{R}$ is an integrable function and L is a nonnegative constant such that

$$L \exp \left(\int_0^T h(s) ds \right) < 1. \quad (4)$$

Then problem (1) has, in the set Δ , a unique solution.

Proof. The existence of a solution of (1) follows from Theorem 1. Thus it remains to prove the uniqueness. Let $x, \bar{x} \in \Delta$ be arbitrary two solutions of (1). We distinguish two cases.

Case 1. $x(t) \neq \bar{x}(t)$ for all $t \in J$. Indeed, without the loss of generality, we can assume that $p(t) = x(t) - \bar{x}(t) > 0$ for $t \in J$. Hence

$$\begin{aligned} p'(t) &= f(t, x(t)) - f(t, \bar{x}(t)) \leq h(t)p(t), \quad t \in J, \\ p(0) &= g(x(T)) - g(\bar{x}(T)) \leq Lp(T), \end{aligned}$$

by assumptions (2) and (3). Therefore,

$$0 < p(0) \leq Lp(T) \leq L \exp \left(\int_0^T h(s) ds \right) p(0).$$

By condition (4), this inequality yields $p(0) = 0$ and thus $p(t) = 0$ on J , which is a contradiction.

Case 2. There exists $t_0 \in J$ such that $x(t_0) = \bar{x}(t_0)$. If $t_0 = T$ or $t_0 = 0$, then

$$x(0) = g(x(T)) = g(\bar{x}(T)) = \bar{x}(0).$$

This and condition (2) prove that $x(t) = \bar{x}(t)$ on J , which is a contradiction. If $t_0 \in (0, T)$, then $x(t) = \bar{x}(t)$ on $[t_0, T]$. Hence $x(T) = \bar{x}(T)$, so $x(0) = \bar{x}(0)$ showing that $x(t) = \bar{x}(t)$ on J . It is a contradiction. This proves that problem (1) has, in Δ , a unique solution. It ends the proof. \square

In the next two sections we are going to construct the solution of problem (1).

3. CASE WHERE g IS NONDECREASING

A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (1) if

$$\begin{cases} u'(t) \leq f(t, u(t)), & t \in J, \\ u(0) \leq g(u(T)), \end{cases}$$

and an upper solution of (1) if the inequalities are reversed.

Let $\Omega = \{u : y_0(t) \leq u \leq z_0(t), t \in J\}$ be a nonempty set.

We introduce the following assumptions for later use.

(H₁) $f \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$;

(H₂) $y_0, z_0 \in C^1(J, \mathbb{R})$ are a lower and an upper solution of (1), respectively, such that $y_0(t) \leq z_0(t)$, $t \in J$;

(H₃) there exists $M \geq 0$ such that $f(t, u) - f(t, v) \leq M[v - u]$ for $t \in J$, $y_0(t) \leq u \leq v \leq z_0(t)$;

(H₄) g is nondecreasing on the interval $[y_0(T), z_0(T)]$.

Lemma 1. *Let Assumption H₁ hold. Assume that $u, v \in \Delta$ are a lower and an upper solution of problem (1), respectively, and $u(t) \leq v(t)$ on J . Let Assumptions H₃, H₄ hold. Let $y, z \in C^1(J, \mathbb{R})$ and*

$$y'(t) = \tilde{f}(t, y(t)) \equiv f(t, u(t)) - M[y(t) - u(t)], \quad t \in J, \quad y(0) = g(u(T)), \quad (5)$$

$$z'(t) = \bar{f}(t, z(t)) \equiv f(t, v(t)) - M[z(t) - v(t)], \quad t \in J, \quad z(0) = g(v(T)). \quad (6)$$

Then

$$u(t) \leq y(t) \leq z(t) \leq v(t), \quad t \in J, \quad (7)$$

and y, z are a lower and an upper solution of problem (1), respectively.

Proof. It is easy to see that problems (5) and (6) have their unique solutions $y, z \in C^1(J, \mathbb{R})$. Put $p = u - y$, so $p(0) \leq g(u(T)) - g(u(T)) = 0$, and $p'(t) \leq f(t, u(t)) - \tilde{f}(t, y(t)) = -Mp(t)$. It gives $p(t) \leq 0$ on J so $u(t) \leq y(t)$, $t \in J$. Similarly, we get $z(t) \leq v(t)$, $t \in J$. Now let $q = y - z$. Then $q(0) = g(u(T)) - g(v(T)) \leq 0$, by Assumption H₄. Moreover, Assumption H₃ yields

$$q'(t) = f(t, u(t)) - f(t, v(t)) - M[q(t) - u(t) + v(t)] \leq -Mq(t).$$

This and the condition for $q(0)$ prove that $q(t) \leq 0$ on J , so (7) holds.

Now we need to show that y, z are a lower and an upper solution of (1), respectively. Indeed, in view of Assumptions H_3 and H_4 , we have

$$\begin{aligned} y'(t) &= \tilde{f}(t, y(t)) - f(t, y(t)) + f(t, y(t)) \\ &\leq f(t, y(t)) + M[y(t) - u(t)] - M[y(t) - u(t)] = f(t, y(t)), \\ z'(t) &= \bar{f}(t, z(t)) - f(t, z(t)) + f(t, z(t)) \\ &\geq f(t, z(t)) - M[v(t) - z(t)] - M[z(t) - v(t)] = f(t, z(t)) \end{aligned}$$

and

$$y(0) = g(u(T)) \leq g(y(T)), \quad z(0) = g(v(T)) \geq g(z(T)).$$

This ends the proof. \square

Note that if Assumptions H_1 to H_4 are satisfied, then by Lemma 1 the sequences $y_n, z_n \in C^1(J, \mathbb{R})$ ($n = 0, 1, \dots$) are defined uniquely so that for every natural n we have

$$\begin{cases} y'_{n+1}(t) = f(t, y_n(t)) - M[y_{n+1}(t) - y_n(t)], & t \in J, \quad y_{n+1}(0) = g(y_n(T)), \\ z'_{n+1}(t) = f(t, z_n(t)) - M[z_{n+1}(t) - z_n(t)], & t \in J, \quad z_{n+1}(0) = g(z_n(T)). \end{cases}$$

Theorem 4. *Assume that Assumptions H_1, H_2, H_3 and H_4 hold. Then*

$$\begin{aligned} y_0(t) \leq y_1(t) \leq \dots \leq y_n(t) \leq z_n(t) \leq \dots \leq z_1(t) \leq z_0(t), \\ t \in J, \quad n = 0, 1, \dots, \end{aligned} \quad (8)$$

and uniformly on J we have

$$\lim_{n \rightarrow \infty} y_n(t) = y(t), \quad \lim_{n \rightarrow \infty} z_n(t) = z(t),$$

where y and z are the minimal and the maximal solution of problem (1) in Δ .

Proof. Using Lemma 1, by mathematical induction we can get (8). Indeed, the limits of sequences $\{y_n\}, \{z_n\}$ are solutions of problem (1). By Theorem 2, problem (1) has, in Δ , extremal solutions. To finish the proof, it is enough to show that y and z are the minimal and the maximal solution of (1). To do it, we need to show that if w is any solution of (1) such that $y_0(t) \leq w(t) \leq z_0(t)$, $t \in J$, then $y_0(t) \leq y(t) \leq w(t) \leq z(t) \leq z_0(t)$, $t \in J$. Assume that for some k , $y_k(t) \leq w(t) \leq z_k(t)$, $t \in J$. Put $p = y_{k+1} - w$, $q = w - z_{k+1}$, so $p(0) = g(y_k(T)) - g(w(T)) \leq 0$, $q(0) = g(w(T)) - g(z_k(T)) \leq 0$. Moreover,

$$\begin{aligned} p'(t) &= f(t, y_k(t)) - M[y_{k+1}(t) - y_k(t)] - f(t, w(t)) \\ &\leq M[w(t) - y_k(t)] - M[y_{k+1}(t) - y_k(t)] = -Mp(t), \\ q'(t) &= f(t, w(t)) - f(t, z_k(t)) + M[z_{k+1}(t) - z_k(t)] \leq -Mq(t), \end{aligned}$$

by Assumptions H_3 . It shows $p(t) \leq 0$, $q(t) \leq 0$ on J , so $y_{k+1}(t) \leq w(t) \leq z_{k+1}(t)$ on J . Hence, by the method of mathematical induction, we have that $y_n(t) \leq w(t) \leq z_n(t)$ for $t \in J$ and for all natural n . Taking the limit $n \rightarrow \infty$, we get the assertion of Theorem 4. \square

4. CASE WHERE g IS NONINCREASING

Functions $u, v \in C^1(J, \mathbb{R})$ are called weakly coupled (w.c.) lower and upper solutions of problem (1) if

$$\begin{cases} u'(t) \leq f(t, u(t)), & t \in J, \quad u(0) \leq g(v(T)), \\ v'(t) \geq f(t, v(t)), & t \in J, \quad v(0) \geq g(u(T)). \end{cases}$$

Let us introduce the following assumptions.

(H_5) $y_0, z_0 \in C^1(J, \mathbb{R})$ are w.c. lower and upper solutions of problem (1) and $y_0(t) \leq z_0(t), t \in J$;

(H_6) g is nonincreasing on the interval $[y_0(T), z_0(T)]$;

(H_7) there exist nonnegative constants A, M and an integrable function $K : J \rightarrow \mathbb{R}$ such that

$$-M[v - u] \leq f(t, v) - f(t, u) \leq K(t)[v - u] \text{ for } t \in J, \quad (9)$$

$$y_0(t) \leq u \leq v \leq z_0(t),$$

$$-A[v - u] \leq g(v) - g(u) \leq 0 \text{ for } y_0(T) \leq u \leq v \leq z_0(T), \quad (10)$$

and

$$A \exp\left(\int_0^T K(s) ds\right) < 1. \quad (11)$$

Lemma 2. Assume that Assumption H_1 is satisfied. Let $u, v \in \Delta$ be w.c. lower and upper solutions of (1) and $u(t) \leq v(t), t \in J$. Let Assumptions H_3, H_6 hold. Let $y, z \in C^1(J, \mathbb{R})$ and

$$\begin{cases} y'(t) = f(t, u(t)) - M[y(t) - u(t)], & t \in J, \quad y(0) = g(v(T)), \\ z'(t) = f(t, v(t)) - M[z(t) - v(t)], & t \in J, \quad z(0) = g(u(T)). \end{cases}$$

Then

$$u(t) \leq y(t) \leq z(t) \leq v(t), \quad t \in J, \quad (12)$$

and y, z are w.c. lower and upper solutions of problem (1).

Proof. Note that y and z are well defined. Put $p = u - y, q = z - v$. Then $p(0) \leq g(v(T)) - g(v(T)) = 0, q(0) \leq g(u(T)) - g(u(T)) = 0$, and

$$p'(t) \leq f(t, u(t)) - f(t, u(t)) + M[y(t) - u(t)] = -Mp(t),$$

$$q'(t) \leq f(t, v(t)) - M[z(t) - v(t)] - f(t, v(t)) = -Mq(t), \quad t \in J.$$

It yields $p(t) \leq 0, q(t) \leq 0, t \in J$, so $u(t) \leq y(t), z(t) \leq v(t), t \in J$. Now let $p = y - z$, so $p(0) = g(v(T)) - g(u(T)) \leq 0$, by Assumption H_6 . In view of Assumption H_3 , we see that

$$p'(t) = f(t, u(t)) - f(t, v(t)) - M[y(t) - u(t) - z(t) + v(t)] \leq -Mp(t).$$

Hence $p(t) \leq 0$ on J , so (12) holds.

By Assumptions H_3 and H_6 , we have

$$\begin{aligned} y'(t) &= f(t, u(t)) - M[y(t) - u(t)] - f(t, y(t)) + f(t, y(t)) \\ &\leq f(t, y(t)) + M[y(t) - u(t)] - M[y(t) - u(t)] = f(t, y(t)), \\ z'(t) &= f(t, v(t)) - M[z(t) - v(t)] - f(t, z(t)) + f(t, z(t)) \geq f(t, z(t)), \end{aligned}$$

and

$$y(0) = g(v(T)) \leq g(z(T)), \quad z(0) = g(u(T)) \geq g(y(T)).$$

It proves that y, z are w.c. lower and upper solutions of problem (1).

The proof is complete. \square

We see that if the assumptions of Lemma 2 are satisfied then the sequences $y_n, z_n \in C^1(J, \mathbb{R})$ ($n = 0, 1, \dots$) are defined uniquely so that for every natural n we have

$$\begin{cases} y'_{n+1}(t) = f(t, y_n(t)) - M[y_{n+1}(t) - y_n(t)], & t \in J, \quad y_{n+1}(0) = g(z_n(T)), \\ z'_{n+1}(t) = f(t, z_n(t)) - M[z_{n+1}(t) - z_n(t)], & t \in J, \quad z_{n+1}(0) = g(y_n(T)). \end{cases}$$

Theorem 5. *Assume that Assumptions H_1, H_5, H_7 hold. Then problem (1) has a unique solution $x \in \Delta$,*

$$\begin{aligned} y_0(t) \leq y_1(t) \leq \dots \leq y_n(t) \leq z_n(t) \leq \dots \leq z_1(t) \leq z_0(t), \\ t \in J, \quad n = 0, 1, \dots, \end{aligned} \quad (13)$$

and uniformly on J we have

$$\lim_{n \rightarrow \infty} y_n(t) = \lim_{n \rightarrow \infty} z_n(t) = x(t).$$

Proof. It is easy to verify that all assumptions of Theorem 3 are satisfied, so problem (1) has a unique solution $x \in \Delta$. To finish the proof, it is enough to show that the sequences y_n, z_n converge to x . Using Lemma 2, we can prove (13), by mathematical induction. It follows from the standard argument that $\{y_n\}, \{z_n\}$ converge uniformly to their respective limit functions. Let $y(t) = \lim_{n \rightarrow \infty} y_n(t), z(t) = \lim_{n \rightarrow \infty} z_n(t)$. Then

$$\begin{cases} y'(t) = f(t, y(t)), & t \in J, \quad y(0) = g(z(T)), \\ z'(t) = f(t, z(t)), & t \in J, \quad z(0) = g(y(T)) \end{cases} \quad (14)$$

and $y_0(t) \leq y(t) \leq z(t) \leq z_0(t), t \in J$. Put $p = z - y$, so $p(t) \geq 0$ on J . Hence $p(0) = g(y(T)) - g(z(T)) \leq Ap(T)$, by Assumption H_6 . Moreover,

$$p'(t) = f(t, z(t)) - f(t, y(t)) \leq K(t)p(t), \quad t \in J.$$

This inequality yields

$$0 \leq p(0) \leq Ap(T) \leq Ap(0) \exp\left(\int_0^T K(s) ds\right)$$

since

$$0 \leq p(t) \leq p(0) \exp \left(\int_0^t K(s) ds \right), \quad t \in J.$$

By condition (11), $p(0) = 0$, and thus $p(t) = 0$, $t \in J$. This proves that $y(t) = z(t)$ on J , so y and z are solutions of problem (1). Since problem (1) has a unique solution x , we have $y = z = x$.

It ends the proof. \square

Remark. The results of this paper remain true if the constant $-M$ is replaced by an integrable on J function M .

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