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# AMD-NUMBERS, COMPACTNESS, STRICT SINGULARITY AND THE ESSENTIAL SPECTRUM OF OPERATORS

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**Abstract**. For an operator $T$ acting from an infinite-dimensional Hilbert space $H$ to a normed space $Y$ we define the upper AMD-number $\overline{\overline{\delta}}(T)$ and the lower AMD-number $\underline{\underline{\delta}}(T)$ as the upper and the lower limit of the net $(\delta(T|_E))_{E\in\mathcal{FD}(H)}$, with respect to the family $\mathcal{FD}(H)$ of all finite-dimensional subspaces of $H$. When these numbers are equal, the operator is called AMD-regular.

It is shown that if an operator $T$ is compact, then $\overline{\overline{\delta}}(T) = 0$ and, conversely, this property implies the compactness of $T$ provided $Y$ is of cotype 2, but without this requirement may not imply this. Moreover, it is shown that an operator $T$ has the property $\overline{\overline{\delta}}(T) = 0$ if and only if it is superstrictly singular. As a consequence, it is established that any superstrictly singular operator from a Hilbert space to a cotype 2 Banach space is compact.

For an operator $T$, acting between Hilbert spaces, it is shown that $\overline{\overline{\delta}}(T)$ and $\underline{\underline{\delta}}(T)$ are respectively the maximal and the minimal elements of the essential spectrum of $|T| := (T^*T)^{\frac{1}{2}}$, and that $T$ is AMD-regular if and only if the essential spectrum of $|T|$ consists of a single point.

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## 1. Introduction

For a given linear operator $T : E \to Y$, where $E$ is an Euclidean space ( i.e., $E$ is a non-zero finite-dimensional Hilbert space) and $Y$ is a normed space, its *mean dilatation number* (briefly *MD-number*), $\delta(T)$ is defined by the equality

$$\delta(T) = \left( \int_{S_E} \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}, \tag{1.1}$$

in which $s$ denotes the uniform distribution on the unit sphere $S_E$ of $E$.

The quantity $\delta(T)$, without giving it any special name, has already been used earlier in the local theory of normed spaces (see, e.g., [61, p. 81], compare also [45, p. 110]).

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The MD-number $\delta(T)$ is related to the $l$-norm or $\gamma$-summing norm of $T$ through the relation (see [61, pp. 81–82])

$$\delta(T) = \frac{1}{\sqrt{\dim E}}\, l(T). \tag{1.2}$$

Recall that

$$l(T) = \left(\int_E \|Tx\|^2 d\gamma(x)\right)^{\frac{1}{2}}, \tag{1.3}$$

where $\gamma$ denotes the standard Gaussian distribution on $E$.

The $l$-norm was introduced in [38] and, independently, in [19] and proved to be a powerful tool of modern Functional Analysis and Operator Theory (see, e.g., [18], [10], [54], [55], [61]). It is clear that thanks to (1.2) these two functionals lead to equivalent approaches. However sometimes it is preferable to work with $\delta$ (cf., e.g., [45], [46]).

When $Y$ is an inner product space, we have

$$l(T) = \|T\|_{HS}, \quad \delta(T) = \frac{1}{\sqrt{\dim E}}\, \|T\|_{HS}, \tag{1.4}$$

where $\|T\|_{HS}$ denotes the Hilbert–Schmidt norm of $T$.

The problem of extension of the functional $l$ to the operators acting from infinite-dimensional domain has been treated by many authors. Recall that a continuous linear operator $T : H \to Y$, where $H$ is an infinite-dimensional Hilbert space and $Y$ is a normed space, is called $\gamma$-summing or Gauss-summing if

$$\tilde{l}(T) := \sup_{M \in \mathcal{FD}(H)} l(T|_M) < \infty, \tag{1.5}$$

where $\mathcal{FD}(H)$ stands for the family of all finite-dimensional vector subspaces $M \subset H$, $\dim(M) \geq 1$. This notion was introduced in [38] (see also [61, p. 82] and [55, p. 38]). It is not hard to see that when $Y$ is a Hilbert space, then an operator $T : H \to Y$ is $\gamma$-summing if and only if it is a Hilbert–Schmidt operator and $\tilde{l}(T) = \|T\|_{HS}$. In general, the study of $\gamma$-summing operators is closely related to the problem of description of Gaussian measures in Banach spaces. In this connection (see [13], [39]), and also directly (see [38], [10], [54], [55], [61]), they were intensively studied and their relations with summing, nuclear, etc., operators were clarified.

The aim of this paper is to study appropriate asymptotic versions of MD-numbers for an operator $T$ from an infinite-dimensional Hilbert space $H$ to any normed space $Y$. Note first that always

$$\sup_{M \in \mathcal{FD}(H)} \delta(T|_M) = \|T\|.$$

Therefore in this case the above described way of extending $l$-norm gives nothing new. Using the fact that the set $\mathcal{FD}(H)$ is upward directed by set-theoretic inclusion and viewing $(\delta(T|_M))_{M\in\mathcal{FD}(H)}$ as a net with this index set, we assign to $T$ two quantities:

$$\overline{\overline{\delta}}(T) := \limsup_{M\in\mathcal{FD}(H)} \delta(T|_M) \tag{1.6}$$

and

$$\underline{\underline{\delta}}(T) := \liminf_{M\in\mathcal{FD}(H)} \delta(T|_M). \tag{1.7}$$

We call these quantities, resp., *upper and lower asymptotic mean dilatation numbers* (briefly upper and lower AMD-numbers) of $T$. In the case, where these numbers are the same, we put $\overline{\delta}(T) := \overline{\overline{\delta}}(T) = \underline{\underline{\delta}}(T)$ and call the operator *asymptotically mean dilatation regular*, briefly AMD-regular.

The notions of AMD-numbers and AMD-regularity were already considered in [12] for Hilbert space-valued operators, where it was observed, in particular, that not every operator is AMD-regular.

In this paper we study the AMD-numbers of Banach space valued operators and apply them for the investigation of compactness-like properties and the essential spectrum of operators.

The paper is organized as follows.

In Section 2 auxiliary material is collected about approximation numbers, compact operators, diagonalizable operators, strictly singular operators, Bernstein numbers and superstrictly singular operators. A proof of the following assertion, which seems not to have appeared earlier in the literature is presented: if $X$ is a Banach space with (infra) type 2 and with an unconditional basis and $Y$ is either an abstract $L$-space or a Banach space with the Orlicz property and with an unconditional basis, then any strictly singular operator $T : X \to Y$ is compact (Theorem 2.18).

Section 3 is dedicated to the MD-numbers. Among other, rather technical, inequalities, it is shown, in particular, that the MD-number $\delta(T)$ and the median of $T$ with respect to the uniform distribution are equivalent quantities (see Proposition 3.7 and the Remark following it). The most delicate result is Proposition 3.9 which in fact is related to the Isoperimetric Inequality.

In Section 4 the general properties of AMD-numbers are analyzed. In the case of diagonalizable operators, i.e., operators having the form $T = \sum \lambda_n e_n \otimes y_n$, where $(\lambda_n)$ is a bounded sequence of scalars and $(e_n)$ and $(y_n)$ are orthonormal bases in $H$ and $Y$, respectively, the following concrete expressions of these numbers are obtained:

$$\overline{\overline{\delta}}(T) = \limsup_n |\lambda_n|, \quad \underline{\underline{\delta}}(T) = \liminf_n |\lambda_n|.$$

From this result it follows that diagonalizable $T$ with diagonal $(\lambda_n)$ is AMD-regular if and only if the sequence $(|\lambda_n|)$ is convergent and when this is the case, we have

$$\overline{\delta}(T) = \lim_n |\lambda_n|.$$

This assertion implies, in particular, that not every operator is AMD-regular (cf. [12], where it is shown directly that for any partial isometry $T$ with infinite-dimensional initial and final spaces one has $\overline{\overline{\delta}}(T) = 1$, while $\underline{\underline{\delta}}(T) = 0$).

In the final part of Section 4 AMD-numbers are applied for characterization of compactness, strict singularity and superstrict singularity of operators. It is shown, e.g., that if $T$ is a compact operator, then $\overline{\overline{\delta}}(T) = 0$ and that the converse statement is also true provided $Y$ has cotype 2, but may not be true without this requirement. More precisely, it turns out that for a given operator $T$, in general, $\overline{\overline{\delta}}(T) \leq d(T, K(H, Y))$ and if $Y$ is of cotype 2, then $\overline{\overline{\delta}}(T) \geq c^{-1} d(T, K(H, Y))$, where $c$ is the cotype 2 constant of $Y$ and $d(T, K(H, Y))$ stands for the distance from $T$ to the space of all compact operators $K(H, Y)$. The core of the section is Theorem 4.11 which establishes, in particular, that in case of an arbitrary $Y$ for a given $T$ the fulfilness of the condition $\overline{\overline{\delta}}(T) = 0$ is equivalent to the superstrict singularity of $T$. This result shows that the upper AMD-number $\overline{\overline{\delta}}(T)$ can also be viewed as a "distance" from the operator $T$ to the set of superstrictly singular operators. The section ends with the following "mean dilatation free" consequence of the previous results: any superstrictly singular operator from a Hilbert space to an *arbitrary* cotype 2 Banach space is compact (Theorem 4.12).

Section 5 deals with the operators between Hilbert spaces. The general case of non-necessarily diagonalizable operators is treated. We prove that for a given operator $T$, $\overline{\overline{\delta}}(T)$ is the maximal and $\underline{\underline{\delta}}(T)$ is the minimal element of the essential spectrum of $|T| := (T^*T)^{\frac{1}{2}}$ and that $T$ is asymptotically mean dilatation regular if and only if the essential spectrum of $|T|$ consists of a single point (Theorem 5.6).

We derive this result from the corresponding statement concerning diagonalizable operators. For this several assertions, about the continuity of a spectrum and spectral radius in a Banach algebra, essential spectrum, etc. are used. Surely, most of these facts are well-known for specialists, we give their precise formulations and outline the proofs only for the reader's convenience.

## 2. Auxiliary Results on Compact and Diagonalizable Operators

**2.1. Notation.** Hereinafter $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The considered normed or inner-product spaces are supposed to be defined over $\mathbb{K}$. The norm of a normed space, resp., the scalar product of an inner-product space is denoted by $\|\cdot\|$, resp., $(\cdot|\cdot)$. Also, $\|T\|$ stands for the ordinary norm of a continuous linear operator $T$ acting between normed spaces.

For a normed space $X$,

- $X^*$ stands for the topological dual space.
- We put:

$$B_X := \{x \in X : \|x\| \leq 1\}, \quad S_X := \{x \in X : \|x\| = 1\}.$$

- For $x \in X$ and a non-empty $A \subset X$

$$d(x, A) := \inf\{\|x - a\| : a \in A\}.$$

- $\mathcal{FD}(X)$ stands for the family of all finite-dimensional non-zero vector subspaces of $X$.

For normed spaces $X$ and $Y$,

- $L(X, Y)$ is the normed space of all continuous linear operators $T : X \to Y$ and $L(X) := L(X, X)$.
- $K(X, Y)$ stands for the set of all compact linear operators $T : X \to Y$ and $K(X) := K(X, X)$.
- $d(T, K(X, Y)) := \inf\{\|T - A\| : A \in K(X, Y), \quad T \in L(X, Y)\}$.
- For an operator $T \in L(X, Y)$ we put

$$m(T) := \inf\{\|Tx\| : x \in S_X\}$$

  and call the quantity the *lower bound* of $T$.
- $\operatorname{rank} T := \dim T(X), \quad T \in L(X, Y)$.

By $H$ we always denote an infinite-dimensional (not necessarily separable) Hilbert space, which is canonically identified with $H^*$. In particular, when $Y$ is also a Hilbert space, for a given operator $T \in L(H, Y)$ its adjoint operator $T^*$ is supposed to act from $Y$ to $H$.

For Hilbert spaces $H, Y$ and an operator $T \in L(H, Y)$ the unique self-adjoint positive (=non-negative) square root of the operator $T^*T \in L(H)$ is denoted by $|T|$.

Let $J$ be an infinite abstract index set and $(\beta_j)_{j\in J}$ be a family of elements of a topological space. Suppose in $J$ a directed partial order $\prec$ is given. Then the family $(\beta_j)_{j\in(J,\prec)}$ is a net with respect to $(J, \prec)$ and, for it, the notion of convergence and the meaning of $\lim\limits_{j\in(J,\prec)} \beta_j$ are clear. If $(\beta_j)_{j\in J}$ is a family of real numbers, then the meaning of notations $\limsup_{j\in(J,\prec)} \beta_j$ and $\liminf_{j\in(J,\prec)} \beta_j$ is also clear:

$$\limsup_{j\in(J,\prec)} \beta_j := \inf_{j\in J} \sup_{i\in J,\ i\succ j} \beta_j, \quad \liminf_{j\in(J,\prec)} \beta_j := \sup_{j\in J} \inf_{i\in J,\ i\succ j} \beta_j.$$

Below we shall use the notation of an 'upper limit', a 'lower limit' and a 'limit' of a given family $(\beta_j)_{j\in J}$ of real numbers, which do not assume that some directed partial order is given in $J$, namely, we put :

$$\text{u-}\limsup_{j\in J} \beta_j := \inf_{\Delta\subset J, \operatorname{card}\Delta<\infty} \sup_{j\notin\Delta} \beta_j, \quad \text{u-}\liminf_{j\in J} \beta_j := \sup_{\Delta\subset J, \operatorname{card}\Delta<\infty} \inf_{j\notin\Delta} \beta_j.$$

The letter 'u' means 'unordered'[1]. When we have that $\text{u-}\limsup_j \beta_j = \text{u-}\liminf_j \beta_j = \beta$, then we say that *the unordered limit of $(\beta_j)_{j\in J}$ exists* and put $\text{u-}\lim\limits_{j\in J} \beta_j = \beta$.

[1] Let $\mathfrak{G}$ be the filter generated by complements of finite subsets of $J$. Following [7, Ch. IV, §5], instead of $\text{u-}\limsup_{j\in J} \beta_j$ we could write $\limsup_{\mathfrak{G}} \beta_j$ and call this quantity "the upper limit with respect to the filter $\mathfrak{G}$".

The following statement is easy to check.

**Lemma 2.1.** *Let $(\beta_j)_{j\in J}$ be any bounded infinite family of real numbers.*
(a) *We have the equalities:*

$$\text{u-}\limsup_{j\in J} \beta_j = \sup_{(j_n)} \lim_n \beta_{j_n} \quad \textit{and} \quad \text{u-}\liminf_{j\in J} \beta_j = \inf_{(j_n)} \lim_n \beta_{j_n},$$

*where* sup, (*resp.* inf), *is taken over all infinite sequences $(j_n)$ of* ***distinct*** *elements of $J$ such that the sequence $(\beta_{j_n})_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$.*
(b) *If $J = \mathbb{N}$, then*

$$\text{u-}\limsup_{j\in\mathbb{N}} \beta_j = \limsup_{j\to\infty} \beta_j \quad \textit{and} \quad \text{u-}\limsup_{j\in\mathbb{N}} \beta_j = \liminf_{j\to\infty} \beta_j.$$

*In particular, $\lim_{j\to\infty} \beta_j$ exists if and only if $\text{u-}\lim_{j\in\mathbb{N}} \beta_j$ exists and in such a case $\lim_{j\to\infty} \beta_j = \text{u-}\lim_{j\in\mathbb{N}} \beta_j$.*

For a given bounded infinite family of scalars $\lambda. := (\lambda_j)_{j\in J}$ and for any fixed natural number $n$ we put

$$a_n(\lambda.) := \inf_{\Delta\subset J,\, \operatorname{card}\Delta<n} \sup\{|\lambda_j| j \in J\setminus\Delta\}, \;\; \tilde{a}_n(\lambda.) := \sup_{\Delta\subset J,\, \operatorname{card}\Delta=n} \min\{|\lambda_j| : j \in \Delta\}.$$

**Lemma 2.2.** *Let $\lambda. := (\lambda_j)_{j\in J}$ be a bounded infinite family of scalars. Then:*
(a) *We have $a_n(\lambda.) = \tilde{a}_n(\lambda.)$, $n = 1, 2, \dots$*
(a′) *If $J_+ := \{j \in J : |\lambda_j| > 0\} \neq \varnothing$ and $\lambda.^+ := (\lambda_j)_{j\in J_+}$, then $a_n(\lambda.) = a_n(\lambda.^+)$, $n = 1, 2, \dots$*
(b) *$(a_n(\lambda.))$ is a decreasing sequence and* $\text{u-}\limsup_{j\in J} |\lambda_j| = \lim_n a_n(\lambda.)$.
(c) *If $J = \mathbb{N}$ and $(|\lambda_j|)_{j\in\mathbb{N}}$ is a decreasing sequence, then $a_n(\lambda.) = |\lambda_n|$, $n = 1, 2, \dots$*

*Proof.* (a) Fix $n$ and let $\tilde{a}_n(\lambda.) < r < \infty$. Then for each $\Delta \subset J$ with $\operatorname{card}(\Delta) = n$ there is $j \in \Delta$ such that $|\lambda_j| < r$. Denote $J_r = \{j \in J : |\lambda_j| \geq r\}$. Then $\operatorname{card} J_r < n$ and $\sup\{|\lambda_j| : j \in J \setminus J_r\} < r$. Hence $a_n(\lambda.) < r$. Since $r$ is arbitrary, we get $a_n(\lambda.) \leq \tilde{a}_n(\lambda.)$. The proof of $a_n(\lambda.) \geq \tilde{a}_n(\lambda.)$ is similar.
(a′) is evident.
We omit a straightforward verification of other statements.[2] □

Denote by $\mathfrak{F}(J)$ the family of all non-empty finite subsets of a set $J$, then $(\mathfrak{F}(J), \subset)$ is a directed set. A family $(x_j)_{j\in J}$ of elements of a Banach space $X$ is called summable with sum $x \in X$, in symbols $\sum_{j\in J} x_j = x$, if the net $(\sum_{j\in\Delta} x_j)_{\Delta\in\mathfrak{F}(J)}$ is convergent to $x$ in the topology of $X$.

It is well-known that a sequence $(x_j)_{j\in\mathbb{N}}$ is summable if and only if the corresponding series is unconditionally convergent and in such a case $\sum_{j\in\mathbb{N}} x_j = \sum_{j=1}^{\infty} x_j$. Note also that if $(x_j)_{j\in J}$ is a summable family, then the set $J_+$ of those indices $j$ for which $x_j \neq 0$ is at most countable and $\sum_{j\in J} x_j = \sum_{j\in J_+} x_j$.

[2] It is not hard to check that for a given bounded family $(\lambda_j)_{j\in J}$ the quantity $a_n(\lambda.)$ coincides with its $n$-th approximation number in the sense of [52, 13.7.3].

A family $(y_j)_{j\in J}$ of elements of a Banach space $Y$ is called *an unconditional basis* of $Y$ if for any $y \in Y$ there exists a unique family $(t_j)_{j\in J}$ of scalars such that $y = \sum_{j\in J} t_j y_j$.

If $(y_j)_{j\in J}$ is an unconditional basis of $Y$, then in $Y^*$ there exists a family of *coordinate functionals* $(y_j^*)_{j\in J}$ such that $\sum_{j\in J} y_j^*(y)y_j = y,\ \forall y \in Y$.

A family $(y_j)_{j\in J}$ of elements of a Banach space $Y$ is called *an unconditional basic family* if it is an unconditional basis for its closed linear span into $Y$.

Note that if $Y$ is a separable infinite-dimensional Banach space and $(y_j)_{j\in J}$ is an infinite unconditional basic family in it, then $J$ is countable.[3]

Let $Y$ be a Hilbert space. Recall that for an operator $T \in L(H, Y)$ the value of the sum $\sum_{j\in J} \|Te_j\|^2$ does not depend on a particular choice of an orthonormal basis $(e_j)_{j\in J}$ of $H$ and when this value is finite, the operator is called a Hilbert–Schmidt operator. It is well known that the equality

$$\|T\|_{HS} = \left(\sum_{j\in J} \|Te_j\|^2\right)^{\frac{1}{2}} \tag{2.1}$$

defines a norm on the vector space of all Hilbert–Schmidt operators. This norm is called the Hilbert–Schmidt norm.

**2.2. Approximation numbers, Bernstein numbers and compact-like operators.** Let $X, Y$ be normed spaces. A linear operator $T : X \to Y$ is called *compact* if $T(B_X)$ is a relatively compact subset of $Y$.

A linear operator $T : X \to Y$ is called *completely continuous* if for any weak-null sequence $(x_n)$ in $X$, $\lim_n \|Tx_n\| = 0$. Any $T \in K(X, Y)$ is completely continuous; the converse is true if $X$ is reflexive (or is 'almost reflexive', see Proposition 2.3), but is not true in general (see the next item).

A normed space $X$ is said to have the *Schur property* if for any weak-null sequence $(x_n)$ in $X$, we have $\lim_n \|x_n\| = 0$ (i.e., if the identity mapping $I : X \to X$ is completely continuous). The space $X = l_1$ is a classical example of an infinite-dimensional Banach space with the Schur property.

**Proposition 2.3 (Rosenthal, Lacey–Whitley).** *Let $X$ be a Banach space which does not contain a subspace isomorphic to $\ell_1$ and $Y$ be another Banach space.*

(a) *Any completely continuous linear operator $T : X \to Y$ is compact.*

(b) *If $Y$ has the Schur property, then $L(X, Y) = K(X, Y)$.*

*Proof.* According to Rosenthal's $\ell_1$-theorem [17, p. 201] each bounded sequence in $X$ has a weak-Cauchy subsequence, i.e., $X$ is almost reflexive in the sense [36]. For almost reflexive $X$ the assertion coincides with Theorem 5 and Corollary 6 in [36]. □

[3] Every infinite-dimensional Banach space contains a basic sequence [40, Theorem 1.a.5]; however, the famous unconditional basic sequence problem "Does every infinite dimensional Banach space contain an unconditional basic sequence?" [40, Problem 1.d.5] has been solved negatively in [24].

*Remark* 2.4. The following converse to Proposition 2.3(a) is also true: suppose $X$ is a Banach space such that any completely continuous linear operator $T : X \to c_0$ is compact, then $X$ does not contain a subspace isomorphic to $\ell_1$ (see [4, Proposition II.6]).

For a given operator $T \in L(X, Y)$ and a natural number $n$, the $n$-th approximation number $a_n(T)$ of $T$ is defined by the equality:

$$a_n(T) = \inf\{\|T - T_0\| : T_0 \in L(X, Y),\ \operatorname{rank} T_0 < n\}.$$

It is easy to see that $(a_n(T))$ is a decreasing sequence and $a_1(T) = \|T\|$ (see [52] and [53] for further properties).

*Remark* 2.5. An operator $T \in L(X, Y)$ is called *approximable* if $\lim_n a_n(T) = 0$. If $Y$ is a Banach space, then any approximable $T \in L(X, Y)$ is compact. The converse is also true provided either $X^*$ or $Y$ has the approximation property (see [40] for the definition and proofs).

**Lemma 2.6** ([55, Lemma 1.8 (p.10)])**.** *Let $Y$ be a normed space and $T \in L(H, Y)$. Then for every $\varepsilon > 0$ there is an orthonormal sequence $(e_n)$ in $H$ such that $\|Te_n\| \geq a_n(T) - \varepsilon$ for all $n \geq 1$.*

The next statement characterizes the compact operators acting from a Hilbert space.

**Corollary 2.7.** *Let $Y$ be a normed space and $T \in L(H, Y)$. Then the following assertions are equivalent:*

(i) *For any infinite orthonormal sequence $(e_n)$ in $H$,*$\lim_n \|Te_n\| = 0$.

(ii) $\lim_n a_n(T) = 0$.

(iii) *The operator $T$ is the norm limit of a sequence of finite-rank operators from $L(H, Y)$.*

(iv) *The operator $T$ is compact.*

(v) *The operator $T$ is completely continuous.*

*Proof.* (i) $\Rightarrow$ (ii) follows from Lemma 2.6. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v)$\Rightarrow$ (i) are evident. (iii) $\Rightarrow$ (iv) is well-known, when $Y$ is complete. In general, (iii) implies that $T(B_H)$ is precompact in $Y$; since $B_H$ is weakly compact and $T$ is weakly continuous too, $T(B_H)$ is also weakly compact in $Y$. These two conditions give that $T(B_H)$ is compact in $Y$ (see [8, IV, §1, Proposition 3]). □

**Corollary 2.8.** *Let $T \in L(H, Y)$. Then*

(a) $\lim_n a_n(T) = d(T, K(H, Y))$.

(b) $d(T, K(H, Y)) = \sup_{(e_n)\in\mathcal{O}(H)} \limsup_n \|Te_n\|$, *where $\mathcal{O}(H)$ denotes the set of all infinite orthonormal sequences $(e_n)$ in $H$.*

*Proof.* (a) follows easily from the implication (iv) $\Rightarrow$ (iii) of Corollary 2.7. (b) The inequality $d(T, K(H, Y)) \leq \sup_{(e_n)\in\mathcal{O}(H)} \limsup_n \|Te_n\|$ follows from (a) and Lemma 2.6. The inequality $d(T, K(H, Y)) \geq \sup_{(e_n)\in\mathcal{O}(H)} \limsup_n \|Te_n\|$ can be shown as follows: fix any $(e_n) \in \mathcal{O}(H)$ and $A \in K(H, Y)$. Clearly $(e_n)$ is weak-null in $H$. Since $A$ is compact, we have $\lim_n \|Ae_n\| = 0$. Consequently, $\|T - A\| \geq \limsup_n(\|Te_n\| - \|Ae_n\|) = \limsup_n \|Te_n\|$. □

*Remark* 2.9. The same proof shows that if $X$ is an arbitrary normed space without the Schur property, then $d(T, K(X, Y)) \geq \limsup_n \|Tx_n\|$ for any normalized weak-null sequence $(x_n)$ in $X$ (cf. [58, Lemma 2(i)]).

An operator $T \in L(X, Y)$ is called *strictly singular* (briefly, SS-) if $m(T|_M) = 0$ for any infinite-dimensional closed vector subspace $M \subset X$. Denote $SS(X, Y)$ the set of all strictly singular operators $T : X \to Y$ and $SS(X) := SS(X, X)$.

We have $K(X, Y) \subset SS(X, Y)$. In general, this inclusion is strict (see Remark 2.11).

Fix $T \in L(X, Y)$ and a natural number $n$. The $n$-th *Bernstein number* $b_n(T)$ of $T$ is defined by the equality (see [48]):

$$b_n(T) = \sup\{m(T|_M) : M \in \mathcal{FD}(X),\ \dim M = n\}. \tag{2.2}$$

Obviously, $\|T\| = b_1(T) \geq b_2(T) \geq \dots$. If $T : X \to Y$ is a compact operator, then $\lim_n b_n(T) = 0$, the converse is not true in general (see the next item and Proposition 2.10(c)).

An operator $T \in L(X, Y)$ is called *superstrictly singular* (briefly, SSS-) if $\lim_n b_n(T) = 0$. Denote $SSS(X, Y)$ the set of all superstrictly singular operators $T : X \to Y$ and $SSS(X) := SSS(X, X)$.

We have $K(X, Y) \subset SS(X, Y) \subset SSS(X, Y)$; in general, the second inclusion also is strict (see Proposition 2.10(e)). It is known that the strictly singular operators and the superstrictly singular form an operator ideal in the sense of Pietsch (see [52, (1.9.4)] and [56]).

In the next proposition are collected mainly the known statements.

**Proposition 2.10.** *Let* $1 \leq r, p < \infty$.

(a) *If* $r < p$, *then* $L(l_p, l_r) = K(l_p, l_r)$ (*Pitt's theorem;* [40, p. 76]).

(b) *If* $X$ *and* $Y$ *are totally incomparable Banach spaces, then any continuous linear operator from* $X$ *to* $Y$ *is strictly singular* [40, p. 75].

*In particular,*

*If* $1 \leq r, p < \infty$ *and* $r \neq p$, *then* $SS(l_r, l_p) = L(l_r, l_p)$ *(see* [23]*).*

(c) *The natural embedding* $T : l_1 \to l_2$ *is a SSS-operator that is not compact* [48].

(d) *If* $1 \leq r < p < \infty$ *and* $T$ *is the natural embedding of* $l_r$ *into* $l_p$, *then* $T \in SSS(l_r, l_p) \setminus K(l_r, l_p)$ *(see* [44, Lemma 1]*).*[4]

(e) *If* $1 < r < p < \infty$, *then* $SSS(l_r, l_p) \neq SS(l_r, l_p)$.[5]

[4] Cf. also [56], where it is shown that in the considered case one has the equalities $b_n(T) = n^{1/p-1/r}$, $n = 1, 2, \dots$, which imply (d) too.

[5] The third author learnt this result, as well as its proof, from Prof. A. Pełczyński.

*Proof.* (e) For a fixed number $s$, $1 < s < \infty$, let us denote by $X_s$ the $l_s$-sum of spaces $l_2^n$, $n = 1, 2, \dots$. According to a well-known result of [49] (see also [40, p. 73]), there exists a bijective linear homeomorphism $U_s : X_s \to l_s$. Let $T_{r,p}$ be the natural embedding of $X_r$ into $X_p$. It is clear that $T_{r,p} \notin SSS(X_r, X_p)$, hence $U_p T_{r,p} U_r^{-1} \notin SSS(l_r, l_p)$, while, by (b), we have $U_p T_{r,p} U_r^{-1} \in SS(l_r, l_p)$. □

*Remark* 2.11. (1) The notion of a strictly singular operator was introduced in [30], where it was established that $SS(X, Y)$ is a closed vector subspace of $L(X, Y)$; in the case of Hilbert spaces it was shown the equality $K(X, Y) = SS(X, Y)$ and a question was posed whether the same is true in the general case (see [30, p. 285]). Later in [21] it was shown that $K(c_0) = SS(c_0)$ and $K(l_p) = SS(l_p)$, $\forall p \in [1, \infty[$ and an example was given (due to M. I. Kadets, see [21, p. 61]) showing that this is not true in general.

(2) The SSS-operators were introduced in fact in [48], where no particular term was given to this notion. The term is taken from [28]. For this notion the term "$\mathfrak{S}_0^*$-operators" or "operators of class $C_0^*$" was used in [42], [43], [44].[6]

(3) Numerous papers were devoted to the study of strictly singular operators, strictly cosingular operators (introduced in [50]), disjointly strictly singular operators (introduced in [26]) and related operators (see, e.g., [3], [9], [11], [14], [15], [20], [23], [27], [36], [31], [42], [43], [44], [47], [48], [50], [51], [56], [64]). Some known facts are already included in the monographs (see [22], [40], [52]). As far as we know, the most recent work which deals with the superstrictly singular operators is [56].

Below we present a result about automatic compactness of strictly singular operators.

**2.3. Orlicz property, type 2 and SS-operators.** In this subsection $X, Y$ will be Banach spaces.

Let us say that an (infinite) family $(y_j)_{j \in J}$ of elements of a Banach space $Y$ is *Besselian* if there exists a constant $b > 0$ such that the inequality

$$b \Big\| \sum_{j \in J} t_j y_j \Big\| \geq \Big( \sum_{j \in J} |t_j|^2 \Big)^{\frac{1}{2}} \tag{2.3}$$

holds for any family $(t_j)_{j \in J}$ of scalars with only a finite number of non-zero terms.[7]

For a Besselian family $(y_j)_{j \in J}$ any constant $b$ for which (2.3) holds is called its *Besselian constant*.

Analogously, let us say that an (infinite) family $(y_j)_{j \in J}$ of elements of a Banach space $Y$ is *Hilbertian* if there exists a constant $b > 0$ such that the inequal-

[6] As the referee has pointed out, the "uniformly strictly singular operators" from [47] are different from SSS-operators.

[7] Sometimes a Besselian sequence is called a 2-*colacunary sequence*, see [1].

ity

$$\Big\| \sum_{j\in J} t_j y_j \Big\| \le b\Big( \sum_{j\in J} |t_j|^2 \Big)^{\frac{1}{2}} \tag{2.4}$$

holds for any family $(t_j)_{j\in J}$ of scalars with only a finite number of non-zero terms.

For a Hilbertian family $(y_j)_{j\in J}$ any constant $b$ for which (2.4) holds is called its *Hilbertian constant*.

In what follows, the set $\{-1,1\}^{\mathbb{N}}$ will be supposed to be equipped with the probability measure $\prod_{n\in\mathbb{N}} \mu_n$, where $\mu_n\{-1\} = \mu_n\{1\} = \frac{1}{2},\ n = 1, 2, \ldots$[8] Recall that a Banach space $Y$ is said to have the

- *Orlicz property* if for any summable sequence $(y_j)_{j\in\mathbb{N}}$ in $Y$, we have $\sum_{j=1}^{\infty} \|y_j\|^2 < \infty$.
- *Rademacher cotype* 2 if for every sequence $(y_j)_{j\in\mathbb{N}}$ in $Y$ for which the series $\sum_{j\in\mathbb{N}} \theta_j y_j$ is convergent in $Y$ for almost all choices of signs $(\theta_j) \in \{-1,1\}^{\mathbb{N}}$, we have $\sum_{j=1}^{\infty} \|y_j\|^2 < \infty$.
- *infratype* 2 if for any sequence $(y_j)_{j\in\mathbb{N}}$ in $Y$, with $\sum_{j=1}^{\infty} \|y_j\|^2 < \infty$ there exists a sequence of signs $(\theta_j) \in \{-1,1\}^{\mathbb{N}}$ such that the series $\sum_{j\in\mathbb{N}} \theta_j y_j$ is convergent in $Y$.
- *Rademacher type* 2 if for any sequence $(y_j)_{j\in\mathbb{N}}$ in $Y$, with $\sum_{j=1}^{\infty} \|y_j\|^2 < \infty$, the series $\sum_{j\in\mathbb{N}} \theta_j y_j$ is convergent in $Y$ for almost all choices of signs $(\theta_j) \in \{-1,1\}^{\mathbb{N}}$.

*Remark* 2.12. Let $Y$ be an infinite-dimensional Banach space.

(1) Clearly, if $Y$ has the Rademacher cotype 2, then $Y$ has the Orlicz property. If $Y$ is a rearrangement invariant (=symmetric) Banach function space on $[0,1]$, then, according to [59, Theorem 1], the converse is also true. However, the converse is not true in general: in [60] an example of a Banach space with an unconditional (even with a symmetric) basis is produced, which satisfies the Orlicz property and fails to be of cotype 2.

(2) Also is clear that if $Y$ has the Rademacher type 2, then $Y$ has infratype 2. Whether the converse is true is not known.

(3) It is standard to see that if $Y$ has the Orlicz property, then in $Y$ any unconditional normalized basic sequence is Besselian.

(4) Also, it is easy to see that if $Y$ has infratype 2, then in $Y$ any unconditional normalized basic sequence is Hilbertian.

(5) If $Y$ has infratype 2, then $Y$ does not contain a subspace isomorphic to $\ell_1$ (since $\ell_1$ has no infratype 2).

(6) If $Y$ has infratype 2, then $Y$ has no Schur property ( this follows from (5) and from the fact that any infinite-dimensional Banach space with the Schur property contains a subspace isomorphic to $\ell_1$ [17, p. 212]).

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[8] This is simply the normalized Haar measure of the compact multiplicative group $\{-1,1\}^{\mathbb{N}}$.

Let us say that a Banach space $Y$ possesses *the Besselian Selection Property*, shortly, *the BS-property*, if in $Y$ any weak-null sequence $(y_n)_{n\in\mathbb{N}}$ with $\inf_{n\in\mathbb{N}} \|y_n\| > 0$ has a Besselian basic subsequence.[9]

We say that a Banach space $Y$ possesses *the Hilbertian Selection Property*, shortly, *the HS-property*, if $Y$ has no Schur property and in $Y$ any weak-null sequence $(y_n)_{n\in\mathbb{N}}$ with $\inf_{n\in\mathbb{N}} \|y_n\| > 0$ has a Hilbertian basic subsequence.

**Proposition 2.13.** *Let $Y$ be an infinite-dimensional Banach space.*

(a) *Suppose that $Y$ has the Orlicz property and has a (not necessarily countable) unconditional basis. Then in $Y$ any weak-null sequence $(y_n)_{n\in\mathbb{N}}$ with $\inf_{n\in\mathbb{N}} \|y_n\| > 0$ has a Besselian* ***unconditional*** *basic subsequence.*

*In particular, $Y$ possesses the BS-property.*

(b) [1, Theorem 6] *If $Y$ is any (abstract) L-space, then $Y$ possesses the BS-property.* [10]

(c) *Suppose $Y$ has infratype* 2 *and has a (not necessarily countable) unconditional basis. Then in $Y$ any weak-null sequence $(y_n)_{n\in\mathbb{N}}$ with* $\inf_{n\in\mathbb{N}} \|y_n\| > 0$ *has a Hilbertian* ***unconditional*** *basic subsequence.*

*In particular, $Y$ possesses the HS-property.*

(d) *If $Y = c_0$, then $Y$ possesses the HS-property.*

*Proof.* (a,c) Fix an unconditional basis $(e_j)_{j\in J}$ of $Y$ and let $(y_n)_{n\in\mathbb{N}}$ be a weak-null sequence in $Y$ with $\inf_{n\in\mathbb{N}} \|y_n\| > 0$. There is a countable $J_0 \subset J$ such that $y_n \in Y_0,\ n = 1, 2, \ldots$, where $Y_0$ is the closed subspace of $Y$ generated by $(e_j)_{j\in J_0}$.

According to Bessaga–Pelczyński's selection principle (see [17, p. 46]) $(y_n)_{n\in\mathbb{N}}$ has a subsequence $(y_{k_n})$, which is equivalent to a block basic sequence of the unconditional basis $(e_j)_{j\in J_0}$ of $Y_0$. Consequently, $(y_{k_n})$ also is an unconditional basic sequence. By Remark 2.12(3) (resp.(4)) we get that $(y_{k_n})$ is a Besselian (resp. Hilbertian) unconditional basic sequence.

(d) is true by the same reasoning as (c) because the natural basis of $c_0$ is Hilbertian and equivalent to any normalized block basic sequence taken with respect to it [40, Proposition 2.a.1]. $\square$

*Remark* 2.14. (1) The statement (a) of Proposition 2.13 for a separable Banach $Y$ space with Rademacher cotype 2 and with an unconditional basis follows at once from [1, Proposition 9].

(2) The statement (b) of Proposition 2.13 is applicable, e.g., for the cotype 2 space $Y = L_1[0, 1]$, while (a) is not because this space has not an unconditional basis.

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[9] Formally, we get that if $Y$ has the Schur property, then $Y$ possesses the BS-property. Actually, if $Y$ has the Schur property, then it possesses a stronger property: any bounded sequence $(y_n)_{n\in\mathbb{N}}$ in $Y$, which has no convergent subsequence, admits a basic subsequence, which is equivalent to the natural basis of $l_1$ (by Rosenthal's theorem [17, p. 201]).

[10] Actually, this is an easy consequence of Theorem 6 in [1], which asserts more: if $Y$ is an L-space and $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence in $Y$, then either $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence or $(y_n)_{n\in\mathbb{N}}$ has a Besselian subsequence. We are grateful to Professors A. Pełczyński and H.P. Rosenthal for paying our attention to the paper [1].

(3) It is not clear whether any $Y$ with the Orlicz property (or with Rademacher cotype 2) possesses the BS-property (cf. [1, p. 304, Remark]).

**Lemma 2.15.** *Let $X$ be a Banach space possessing the HS-property, $Y$ be a Banach space possessing the BS-property and $T \in L(X,Y)$ be an operator which* ***is not completely continuous****. Then*

(a) *there exists a Hilbertian basic sequence $(x_n)$ in $X$ such that $(Tx_n)$ is a Besselian basic sequence in $Y$;*

(b) *there exists an infinite-dimensional closed vector subspace $M \subset X$ such that $M$ is isomorphic to $\ell_2$ and $m(T|_M) > 0$.*

*Proof.* (a) Since $T$ is not completely continuous, there exists a normalized weak-null sequence $(z_n)$ in $X$ such that $\inf_n \|Tz_n\| := r > 0$. Clearly, $(Tz_n)$ is a weak-null sequence in $Y$. Since $Y$ possesses the BS-property, $(Tz_n)$ has a Besselian basic subsequence $(Tz_{k_n})$. Let $a_n := z_{k_n}$, $n = 1, 2, \ldots$. Now, since $X$ has the HS-property, the sequence $(a_n)$ has a Hilbertian basic subsequence $(a_{m_n}) = (x_n)$. Evidently, the sequence $(x_n)$ has the needed properties.

(b) Let $(x_n)$ be a sequence from (a). Denote $M$ the closed vector subspace of $X$ generated by $(x_n)$. Let also $b_1$ (resp. $b_2$) be a Hilbertian (resp. a Besselian) constant of $(x_n)$ (resp. of $(Tx_n)$). Since $(Tx_n)$ is Besselian, we get that $(x_n)$ is Besselian too (with constant $b_2\,\|T\|$). Therefore the basis $(x_n)$ of $M$ is both Hilbertian and Besselian. Consequently $M$ is isomorphic to $\ell_2$.

Fix $x \in M$. Then for a sequence $(t_j)$ of scalars we have $x = \sum_{j=1}^{\infty} t_j x_j$ and

$$\|Tx\| = \left\| \sum_{j=1}^{\infty} t_j T x_j \right\| \geq b_2^{-1} \left( \sum_{j=1}^{\infty} |t_j|^2 \right)^{1/2} \geq b_2^{-1} b_1^{-1} \|x\|.$$

Hence, $m(T|_M) \geq b_2^{-1} b_1^{-1} > 0$. □

**Corollary 2.16.** *Let $X$ be a Banach space possessing the HS-property and $Y$ be a Banach space possessing the BS-property. Then we have:*

(a) *Any $T \in SS(X,Y)$ is completely continuous.*

(b) *If either $X$ or $Y$ does not contain a closed vector subspace isomorphic to $\ell_2$, then any $T \in L(X,Y)$ is completely continuous.*

*Proof.* (a,b) Suppose there exists $T \in L(X,Y)$ which is not completely continuous. Then by Lemma 2.15(b) there is an infinite-dimensional closed subspace $M \subset X$ such that $M$ is isomorphic to $\ell_2$ and $m(T|_M) > 0$. Hence, $T \notin SS(X,Y)$ and we get (a). Since $m(T|_M) > 0$, the subspace $T(M)$ of $Y$ is isomorphic to $M$. Therefore $X$ and $Y$ both have a subspace isomorphic to $\ell_2$ and we get (b). □

*Remark* 2.17. The following example shows that in Corollary 2.16(a) it cannot be asserted that any $T \in SS(X,Y)$ is compact. Let $X = \ell_1 \times c_0$ and $Y = \ell_2$. Then $X$ possesses the HS-property (this is easy to see). Let now $T = T_{1,2} P_1$, where $T_{1,2} : \ell_1 \to \ell_2$ is the natural embedding and $P_1 : \ell_1 \times c_0 \to \ell_1$ is the natural projection. Then $T$ is even superstrictly singular (as $T_{1,2}$ has this property), but $T$ is not compact.

We can formulate the result which seems not to have been noticed in the literature.

**Theorem 2.18.** *Let $X$ be a Banach space of infratype 2 with an unconditional basis, $Y$ be either a Banach space having the Orlicz property and an unconditional basis or any abstract L-space. Then any strictly singular operator $T : X \to Y$ is compact.*

*Proof.* Observe that by Proposition 2.13(c) $X$ possesses the HS-property and by Proposition 2.13(b,c) $Y$ possesses the BS-property. Let now $T \in SS(X, Y)$. Then by Corollary 2.16(a), $T$ is completely continuous. Since $X$ is of infratype 2, it does not contain a closed vector subspace isomorphic to $\ell_1$ (see Remark 2.12(5)). Then by Proposition 2.3(a) we get that $T$ is a compact operator. □

*Remark* 2.19. (1) Theorem 2.18 implies, in particular, that $K(H) = SS(H)$.

(2) When $X = \ell_2$ and $Y$ is a cotype 2 space with unconditional basis, Theorem 2.18 was pointed out to us by Prof. A. Plichko.

(3) In [11] a general result is obtained, from which it follows that if $1 < r \leq p < \infty$, then any regular strictly singular $T : \mathbb{L}_p[0, 1]) \to \mathbb{L}_r[0, 1]$ is compact; Theorem 2.18 implies that if $1 \leq r \leq 2 \leq p \leq \infty$, then the same conclusion is true without assuming additionally that $T$ is regular.

(4) Theorem 2.18 leaves the following question open: let $Y$ be an arbitrary Banach space with Orlicz property (or with Rademacher cotype 2); is then any strictly singular operator $T : l_2 \to Y$ compact? Below, through AMD-numbers, we obtain a positive answer for *SSS-operators* in *cotype* 2 case (see Theorem 4.12).

(5) *In general, for a Banach space $Y$ the validity of the inclusion $SS(\ell_2, Y) \subset K(\ell_2, Y)$ may not imply that $Y$ has the Orlicz property.* Indeed, let $Y$ be a Banach space with the Schur property, then $L(\ell_2, Y) = K(\ell_2, Y) = SS(\ell_2, Y)$. Take now as $Y$ the $\ell_1$−sum of spaces $\ell_\infty^n$, $n = 1, 2, \ldots$. Then $Y$ has no Orlicz property (this is evident) and has the Schur property (see [5, Corollary 2.4(c)]).

**2.4. Diagonalizable operators and their approximation numbers.** Fix a family of scalars $(\lambda_j)_{j\in J}$.

Let $X$ be a Banach space with an unconditional basis $(x_j)_{j\in J}$ and $Y$ be a Banach space with an unconditional basis $(y_j)_{j\in J}$. An operator $T \in L(X, Y)$ is called *diagonal* and the family $(\lambda_j)_{j\in J}$ is called *the diagonal* of $T$ with respect to $(x_j)_{j\in J}$ and $(y_j)_{j\in J}$ if $Tx_j = \lambda_j y_j, \ \forall j \in J$.

If $X, Y$ are Banach spaces with unconditional bases (with equal cardinalities), then an operator $T \in L(X, Y)$ is called *diagonalizable* (resp., *diagonalizable with the diagonal* $(\lambda_j)_{j\in J}$) if there exists an unconditional basis $(x_j)_{j\in J}$ in $X$ and an unconditional basis $(y_j)_{j\in J}$ in $Y$, such that $T$ is diagonal with respect to $(x_j)_{j\in J}$ and $(y_j)_{j\in J}$ (resp., with the diagonal $(\lambda_j)_{j\in J}$).

If $X$ is a Banach space with an unconditional basis, then an operator $T \in L(X, X)$ is called *strictly diagonalizable* if there exists an unconditional basis $(x_j)_{j\in J}$ in $X$ such that $T$ is diagonal with respect to $(x_j)_{j\in J}$ and $(x_j)_{j\in J}$.

If $X, Y$ are Hilbert spaces, then an operator $T \in L(X,Y)$ is called *ortho-diagonalizable* if there exists an orthonormal basis $(x_j)_{j\in J}$ in $X$ and an orthonormal basis $(y_j)_{j\in J}$ in $Y$, such that $T$ is diagonal with respect to $(x_j)_{j\in J}$ and $(y_j)_{j\in J}$; an operator $T \in L(X,X)$ is called *strictly ortho-diagonalizable* if there exists an orthonormal basis $(x_j)_{j\in J}$ in $X$ such that $T$ is diagonal with respect to $(x_j)_{j\in J}$ and $(x_j)_{j\in J}$.

It is well-known that if $X, Y$ are Hilbert spaces, then any compact operator $T : X \to Y$ is ortho-diagonalizable, with the diagonal tending to zero (Schmidt's theorem, see, e.g.,[53, p. 119]).

It is easy to see that if $H, Y$ are Hilbert spaces and $T \in L(H,Y)$ is ortho-diagonalizable, then $T^*T$ is strictly ortho-diagonalizable and conversely. Notice that if $T \in L(H)$ is an ortho-diagonalizable positive self-adjoint operator, then it is strictly ortho-diagonalizable, but we shall not use this further.

**Lemma 2.20.** *Let $X$ be a Banach space with an unconditional basis $(e_j)_{j\in J}$, $Y$ be a Banach space, $T \in L(X,Y)$ be a non-zero operator, $\lambda_j := \|Te_j\|$, $\forall j \in J$ and $J_+ := \{j \in J : \|Te_j\| > 0\}$.*

(a) *If $(e_j)_{j\in J}$ is a Besselian family with a constant $b$ and $(\frac{Te_j}{\|Te_j\|})_{j\in J_+}$ is a Hilbertian family with a constant $h$ , then*

$$a_n(T) \le b\,h\,a_n(\lambda.) \quad \forall n \in \mathbb{N}.$$

(b) *If $(e_j)_{j\in J}$ is a Hilbertian family with a constant $h$ and $(\frac{Te_j}{\|Te_j\|})_{j\in J_+}$ is a Besselian family in $Y$ with a constant $b$, then*

$$a_n(T) \ge b^{-1}\,h^{-1}\,a_n(\lambda.) \quad \forall n \in \mathbb{N}.$$

*Proof.* Put $y_j := \frac{Te_j}{\|Te_j\|}$, $\forall j \in J_+$ and $\lambda^+_. := (\lambda_j)_{j\in J_+}$.

(a) Fix a non-empty finite $\Delta \subset J_+$. Consider the operator $T_\Delta \in L(X,Y)$ defined by the equality:

$$T_\Delta\, x = \sum_{j\in\Delta} \lambda_j e_j^*(x) y_j, \quad x \in X.$$

Since $(y_j)_{j\in J_+}$ is a Hilbertian family with a constant $h$ and $(e_j)_{j\in J_+}$ is a Besselian family with a constant $b$, we can write:

$$\begin{aligned}\|T - T_\Delta\| &= \sup_{\|x\|\le 1} \left\| \sum_{j\in J_+\setminus\Delta} \lambda_j e_j^*(x) y_j \right\| \le h \sup_{\|x\|\le 1} \left( \sum_{j\in J_+\setminus\Delta} |\lambda_j e_j^*(x)|^2 \right)^{\frac{1}{2}} \\ &\le h \sup_{j\in J_+\setminus\Delta} \lambda_j \sup_{\|x\|\le 1} \left( \sum_{j\in J} |e_j^*(x)|^2 \right)^{\frac{1}{2}} \le h\,b \sup_{j\in J_+\setminus\Delta} \lambda_j.\end{aligned}$$

Fix now $n \in \mathbb{N}$ and take a $\Delta \subset J_+$ with $\operatorname{card}\Delta < n$. Then since $\operatorname{rank} T_\Delta < n$, the definition of $a_n(T)$ and the above inequality imply:

$$a_n(T) \le \|T - T_\Delta\| \le h\,b \sup_{j\in J_+\setminus\Delta} \lambda_j.$$

Hence,

$$a_n(T) \le hb \inf_{\Delta \subset J_+,\, \mathrm{card}\,\Delta < n} \sup_{j \in J_+ \setminus \Delta} \lambda_j = h\, b\, a_n(\lambda_\cdot^+) = h\, b\, a_n(\lambda_\cdot)$$

and (a) is proved.

(b) Fix any $\Delta \subset J_+$ with $\mathrm{card}\,\Delta = n$ and denote $E$ the vector subspace of $X$ generated by $\{e_j : j \in \Delta\}$. Take now any $T_0 \in L(X,Y)$ with $\mathrm{rank}\,T_0 < n$. Since $\dim(E) = n$, the restriction of $T_0$ to $E$ cannot be injective, hence $\ker T_0 \cap E \ne \{0\}$. Then we can find $x_0 \in \ker T_0$ such that $\|x_0\| = 1$ and $x_0 = \sum_{j\in\Delta} t_j e_j$ for some scalars $t_1, \dots, t_n$. Since $(y_j)_{j\in J_+}$ is a Besselian family with the constant $b$ and $(e_j)_{j\in J}$ is a Hilbertian family with the constant $h$, we can write:

$$\begin{aligned}
\|T - T_0\| \ge \|Tx_0 - T_0 x_0\| = \|Tx_0\| = \Big\| \sum_{j\in\Delta} t_j T e_j \Big\| = \Big\| \sum_{j\in\Delta} t_j \lambda_j y_j \Big\| \\
\ge b^{-1} \Big( \sum_{j\in\Delta} |t_j \lambda_j|^2 \Big)^{\frac{1}{2}} \ge b^{-1} \min_{j\in\Delta} \lambda_j \big( \sum_{j\in\Delta} |t_j|^2 \big)^{\frac{1}{2}} \\
\ge b^{-1} h^{-1} \min_{j\in\Delta} \lambda_j \|x_0\| = b^{-1} h^{-1} \min_{j\in\Delta} \lambda_j .
\end{aligned}$$

This, as $\Delta \subset J_+$ is arbitrary, gives: $\|T - T_0\| \ge b^{-1} \tilde{a}_n(\lambda_\cdot^+)$ and, by Lemma 2.2(a,a′), we get: $a_n(T) \ge b^{-1} h^{-1} a_n(\lambda_\cdot)$. □

**Corollary 2.21.** *If $H, Y$ are Hilbert spaces and $T \in L(H,Y)$ is an ortho-diagonalizable operator with the diagonal $(\lambda_j)_{j\in J}$, then*

$$a_n(T) = a_n(\lambda_\cdot) \quad \forall n \in \mathbb{N}, \tag{2.5}$$

*and*

$$d(T, K(H,Y)) = \text{u-}\limsup_{j\in J} |\lambda_j|. \tag{2.6}$$

*Proof.* Since any orthonormal family is both Hilbertian and Besselian with constant 1, from (a) and (b) we get (2.5). Then by Corollary 2.8(a) and Lemma 2.2(b) we can write

$$d(T, K(H,Y)) = \lim_n a_n(T) = \lim_n a_n(\lambda_\cdot) = \text{u-}\limsup_{j\in J} |\lambda_j|,$$

i.e., (2.6) holds. □

*Remark* 2.22. (1) Equality (2.5) remains valid also for diagonal operators acting in $l_\infty$ (see [29, p. 510, Lemma 27.10.3] from where the method of the above given proof is taken partially). The same equality is proved in [52, (11.11.3)] for a diagonal operator $T : \ell_p \to \ell_p$ with *decreasing* nonnegative diagonal.

(2) Let $1 < q < 2$ and $T : l_2 \to l_q$ be a diagonal operator with respect to natural bases with the diagonal $(\lambda_n)$ such that $(|\lambda_n|)$ is a decreasing sequence. Then since the natural basis of $l_q$ is Besselian with constant 1, from Lemma 2.20(a) we get $a_n(T) \ge |\lambda_n|$, $n = 1, 2, \dots$. In this case it is known that $a_n(T) = (\sum_{k=n}^{\infty} |\lambda_k|^r)^{\frac{1}{r}}$, $n = 1, 2, \dots$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$ [52, (11.11.4)].

## 3. Inequalities for $\delta$ and $l$-norm

**3.1. General inequalities.** In this subsection $E$ will denote a non-zero finite-dimensional Hilbert space with $\dim(E) = n$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $Y$ will stand for a general normed space over $\mathbb{K}$ and $T \in L(E, Y)$ will be a fixed operator.

It is well-known that on the Borel $\sigma$-algebra of the unit sphere $S_E$ of $E$ there exists a unique isometrically invariant probability measure $s_E$ called a uniform distribution on $S_E$, which we simply denote by $s$. This measure is used for the definition of *MD-number* $\delta(T)$ in the introduction (see equality (1.1)).

Clearly $E$ carries the Lebesgue measure $\lambda$ obtained from any realization of $E$ as $\mathbb{K}^n$.

In what follows, to simplify the notation, we introduce a parameter $\kappa$, which equals $1/2$ when the considered spaces are real, and equals 1 in the complex case.

Recall now that the standard Gaussian measure $\gamma_E$ on $E$ is a measure such that

$$\gamma_E(B) = \left(\frac{\kappa}{\pi}\right)^{\kappa n} \int\limits_B \exp(-\kappa\|x\|_E^2) d\lambda(x)$$

for any Borel subset $B$ of $E$.

Observe that $\gamma_E$ is also isometrically invariant Borel probability measure on $E$, which is also uniquely defined through its Fourier transform

$$\hat{\gamma_E}(h) := \int\limits_E \exp\{i \operatorname{Re}(x|h)\}\, d\,\gamma_E(x) = \exp\{-\kappa\|h\|_E^2\} \;\; \forall\, h \in E. \tag{3.1}$$

Let

$$a_p(\gamma_E) := \left(\int\limits_E \|x\|^p \, d\,\gamma_E(x)\right)^{\frac{1}{p}}, \;\; 0 < p < \infty. \tag{3.2}$$

The exact value of $a_p(\gamma_E)$ will not be important in the sequel. It is easy to see that

$$a_p(\gamma_E) \le a_2(\gamma_E) = \sqrt{\dim(E)} \quad \forall p \in ]0, 2] \tag{3.3}$$

while,

$$a_1(\gamma_{\mathbb{R}}) = \sqrt{\frac{2}{\pi}}, \;\; a_1(\gamma_{\mathbb{C}}) = \frac{\sqrt{\pi}}{2}. \tag{3.4}$$

The following statement relates $\gamma_E$ with $s_E$ and is not difficult to check (using the uniqueness of $s_E$): if $f : E \to \mathbb{C}$ is a positively homogeneous measurabe function, then

$$\left(\int\limits_E |f(x)|^p \, d\,\gamma_E(x)\right)^{\frac{1}{p}} = a_p(\gamma_E) \left(\int\limits_{S_E} |f(x)|^p \, d\,s_E(x)\right)^{\frac{1}{p}}, \;\; 0 < p < \infty. \tag{3.5}$$

The measure $\gamma_E$ is used for the definition of $l(T)$ in the Introduction (see equality (1.3)).

We shall use frequently formula (1.2)(see Introduction) which follows at once from equality in (3.3) and (3.5) applied for $p = 2$ to $x \to f(x) := \|Tx\|$.

Evidently, we have

$$m(T) \leq \delta(T) \leq \|T\|, \quad \forall T \in L(E, Y)$$

where $m(T)$ is the lower bound of $T$.

To get some other more useful observations, some facts about Gaussian measures and Gaussian random variables are necessary.

Fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence $(g_k)_{k\in\mathbb{N}}$ of independent standard $\mathbb{K}$-valued Gaussian random variables on this space.[11] We denote by $\mathbb{E}$ the integral with respect to probability measure $\mathbb{P}$.

**Theorem 3.1** ([37, p. 1925, Corollary 5])**.** *Let $Y$ be a normed space over $\mathbb{K}$, $(g_k)_{k\in\mathbb{N}}$ be a sequence of independent standard $\mathbb{K}$-valued random variables, $n \in \mathbb{N}$, $y_1, y_2, \ldots, y_n \in Y$ and $0 < r < p < \infty$. Then we have*

$$\left(\mathbb{E}\|\sum_{k=1}^{n} g_k y_k\|^p\right)^{\frac{1}{p}} \leq K_{p,r} \left(\mathbb{E}\|\sum_{k=1}^{n} g_k y_k\|^r\right)^{\frac{1}{r}}$$

*with the constant*

$$K_{p,r} = \frac{a_p(\gamma_{\mathbb{R}})}{a_r(\gamma_{\mathbb{R}})},$$

*which is the best possible in the real case. In particular,*

$$\left(\mathbb{E}\|\sum_{k=1}^{n} g_k y_k\|^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{\pi}{2}}\mathbb{E}\|\sum_{k=1}^{n} g_k y_k\|.$$

This theorem with some universal constant $C_{p,r}$ depending only on $p$ and $r$, was well-known earlier[12]. We also note that in [37] this statement is proved in the real case; the complex case (with the same $K_{p,r}$ as in the real case) follows easily from this.

**Lemma 3.2.** *Let $E$ be a finite-dimensional Hilbert space with* $\dim E = n \geq 1$*, $Y$ be a normed space, $T \in L(E, Y)$ and $e_k$, $k = 1, \ldots, n$ be any fixed orthonormal basis of $E$. Then $\gamma_E$ coincides with the distribution*[13] *of the mapping $\omega \to \sum_{k=1}^{n} g_k(\omega)e_k$ in $E$ and*

$$l(T) = \left(\mathbb{E}\|\sum_{k=1}^{n} g_k T e_k\|^2\right)^{\frac{1}{2}}. \tag{3.6}$$

[11] This means that the random variables $g_k : \Omega \to \mathbb{K}$, $k = 1, 2, \ldots$, are stochastically independent and $g_k$ has the distribution $\gamma_{\mathbb{K}}$ for any $k \in \mathbb{N}$. E.g., it is possible to take $\Omega = \mathbb{K}^{\mathbb{N}}$, $\mathbb{P} = \prod_{k\in\mathbb{N}} \mu_k$, where $\mu_k = \gamma_{\mathbb{K}}$, $k = 1, 2, \ldots$, and $(g_k)$ is the sequence of coordinate functionals.

[12] See, e.g., [62, Theorem 5.5.3(d)]; the proof of [55, Corollary 4.9, p.48] gives the estimation $C_{2,1} \leq \pi\sqrt{\frac{\pi}{2}}$.

[13] i.e., with the image of $\mathbb{P}$.

*Moreover, we have*

$$\int_E \|Tx\| d\gamma_E(x) \le l(T) \le \sqrt{\frac{\pi}{2}} \int_E \|Tx\| d\gamma_E(x) \tag{3.7}$$

*and*

$$\int_{S_E} \|Tx\| ds_E(x) \le \delta(T) \le \sqrt{\frac{\pi}{2}} \int_{S_E} \|Tx\| ds_E(x). \tag{3.8}$$

*Proof.* Denote by $\mu$ the image of $\mathbb{P}$ under the mapping $\omega \to \sum_{k=1}^n g_k(\omega)e_k$. Then $\mu$ is a probability measure on $E$. It is easy to observe (using the independence of $g_k$, $k = 1, \dots, n$ and equality (3.1)) that $\hat{\mu} = \hat{\gamma}_E$. Hence $\mu = \gamma_E$ (in view of the uniqueness theorem for the Fourier transform).

Equality (3.6) now follows from the definition of $l$ and from the change of variable formula.

The left inequalities in (3.7) and (3.8) are evident. The right inequality in (3.7) follows from (3.6) and Theorem 3.1. Applying then (1.2), (3.7), (3.5)(for $p = 1$) and (3.3), we get the relation

$$\sqrt{n}\,\delta(T) = l(T) \le \sqrt{\frac{\pi}{2}} \int_E \|Tx\| d\gamma_E(x)$$

$$= \sqrt{\frac{\pi}{2}}\, a_1(\gamma_E) \int_{S_E} \|Tx\| ds_E(x) \le \sqrt{\frac{\pi}{2}}\,\sqrt{n} \int_{S_E} \|Tx\| ds_E(x),$$

which gives the right inequality in (3.8). □

Relation (3.8) shows that for a given operator $T$ its MD-number $\delta(T)$ and its “Levy’s mean number” $\int_{S_E} \|Tx\| ds_E(x)$ are equivalent quantities (the latter quantity is used, e.g., in [45]).

**Corollary 3.3.** *We have*

$$\|Te\| \le l(T), \quad \frac{1}{\sqrt{\dim E}} \|Te\| \le \delta(T) \quad \forall e \in S_E \tag{3.9}$$

*Moreover, the functionals $l$ and $\delta$ are norms on $L(E, Y)$ with properties*

$$\|T\| \le l(T) \le \sqrt{\dim E}\, \|T\|, \tag{3.10}$$

$$\frac{1}{\sqrt{\dim E}} \|T\| \le \delta(T) \le \|T\|. \tag{3.11}$$

*Proof.* Let $e_k$, $k = 1, \dots, n$ be any orthonormal basis of $E$. Then via (3.6)

$$\|Te_1\| \le l(T) \le \|T\| \left( \mathbb{E} \left\| \sum_{k=1}^n g_k e_k \right\|_E^2 \right)^{\frac{1}{2}} = \sqrt{n} \|T\|.$$

Since any element of $S_E$ can be taken as $e_1$, this relation implies (3.10). Evidently, (3.10) and (1.2) imply (3.11). It is clear that the functionals $l$ and $\delta$ are seminorms on $L(E, Y)$. From (3.10) and (3.11) we get that they actually are norms equivalent to the usual operator norm. $\square$

**Corollary 3.4.** *Let $M \subset E$ be a fixed non-zero vector subspace. Then:*

$$l(T|_M) \le l(T), \tag{3.12}$$

$$\delta(T|_M) \le \sqrt{\frac{\dim E}{\dim M}}\, \delta(T). \tag{3.13}$$

*Moreover, if $M_1, M_2 \subset E$ are any (i.e., not necessarily algebraically complementary to each other) vector subspaces with $M_1 + M_2 = E$, then we have also*

$$\delta(T) \ge \frac{\delta(T|_{M_2})}{\sqrt{\frac{\dim M_1}{\dim M_2} + 1}}, \tag{3.14}$$

$$\delta(T) \le \sqrt{\frac{\dim M_1}{\dim M_1 + \dim M_2}}\, \|T\| + \delta(T|_{M_2}). \tag{3.15}$$

*Proof.* Let $\dim M = m$. We can suppose that the basis $e_k$, $k = 1, \dots, n$ is chosen in such a way that $e_k$, $k = 1, \dots, m$ is an orthonormal basis of $M$. Now (3.12) follows from (3.6) via the inequality

$$\left(\mathbb{E}\left\|\sum_{k=1}^{m} g_k T e_k\right\|^2\right)^{\frac{1}{2}} \le \left(\mathbb{E}\left\|\sum_{k=1}^{n} g_k T e_k\right\|^2\right)^{\frac{1}{2}},$$

which is in turn an easy consequence of the independence and symmetry of $g_k$, $k = 1, \dots, n$ (see, e.g., [62, Lemma 5.3.4(a), p.281]).

Inequality (3.13) follows from (3.12) and (1.2).

Using (1.2) and (3.12) for $M = M_2$, we can write $\delta(T) \ge \frac{\ell(T|_{M_2})}{\sqrt{\dim E}}$. Since $\dim E \le \dim M_1 + \dim M_2$, from this we get (3.14).

The inequality (3.15) needs a little more work (and linear algebra). Denote by$M'$ the orthogonal complement of $M_2$ into $E$ and $n' = \dim M'$. We can suppose now that the basis $e_k$, $k = 1, \dots, n$ is chosen in such a way that $e_k$, $k = 1, \dots, n'$ is an orthonormal basis of $M'$. Then using the subadditivity of the $\mathbb{L}_2(\Omega, \mathcal{A}, \mathbb{P})$ norm and (3.6) we get

$$l(T) \le \left(\mathbb{E}\left\|\sum_{k=1}^{n'} g_k T e_k\right\|^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\left\|\sum_{k=n'+1}^{n} g_k T e_k\right\|^2\right)^{\frac{1}{2}} = l(T|_{M'}) + l(T|_{M_2}).$$

This and (1.2) give

$$\delta(T) \le \sqrt{\frac{n'}{n}}\delta(T|_{M'}) + \sqrt{\frac{\dim(M_2)}{n}}\delta(T|_{M_2}) \le \sqrt{\frac{n'}{n}}\delta(T|_{M'}) + \delta(T|_{M_2}).$$

Since by (3.11) we have $\delta(T|_{M'}) \leq \|T|_{M'}\| \leq \|T\|$ and $n' \leq \dim(M_1)$ (as $n' + \dim(M_2) = \dim(M) \leq \dim(M_1) + \dim(M_2)$, hence $n' \leq \dim(M_1)$), we get

$$\delta(T) \leq \sqrt{\frac{n'}{n' + \dim M_2}}\|T\| + \delta(T|_{M_2}) \leq \sqrt{\frac{\dim(M_1)}{\dim(M_1) + \dim(M_2)}}\|T\| + \delta(T|_{M_2}),$$

i.e., (3.15) holds. □

The next statement deals with the ideal properties of the norms $l$ and $\delta$.

**Proposition 3.5.** *Let $E$ be a finite-dimensional Hilbert space with $\dim E \geq 1$, $Y$ be a normed space and $T \in L(E, Y)$. Suppose further that $E_1$ is another finite-dimensional Hilbert space with $\dim E_1 \geq 1$, $Y_1$ another normed space, $u \in L(E_1, E)$ and $v \in L(Y, Y_1)$. Then:*

$$l(vT) \leq \|v\| l(T), \quad \delta(vT) \leq \|v\|\delta(T), \tag{3.16}$$

$$l(Tu) \leq \|u\| l(T) \tag{3.17}$$

*and*

$$\delta(Tu) \leq \sqrt{\frac{\dim E}{\dim E_1}}\, \delta(T)\|u\|. \tag{3.18}$$

*Proof.* The inequality $l(vT) \leq \|v\| l(T)$ is evident. The inequality $l(Tu) \leq l(T)\|u\|$ can be shown as follows. Let $n := \dim E$ and $n_1 := \dim E_1$. We have

$$ux = \sum_{k=1}^{n_1} \lambda_k (x|e'_k) e_k, \quad x \in E_1,$$

where $e'_k$, $k = 1, \ldots, n_1$ is an orthonormal basis of $E_1$, $e_k$, $k = 1, \ldots, n_1$ is a finite sequence in $E$ whose non-zero members are orthonormal and $\lambda_k$, $k = 1, \ldots, n_1$ are non-negative numbers such that $\|u\| = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}$. Then

$$Tux = \sum_{k=1}^{n_1} \lambda_k (x|e'_k) T e_k, \quad \forall x \in H_1.$$

Using this equality, (3.6) and the contraction principle (see [62, Lemma 5.4.1(c), p.298]), we get

$$l(Tu) = \left(\mathbb{E}\left\|\sum_{k=1}^{n_1} g_k \lambda_k T e_k\right\|^2\right)^{\frac{1}{2}} \leq (\max_{1\leq k\leq n_1} \lambda_k)\left(\mathbb{E}\left\|\sum_{k=1}^{n_1} g_k T e_k\right\|^2\right)^{\frac{1}{2}} \leq l(T)\|u\|.$$

(3.18) follows from (3.17) and (1.2). □

**3.2. Inequalities involving medians.** Let $\mu$ be a probability measure given on a $\sigma$-algebra $\mathcal{A}$ of subsets of a set $\Omega$ and $f : \Omega \to \mathbb{R}$ be a $\mathcal{A}$-measurable function. There exists at least one $t \in \mathbb{R}$ such that

$$\mu\{\omega \in \Omega : f(\omega) \geq t\} \geq \frac{1}{2} \leq \mu\{\omega \in \Omega : f(\omega) \leq t\}. \tag{3.19}$$

The set $M_{f,\mu}$ of all $t \in \mathbb{R}$ for which (3.19) holds either consists of one single point or is a nondegenerate closed segment. Any member of $M_{f,\mu}$ is called a *median* of $f$ with respect to $\mu$.

**Lemma 3.6.** *Let $(\Omega, \mathcal{A}, \mu)$ be probability space and $f : \Omega \to \mathbb{R}$ be a $\mathcal{A}$-measurable function.*

(a) *If $f \geq 0$ $\mu$-a.e., $f \in \mathbb{L}_2(\Omega, \mathcal{A}, \mu)$, $\|f\|_2 > 0$ and $\alpha := \frac{\|f\|_1}{\|f\|_2}$, then*

$$(\alpha - \sqrt{1-\alpha^2})\|f\|_2 \leq t \leq \sqrt{2}\|f\|_2 \quad \forall t \in M_{f,\mu}. \tag{3.20}$$

*Moreover, if $B \subset \Omega$ is a measurable set such that $\mu(B) = \frac{1}{2}$ and $f := 1_B$, then* $\mathrm{Med}_\mu(f) = [0, 1]$ *and in* (3.20) *we have equalities for some medians.*

(b) *If $\Omega$ is a connected topological space, $\mu$ is a Borel probability measure, which is strictly positive on any non-void open subset of $\Omega$ and $f$ is continuous, then the median of $f$ is unique.*

*Proof.* (a) Fix $t \in M_{f,\mu}$. An application of the (Markov) inequality

$$\mu\{\omega \in \Omega : f(\omega) \geq r\} \leq r^{-2}\|f\|_2, \quad r > 0,$$

for $r = \sqrt{2}\|f\|_2$ gives: $\mu\{\omega \in \Omega : f(\omega) \geq \sqrt{2}\|f\|_2\} \leq \frac{1}{2}$. Hence $t \leq \sqrt{2}\|f\|_2$.

The left estimation in (3.20) is derived from the following remarkable known property of a median:

$$\|f - t\|_1 \leq \|f - x\|_1, \ \forall \, x \in \mathbb{R}.$$

For $x = \|f\|_1$ this inequality gives: $t \geq \|f\|_1 - \|f - \|f\|_1\|_1$. Since

$$\|f - \|f\|_1\|_1 \leq \|f - \|f\|_1\|_2 = \sqrt{\|f\|_2^2 - \|f\|_1^2},$$

we get

$$t \geq \|f\|_1 - \|f - \|f\|_1\|_1 \geq \|f\|_1 - \sqrt{\|f\|_2^2 - \|f\|_1^2} = (\alpha - \sqrt{1-\alpha^2})\|f\|_2.$$

The rest is evident.

(b) Suppose that there are $t_1, t_2 \in M_{f,\mu}$ such that $t_1 < t_2$. Then $\mu\,(f^{-1}]t_1, t_2[) = 0$. Since $t_1, t_2$ are medians of $f$, we can find real numbers $r_1, r_2 \in f(\Omega)$ such that $r_1 \leq t_1$ and $r_2 \geq t_2$. Since $\Omega$ is connected and $f$ is continuous, $f(\Omega)$ is a connected subset of $\mathbb{R}$. Hence $f(\Omega)$ is convex and then $[r_1, r_2] \subset f(\Omega)$. In particular, $[t_1, t_2] \subset f(\Omega)$. This implies that $f^{-1}(]t_1, t_2[)$ is a non-void open set and because of the required property of $\mu$, it must have a strictly positive measure. A contradiction. $\square$

Let $T \in L(E,Y)$. Consider the function $f_T : E \to \mathbb{R}_+$ defined by the equality $f_T(x) = \|Tx\|$, $x \in E$. By Lemma 3.6(b) the function $f_T|_{S_E}$ has a unique median with respect to the uniform distribution $s = s_E$, which we denote by $\mathrm{Med}_s(T)$ and call *the median of* $T$ *with respect to* $s$. Analogously, $f_T$ has a unique median with respect to the standard Gaussian measure $\gamma_E$, which we denote by $\mathrm{Med}_\gamma(T)$ and call it *the median of* $T$ *with respect to* $\gamma$.

**Proposition 3.7.** *Let* $E$ *be a finite-dimensional Hilbert space with* $\dim E = n \geq 1$, $Y$ *be a normed space and* $T \in L(E,Y)$. *Then we have*

$$\frac{1}{3\sqrt{\pi}}\delta(T) \leq \frac{\sqrt{2}-\sqrt{\pi-2}}{\sqrt{\pi}}\,\delta(T) \leq \mathrm{Med}_s(T) \leq \sqrt{2}\,\delta(T) \tag{3.21}$$

*and*

$$\frac{1}{3\sqrt{\pi}}l(T) \leq \frac{\sqrt{2}-\sqrt{\pi-2}}{\sqrt{\pi}}\,l(T) \leq \mathrm{Med}_\gamma(T) \leq \int\limits_E \|Tx\|\,d\,\gamma_E(x) \leq \, l(T). \tag{3.22}$$

*Proof.* Inequalities (3.21) and the left-hand side of (3.21) follow from Lemma 3.6 applied to $f = f_T|_{S_E}$ in the first case and to $f = f_T$ in the second case and taking into account that in either cases, according to Lemma 3.2 we have the inequality $\alpha = \frac{\|f\|_1}{\|f\|_2} \geq \sqrt{\frac{2}{\pi}}$. The inequality

$$\mathrm{Med}_\gamma(T) \leq \int\limits_E \|Tx\|\,d\,\gamma_E(x)$$

is a particular case of a general result of [34], which states that the same is true for any Gaussian measure and any convex functional. □

*Remark* 3.8. (1) This proposition shows that for a given operator $T$ the median $\mathrm{Med}_s(T)$ and the MD-number $\delta(T)$ are equivalent quantities.

(2) Proposition 3.7 for $\mathrm{Med}_s(T)$ coincides with [61, p. 26, Lemma 7.2], where it is proved in a different way for some numerical constant in place of $\frac{1}{3\sqrt{\pi}}$ (of course, without having the exact estimate (3.8)).

The next assertion demonstrates that the evident inequalities $m(T) \leq \delta(T)$ and $\delta(T) \leq \|T\|$ can be inverted. The proof involves in fact the Isoperimetric Inequality.

**Proposition 3.9.** *There exists universal constant* $C$ *such that the following is true:*

*Let* $E$ *be a finite-dimensional Hilbert space with* $\dim E \geq 1$, $Y$ *be a normed space,* $T \in L(E,Y)$ *and* $\tau \in [\|T\|, \infty[$. *Then for each* $\varepsilon \in ]0, \frac{\tau}{2}[$ *there exists a vector subspace* $F \subset E$ *with*

$$\dim(F) \geq C\,\frac{\varepsilon^2 \dim(E)}{|\log\frac{\varepsilon}{\tau}|\tau^2}, \tag{3.23}$$

*such that*

$$\delta(T) \leq \, 3\sqrt{\pi}\,(m(T|_F) + \varepsilon) \tag{3.24}$$

*and*

$$\sqrt{2}(\|T|_F\| - \varepsilon) \leq \delta(T). \tag{3.25}$$

*Proof.* Let $f(x) = \|Tx\|$, $x \in S_E$. Then $f$ is a $\tau$-Lipschitz function on $S_E$ with respect to the Euclidean norm and its median coincides with $\mathrm{Med}_s(T)$. An application of fundamental Proposition 12.3 from [2, p. 284] to $f$ gives the existence of $F$ satisfying (3.23) and having the property

$$\mathrm{Med}_s(T) + \varepsilon \geq \|Tx\| = f(x) \geq \mathrm{Med}_s(T) - \varepsilon, \quad \forall x \in S_F.$$

Hence

$$m(T|_F) := \inf\{\|Tx\| : x \in S_F\} \geq \mathrm{Med}_s(T) - \varepsilon \tag{3.26}$$

and

$$\mathrm{Med}_s(T) + \varepsilon \geq \sup\{\|Tx\| : x \in S_F\} := \|T|_F\|. \tag{3.27}$$

Now, (3.26) and Proposition 3.7 imply (3.24); analogously, (3.27) and Proposition 3.7 imply (3.25). □

**3.3. Inequalities related with type and cotype 2.** In Section 2 we have already used the notions of Rademacher type 2 and Rademacher cotype 2 Banach spaces. In what follows it would be more convenient to deal with the "Gaussian" versions of these notions.

A non-trivial normed space $Y$ is called:

- *of Gaussian type* 2 if there is a constant $c > 0$ such that for each $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in Y$,

$$\left(\mathbb{E}\|\sum_{k=1}^{n} g_k y_k\|^2\right)^{\frac{1}{2}} \leq c\left(\sum_{k=1}^{n} \|y_k\|^2\right)^{\frac{1}{2}}. \tag{3.28}$$

  If $Y$ is of Gaussian type 2, then the best possible constant $c$ for which (3.28) holds is called the *Gaussian type* 2 *constant* of $Y$.
- *of Gaussian cotype* 2 if there is a constant $c > 0$ such that for each $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in Y$,

$$c\left(\mathbb{E}\|\sum_{k=1}^{n} g_k y_k\|^2\right)^{\frac{1}{2}} \geq \left(\sum_{k=1}^{n} \|y_k\|^2\right)^{\frac{1}{2}}. \tag{3.29}$$

  If $Y$ is of Gaussian cotype 2, then the best possible constant $c$ for which (3.28) holds is called the *Gaussian cotype* 2 *constant* of $Y$.

It is known and it is not hard to show that the Gaussian type 2 and Rademacher type 2 are equivalent notions (see, e.g.,[18, Theorem 12.26]). It is also known and it is a highly delicate result that the Gaussian cotype 2 and Rademacher cotype 2 are equivalent notions (see, [18, Corollary 12.28] and the text following it). If $Y$ is an inner-product space, then $Y$ is both of type 2 and of cotype 2 with constant one (this is easy to see). The converse is also true

[33]: if a normed space $Y$ is simultaneously of type 2 and cotype 2, then $Y$ is isomorphic to an inner-product space.

*Remark* 3.10. Let $Y$ be a non-trivial normed space.

(a) $Y$ is of Gaussian type 2, if and only if there is a constant $c > 0$ such that for any finite-dimensional Hilbert space $E$, $n := \dim E \geq 1$, any orthonormal basis $e_1, \ldots, e_n$ and any $T \in L(E, Y)$ we have the inequality

$$l(T) \leq c \left( \sum_{k=1}^{n} \|Te_k\|^2 \right)^{1/2} . \tag{3.30}$$

(b) $Y$ is of Gaussian cotype 2, if and only if there is a constant $c > 0$ such that for any finite-dimensional Hilbert space $E$, $n := \dim(M)$, any orthonormal basis $e_1, \ldots, e_n$ and any $T \in L(E, Y)$ we have the inequality

$$l(T) \geq c^{-1} \left( \sum_{k=1}^{n} \|Te_k\|^2 \right)^{1/2} . \tag{3.31}$$

(c) It is worthwhile to note that without any further supposition about $Y$, the following is true: for any finite-dimensional Hilbert space $E$, $n := \dim E$, any $T \in L(E, Y)$ *there are* orthonormal bases $e'_1, \ldots, e'_n$ and $e''_1, \ldots, e''_n$ of $E$ such that we have the inequality

$$\left( \sum_{k=1}^{n} \|Te'_k\|^2 \right)^{1/2} \leq l(T) \leq \left( \sum_{k=1}^{n} \|Te''_k\|^2 \right)^{1/2} .$$

This fact we shall not use in what follows, its proof in fact is given in [35].

From Remark 3.10(a), (b) we get the following statement.

**Lemma 3.11.** *Let $E$ be a finite-dimensional Hilbert space with $\dim E = n \geq 1$, $Y$ a normed space, $T \in L(E, Y)$ and $(e_1, \ldots, e_n)$ any fixed orthonormal basis of $E$. Then:*

(a) *If $Y$ is of Gaussian cotype 2, then*

$$\delta(T) \geq c^{-1} \min_{k \leq n} \|Te_k\|, \tag{3.32}$$

*where $c$ is the cotype 2 constant of $Y$.*

(b) *If $Y$ is of Gaussian type 2, then*

$$\delta(T) \leq c \max_{k \leq n} \|Te_k\|, \tag{3.33}$$

*where $c$ is the type 2 constant of $Y$.*

*Proof.* (a) Taking into account (1.2), (3.6) and (3.29), we get

$$\begin{aligned} \sqrt{n}\delta(T) = l(T) = \left( \mathbb{E} \left\| \sum_{k=1}^{n} g_k Te_k \right\|^2 \right)^{\frac{1}{2}} &\geq c^{-1} \left( \sum_{k=1}^{n} \|Te_k\|^2 \right)^{1/2} \\ &\geq c^{-1} \sqrt{n} \min_{k \leq n} \|Te_k\|. \end{aligned}$$

(b) follows in a similar way from (1.2), (3.6) and (3.28). □

## 4. General Properties of AMD-Numbers

**4.1. Continuity properties of AMD-numbers.** Let now $H$ be an infinite dimensional Hilbert space and $Y$ be a normed space. In this case for a given operator $T \in L(H,Y)$ we have already defined its upper AMD-number $\overline{\overline{\delta}}(T)$, the lower AMD-number $\underline{\underline{\delta}}(T)$ and the notion of an AMD-regular operator in the Introduction.

It is possible to rewrite these definitions in a more direct form without using upper and lower limits. For this fix an $M \in \mathcal{FD}(H)$ and put

$$\beta_M(T) := \sup\{\delta(T|_N) : N \supset M,\ N \in \mathcal{FD}(H)\}$$

and

$$\alpha_M(T) := \inf\{\delta(T|_N) : N \supset M,\ N \in \mathcal{FD}(H)\}.$$

Using these notations we get:

$$\overline{\overline{\delta}}(T) = \inf_{M \in \mathcal{FD}(H)} \beta_M(T) \tag{4.1}$$

and

$$\underline{\underline{\delta}}(T) = \sup_{M \in \mathcal{FD}(H)} \alpha_M(T). \tag{4.2}$$

If $\overline{\overline{\delta}}(T) = \underline{\underline{\delta}}(T)$, then $T$ is named *AMD-regular*. Clearly, for an AMD-regular $T$ the net $(\delta(T|_M))_{M \in \mathcal{FD}}$ is convergent and $\overline{\delta}(T) := \lim\limits_{M \in \mathcal{FD}} \delta(T|_M)$. Let us denote by $\mathrm{AMDR}(H,Y)$ the subset of all AMD-regular operators from $L(H,Y)$. This is a proper subset of $L(H,Y)$ (provided $H$ and $Y$ are infinite-dimensional Hilbert spaces, see Remark 4.5).

The next assertion collects the continuity properties of asymptotic mean dilatation numbers.

**Proposition 4.1.** *Let $Y$ be a normed space. Then:*

(a) *$T \to \overline{\overline{\delta}}(T)$ is a continuous seminorm on the normed space $L(H,Y)$ with property $\overline{\overline{\delta}}(T) \le \|T\|,\ \forall\, T \in L(H,Y)$.*

*In particular,* $\ker(\overline{\overline{\delta}}) := \{T \in L(H,Y) : \overline{\overline{\delta}}(T) = 0\}$ *is a closed vector subspace of $L(H,Y)$.*

(a′) *If $Y_1$ is a normed space, then*

$$\overline{\overline{\delta}}(vT) \le \|v\| \overline{\overline{\delta}}(T) \quad \forall\, T \in L(H,Y) \quad \forall\, v \in L(Y,Y_1). \tag{4.3}$$

*In particular,* $\ker(\overline{\overline{\delta}})$ *is a left ideal.*

(b) *$\underline{\underline{\delta}}$ has property*

$$|\underline{\underline{\delta}}(T_1) - \underline{\underline{\delta}}(T_2)| \le \|T_1 - T_2\| \quad \forall\, T_1, T_2 \in L(H,Y).$$

*In particular, $\underline{\underline{\delta}}$ is continuous on $L(H,Y)$ and* $\ker(\underline{\underline{\delta}})$ *is a closed subset of $L(H,Y)$ (which is not a vector subspace when $Y = H$, see Remark* 4.5*).*

(c) *$AMDR(H,Y)$ is a closed subset in $L(H,Y)$ ($AMDR(H,H)$ is not a vector subspace of $L(H,H)$, see Remark* 4.5) *and $\overline{\delta}$ is a continuous functional on it.*

*Proof.* (a) follows easily from the fact that for any fixed $M \in \mathcal{FD}$ the functional $\delta$ is a norm on $L(M,Y)$ and from the evident inequality $\delta(T|_M) \le \|T|_M\| \le \|T\|$.

(b) Fix $T_1, T_2 \in L(H,Y)$ and a finite dimensional $M \subset H$. We can write

$$\delta(T_2|_M) \le \delta((T_2 - T_1)|_M) + \delta(T_1|_M) \le \|T_2 - T_1\| + \delta(T_1|_M).$$

This inequality implies at once $\underline{\underline{\delta}}_1(T_2) \le \|T_2 - T_1\| + \underline{\underline{\delta}}_1(T_1)$. Consequently, $\underline{\underline{\delta}}_1(T_2) - \underline{\underline{\delta}}_1(T_1) \le \|T_2 - T_1\|$. Similarly, we have $\underline{\underline{\delta}}_1(T_1) - \underline{\underline{\delta}}_2(T_1) \le \|T_2 - T_1\|$. Hence we have (b).

(c) follows from (a) and (b). □

The next statement provides, in particular, a "sequential" way for computation of AMD-mumbers.

**Proposition 4.2.** *Let $Y$ be a normed space and $T \in L(H,Y)$.*

(a) *We have:*

$$\overline{\overline{\delta}}(T) = \sup_{(E_n)} \limsup_n \delta(T|_{E_n}) \tag{4.4}$$

*and*

$$\underline{\underline{\delta}}(T) = \inf_{(E_n)} \liminf_n \delta(T|_{E_n}), \tag{4.5}$$

*where supremum in* (4.4), *resp., infimum in* (4.5) *is taken with respect to all sequences $(E_n)_{n\in\mathbb{N}}$ of non-trivial finite-dimensional subspaces of $H$ such that $\lim_n \dim E_n = \infty$.*

(b) *If $T$ is AMD-regular and $(E_n)_{n\in\mathbb{N}}$ an arbitrary sequence of non-trivial finite-dimensional subspaces of $H$ such that $\lim_n \dim E_n = \infty$, then $\lim_n \delta(T|_{E_n})$ exists and $\overline{\delta}(T) = \lim_n \delta(T|_{E_n})$.*

*Proof.* Let $(E_n)_{n\in\mathbb{N}}$ be any sequence of non-trivial finite-dimensional subspaces of $H$ such that $\lim_n \dim E_n = \infty$.

(a) (1) Let us verify that

$$\overline{\overline{\delta}}(T) \ge \limsup_n \delta(T|_{E_n}). \tag{4.6}$$

Let $M_0$ be any finite-dimensional subspace of $H$. Fix a natural $n$ and put $M_n = M_0 + E_n$. We can write using (3.14) (for $E = M_n, M_2 = E_n, M_0 = M_1$)

$$\beta_{M_0}(T) \ge \delta(T|_{M_n}) \ge \frac{\delta(T|_{E_n}}{\sqrt{\frac{\dim M_0}{\dim E_n} + 1}}. \tag{4.7}$$

Since $n$ is arbitrary, this inequality implies $\beta_{M_0}(T) \ge \limsup_n \delta(T|_{E_n})$. Since $M_0$ is also arbitrary, we get (4.6). The inequality $\overline{\overline{\delta}}(T) \le \sup_{(E_n)} \limsup_n \delta(T|_{E_n})$ is a direct consequence of the definition of $\overline{\overline{\delta}}(T)$.

(2) Let us verify now that

$$\underline{\underline{\delta}}(T) \leq \liminf_n \delta(T|_{E_n}). \tag{4.8}$$

Let $M_0$ be any finite-dimensional subspace of $H$. Fix a natural $n$ and put $M_n = M_0 + E_n$. We can write using (3.15):

$$\begin{aligned} \alpha_{M_0}(T) \leq \delta(T|_{M_n}) &\leq \sqrt{\frac{\dim M_0}{\dim M_0 + \dim E_n}} \|T|_{M_n}\| + \delta(T|_{E_n}) \\ &\leq \sqrt{\frac{\dim M_0}{\dim M_0 + \dim E_n}} \|T|\| + \delta(T|_{E_n}). \end{aligned}$$

Since $n$ is arbitrary, from this inequality we obtain $\alpha_{M_0}(T) \leq \liminf_n \delta(T|_{E_n})$. Since $M_0$ is also arbitrary, we get (4.8). The inequality

$$\underline{\underline{\delta}}(T) \geq \inf_{(E_n)} \liminf_n \delta(T|_{E_n})$$

is a direct consequence of the definition of $\underline{\underline{\delta}}(T)$.

(b) follows from (a). □

*Remark* 4.3. Let $H$ be a separable Hilbert space, $(e_n)$ be an orthonormal basis of $H$ and for a fixed $n$, $E_n$ be the vector subspace generated by $\{e_1, \dots, e_n\}$. One can expect that in this case for any $T \in L(H)$ the equality $\overline{\overline{\delta}}(T) = \limsup_n \delta(T|_{E_n})$ should be true. But this is not so. Using Proposition 4.7(a) it is easy to see that for the positive diagonal operator $T$ from Remark 4.5(1) we have

$$\overline{\overline{\delta}}(T) = 1 > \frac{1}{\sqrt{2}} = \limsup_n \delta(T|_{E_n}).$$

It is not clear whether there exists a "universal" sequence $(E_n)$ which could be used for the exact computation of $\overline{\overline{\delta}}(T)$ for all possible operators $T$.

**4.2. Orthonormal sequences and AMD-number.** In this subsection we show that in certain cases AMD-numbers control and are controlled by the behaviour of an operator on orthonormal sequences.

**Proposition 4.4.** *Let $Y$ be a normed space, $T \in L(H, Y)$ and $(e_n)$ any infinite orthonormal sequence in $H$. The following assertions are valid:*

(a) *If $Y$ is of cotype 2, then*

$$\overline{\overline{\delta}}(T) \geq \frac{1}{c} \limsup_n \|Te_n\|,$$

*where $c$ is the cotype-2 constant of $Y$.*

(b) *If $Y$ is of type 2, then*

$$\underline{\underline{\delta}}(T) \leq c \liminf_n \|Te_n\|,$$

*where $c$ is the type-2 constant of $Y$.*

(c) *If $Y$ is an inner-product space and $T$ is AMD-regular, then $\lim_n \|Te_n\|$ exists and $\overline{\delta}(T) = \lim_n \|Te_n\|$.*

*Proof.* (a) Put $\limsup_n \|Te_n\| = \lambda$. We can assume $\lambda > 0$. There is a subsequence $(\|Te_{k_n}\|)$ such that

$$\lim_n \|Te_{k_n}\| = \lambda.$$

Fix $\varepsilon, 0 < \varepsilon < \lambda$. Then there is $j_\varepsilon \in \mathbb{N}$ such that

$$\|Te_{k_j}\| > \lambda - \varepsilon \quad \forall j > j_\varepsilon.$$

Fix now $n \in \mathbb{N}$ arbitrarily and consider the vector subspace $E_n$ of $H$ generated by $\{e_{k_{j_\varepsilon+1}}, \dots, e_{k_{j_\varepsilon+n}}\}$. Since $Y$ is of cotype 2, we can apply (3.32) for $E = E_n$ and $T|_{E_n}$ and write

$$\delta(T|_{E_n}) \geq c^{-1} \min_{j_\varepsilon < j \leq n} \|Te_{k_j}\| > c^{-1}(\lambda - \varepsilon).$$

Since $n$ is arbitrary, this inequality together with Proposition 4.2(a) implies $\overline{\overline{\delta}}(T) \geq c^{-1}(\lambda - \varepsilon)$. But $\varepsilon > 0$ is arbitrary. Consequently, $\overline{\overline{\delta}}(T) \geq \frac{1}{c}\lambda$ and (a) is proved.

(b) Put $\liminf_n \|Te_n\| = \lambda$. There is a subsequence $(\|Te_{k_n}\|)$ such that

$$\lim_n \|Te_{k_n}\| = \lambda.$$

Fix $\varepsilon, 0 < \varepsilon$. Then there is $j_\varepsilon \in \mathbb{N}$ such that

$$\|Te_{k_j}\| < \lambda + \varepsilon \quad \forall j > j_\varepsilon.$$

Fix now $n \in \mathbb{N}$ arbitrarily and consider the vector subspace $E_n$ of $H$ generated by $\{e_{k_{j_\varepsilon+1}}, \dots, e_{k_{j_\varepsilon+n}}\}$. Since $Y$ is of type 2, we can apply (3.33) for $E = E_n$ and $T|_{E_n}$ and write

$$\delta(T|_{E_n}) \leq c \max_{j_\varepsilon < j \leq n} \|Te_{k_j}\| < c(\lambda + \varepsilon).$$

Since $n$ is arbitrary, this inequality together with Proposition 4.2(a) implies $\underline{\underline{\delta}}(T) \leq c(\lambda + \varepsilon)$. But $\varepsilon > 0$ is arbitrary. Consequently $\underline{\underline{\delta}}(T) \leq c\lambda$ and (b) is proved.

(c) follows from (a) and (b) since any inner-product space is both of type 2 and of cotype 2 with constant 1. □

*Remark* 4.5. Using Proposition 4.4 it is easy to provide several examples.

(1)*An example of an operator $T$ which is not AMD-regular.* Let $H$ be an infinite-dimensional separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$, consider the operator $T : H \to H$ defined: $Te_1 = e_1; Te_2 = 0; Te_3 = e_3; Te_4 = 0 \dots$. Then for such an operator $\overline{\overline{\delta}}(T) = 1$ (by Proposition 4.4(a) and $\underline{\underline{\delta}}(T) = 0$ (by Proposition 4.4(b)). Hence $T$ is not AMD-regular.

(2) AMDR $(H, H)$ *is not a vector space.* Evidently, for any isometric operator $T : H \to H$ we have $\overline{\delta}(T) = 1$. However there are two unitary operators $T_1$ and $T_2$ such that $\overline{\delta}(T_1 + T_2)$ does not exist. To provide an example, take $T_1 = I$ identity operator, and define $T_2$ by the relations $T_2(e_{2n}) = e_{2n}$ and $T_2(e_{2n-1}) = -e_{2n-1}\ \forall n$. Clearly, $T_2$ is a unitary operator but $(T_1 + T_2)(e_{2n}) = 2e_{2n}$ and $(T_1 + T_2)(e_{2n-1}) = 0$ for all $n$. According to Proposition 4.4(a,b), we have $\overline{\overline{\delta}}(T_1 + T_2) = 2$ and $\underline{\underline{\delta}}(T_1 + T_2) = 0$.

(3) $\ker(\underline{\underline{\delta}})$ *is not a vector space.* The same idea as in (2).

The next assertion will allow us to compute exactly AMD-numbers for the diagonalizable operators.

**Proposition 4.6.** *Let $Y$ be a Banach space, $T \in L(H,Y)$ be a non-zero operator and $(e_j)_{j\in J}$ an orthonormal basis in $H$. Put $J_+ := \{j \in J : \|Te_j\| > 0\}$.*

(a) *If $(\frac{Te_j}{\|Te_j\|})_{j\in J_+}$ is a Hilbertian family in $Y$ with a constant $c$, then*

$$\overline{\overline{\delta}}(T) \le c \ u\text{-}\limsup_{j\in J} \|Te_j\|.$$

(b) *If $(\frac{Te_j}{\|Te_j\|})_{j\in J_+}$ is a Besselian family in $Y$ with a constant $c$ , then*

$$\underline{\underline{\delta}}(T) \ge c^{-1} \ u\text{-}\liminf_{j\in J} \|Te_j\|.$$

*Proof.* (a) Since $\text{u-}\limsup_{j\in J} \|Te_j\| = \text{u-}\limsup_{j\in J_+} \|Te_j\|$, we can suppose that $J = J_+$. Fix a finite $\Delta \subset J$ and consider the vector subspace $M_\Delta$ generated by $\{e_j : j \in \Delta\}$.

Since $(\frac{Te_j}{\|Te_j\|})_{j\in J}$ is a Hilbertian family in $Y$ with the constant $c$, we have

$$\|Th\| \le c \sup_{j\notin\Delta} \|Te_j\|, \ \forall h \in S_H, \ h \text{ orthogonal to } M_\Delta. \tag{4.9}$$

Take now any finite-dimensional $M \supset M_\Delta$. We have $M = M_\Delta + M_2$, where $M_2$ is the orthogonal complement of $M_\Delta$ in $M$; then applying (3.15) and (4.9) to $T|_M$, we get

$$\delta(T|_M) \le \sqrt{\frac{\dim M_\Delta}{\dim M}}\|T\| + c \sup_{j\notin\Delta} \|Te_j\|. \tag{4.10}$$

Fix $\varepsilon > 0$, take a finite-dimensional $M_\varepsilon \supset M_\Delta$ in such a way that

$$\sqrt{\frac{\dim M_\Delta}{\dim M_\varepsilon}}\|T\| < \varepsilon$$

From (4.10), according to the choice of $M_\varepsilon$, we obtain:

$$\begin{aligned}\delta(T|_M) &\le \sqrt{\frac{\dim M_\Delta}{\dim M_\varepsilon}}\|T\| + c \sup_{j\notin\Delta} \|Te_j\| \\ &< \varepsilon + c \sup_{j\notin\Delta} \|Te_j\| \quad \forall M \in \mathcal{FD}(H), \quad M \supset M_\varepsilon.\end{aligned}$$

Consequently, $\beta_{M_\varepsilon}(T) \le \varepsilon + c \sup_{j\notin\Delta} \|Te_j\|$. Therefore

$$\overline{\overline{\delta}}(T) \le \beta_{M_\varepsilon}(T) \le \varepsilon + c \sup_{j\notin\Delta} \|Te_j\|.$$

So $\overline{\overline{\delta}}(T) \leq \varepsilon + c \sup_{j\notin\Delta} \|Te_j\|$. Since the latter inequality holds for all finite subsets $\Delta \subset J$, it implies

$$\overline{\overline{\delta}}(T) \leq \varepsilon + c \inf_{\Delta\subset J,\, \mathrm{card}\,\Delta<\infty} \sup_{j\notin\Delta} \|Te_j\| = \varepsilon + c\, \text{u-}\limsup_{j\in J} \|Te_j\|.$$

Since $\varepsilon$ is arbitrary, we obtain the required inequality.

The proof of (b) is similar. $\square$

**Proposition 4.7.** *Let $Y$ be a Hilbert space and $T \in L(H,Y)$ be an ortho-diagonalizable operator with diagonal $(\lambda_j)_{j\in J}$. Then:*

(a) $\overline{\overline{\delta}}(T) = \text{u-}\limsup_{j\in J} |\lambda_j|$.

(b) $\underline{\underline{\delta}}(T) = \text{u-}\liminf_{j\in J} |\lambda_j|$.

(c) *The operator $T$ is AMD-regular if and only if* $\text{u-}\lim\limits_{j\in J} |\lambda_j|$ *exists and, in case of AMD-regularity, the equality*

$$\overline{\delta}(T) = \text{u-}\lim_{j\in J} |\lambda_j|$$

*holds.*

*Proof.* (a) $T$ is ortho-diagonalizable with diagonal $(\lambda_j)_{j\in J}$ means that there are an orthonormal basis $(e_j)_{j\in J}$ in $H$ and an orthonormal basis $(y_j)_{j\in J}$ in $Y$ such that $Te_j = \lambda_j y_j,\ \forall j \in J$. The inequality $\overline{\overline{\delta}}(T) \geq \limsup_j |\lambda_j|$ follows from Proposition 4.4(a) and Lemma 2.1, since $Y$ is of type 2 with constant one. The inequality $\overline{\overline{\delta}}(T) \leq \limsup_j |\lambda_j|$ follows from Proposition 4.6(a) since any orthonormal family in a Hilbert space is Hilbertian with constant one.

The proof of (b) is similar, (c) follows from (a) and (b). $\square$

*Remark* 4.8. (1) Evidently, if $T : H \to Y$ is a linear isometry, then $\overline{\delta}(T) = 1$. From Proposition 4.7 we get that the converse is not true. Indeed, let $H$ be separable and $\lambda_n = 1 - \frac{1}{n+1}$, $n = 1, 2, \ldots$, then we have $\|T\| = 1, \overline{\delta}(T) = 1$, but $T$ is not an isometry, even more, if $x \in H$ and $x \neq 0$, then $\|T(x)\| < \|x\|$.

(2) Proposition 4.6(a,b) can give some information also in the case where $Y$ is not necessarily a Hilbert space, e.g., if $Y = l_p,\ 2 < p < \infty$ and $T : l_2 \to l_p$ is a diagonal operator with the diagonal $(\lambda_n)$ with respect to natural bases, and since the natural basis of $l_p$ is Hilbertian, we get $\overline{\overline{\delta}}(T) \leq \limsup_j |\lambda_j|$. This estimate is slightly better than the trivial one: $\overline{\overline{\delta}}(T) \leq \|T\|$.

**4.3. AMD-numbers and compactness-like properties of operators.** In this subsection we show that the AMD-numbers can be used as tools for the study of compactness-like properties of the operators acting from a Hilbert space. We begin with a proposition.

**Proposition 4.9.** *Let $Y$ be a Banach space.*

(a) *Always $K(H,Y) \subset \{T \in L(H,Y) : \overline{\overline{\delta}}(T) = 0\}$.*[14]

[14] This follows also from more delicate Theorem 4.11(b).

(b) *If $Y$ is of cotype 2, then* $K(H,Y) = \{T \in L(H,Y) : \overline{\overline{\delta}}(T) = 0\}$ (*the inclusion* $\{T \in L(H,Y) : \overline{\overline{\delta}}(T) = 0\} \subset K(H,Y)$ *is not true in general, see Remark* 4.13).

*Proof.* (a) Let $T \in K(H,Y)$. It is clear that if $T$ is a finite-rank operator, then $\overline{\overline{\delta}}(T) = 0$. If $T$ is arbitrary, then (by Corollary 2.7) $T$ is the norm limit of a sequence of finite-rank operators. This and the continuity of $\overline{\overline{\delta}}$ imply $\overline{\overline{\delta}}(T) = 0$.

(b) Let $T \in L(H,Y)$ and $\overline{\overline{\delta}}(T) = 0$. Since $Y$ is of cotype 2, by Proposition 4.4(a), we have that $\lim_n \|Te_n\| = 0$ for any infinite orthonormal sequence $(e_n)$ in $H$. Hence, by Corollary 2.7, $T \in K(H,Y)$. $\square$

The next quantitative version of this proposition will be used in the last section.

**Theorem 4.10.** *Let $T \in L(H,Y)$. The following statements are valid:*

(a) $\overline{\overline{\delta}}(T) \le d(T, K(H,Y))$.

(b) *If $Y$ is of cotype 2, then*

$$\overline{\overline{\delta}}(T) \ge \frac{1}{c} d(T, K(H,Y)),$$

*where $c$ is the cotype 2 constant of $Y$.*

(c) *If $Y$ is a Hilbert space, then* $\overline{\overline{\delta}}(T) = d(T, K(H,Y))$.

*Proof.* (a) Fix any $T_0$ in $K(H,Y)$. By Proposition 4.9(a) $\overline{\overline{\delta}}(T_0) = 0$. Hence

$$\overline{\overline{\delta}}(T) = \overline{\overline{\delta}}(T - T_0 + T_0) \le \overline{\overline{\delta}}(T - T_0) + \overline{\overline{\delta}}(T_0) = \overline{\overline{\delta}}(T - T_0) \le \|T - T_0\|.$$

Since $T_0$ is arbitrary, this inequality implies (a).

(b) According to Proposition 4.4(a) and Corollary 2.8(c) we have

$$c\overline{\overline{\delta}}(T) \ge \sup_{(e_n) \in \mathcal{O}(H)} \limsup_n \|Te_n\| \ge d(T, K(H,Y)).$$

(c) follows from (a) and (b). $\square$

The next theorem implies, in particular, that the condition $\overline{\overline{\delta}}(T) = 0$ is satisfied if and only if $T$ is a superstrictly singular operator.

**Theorem 4.11.** *Let $Y$ be a Banach space and $T \in L(H,Y)$.*

(a) *We have:*

$$\lim_n b_n(T) \le \overline{\overline{\delta}}(T) \le 3\sqrt{\pi} \lim_n b_n(T).$$

(b) $\overline{\overline{\delta}}(T) = 0$ *if and only if $T$ is superstrictly singular.*

(c) *If the restriction of $T$ on some infinite-dimensional closed vector subspace of $H$ is strictly singular, then* $\underline{\underline{\delta}}(T) = 0$.

*Proof.* (a) Let us show first that $\overline{\overline{\delta}}(T) \geq \lim_n b_n(T)$. Assume that $\lim_n b_n(T) > r > 0$. Fix a natural $n$. Then there is $E_n \in \mathcal{FD}(X)$, $\dim E_n = n$ such that $m(T|_{E_n}) > r$. Take now any non-zero finite-dimensional $M_0 \subset H$. An application of inequality (3.14) to $M := M_0 + E_n$, $M_1 = M_0$ and $M_2 = E_n$ gives

$$\beta_{M_0}(T) \geq \delta(T|_{M_0+E_n}) \geq \frac{\delta(T|_{E_n})}{\sqrt{\frac{\dim M_0}{\dim E_n} + 1}} \geq \frac{m(T|_{E_n})}{\sqrt{\frac{\dim M_0}{n} + 1}} > \frac{r}{\sqrt{\frac{\dim M_0}{n} + 1}}.$$

Since $n$ is arbitrary, this inequality implies $\overline{\overline{\delta}}(T) \geq r$. As $r$ is arbitrary, we get the needed inequality.

Let us show now $\overline{\overline{\delta}}(T) \leq 3\sqrt{\pi} \lim_n b_n(T)$. For this we use Proposition 3.9. We can suppose that $\|T\| = 1$. Fix $\varepsilon \in ]0, 1/2[$ and a sequence $(E_n)$ of finite-dimensional vector subspaces of $H$ such that $\lim_n \dim(E_n) = \infty$. Let also $C$ be the universal constant from Proposition 3.9. Fix then a natural $n$. Then by means of Proposition 3.9, applied to $E = E_n$, $T|_{E_n}$ and $\tau = 1$, we can find and fix a vector subspace $F_n \subset E_n$ with

$$\dim(F_n) \geq C \, \frac{\varepsilon^2 \dim(E_n)}{|\log \varepsilon|} \tag{4.11}$$

such that

$$m(T|_{F_n}) \geq \frac{1}{3\sqrt{\pi}} \delta(T|_{E_n}) - \varepsilon. \tag{4.12}$$

In this way we obtain a sequence $(F_n)$ of finite-dimensional subspaces of $H$, which depends on $T$ and on $\varepsilon$. Put $k_n := \dim(F_n)$, $n = 1, 2, \ldots$ Clearly, (4.12) implies

$$b_{k_n}(T) \geq \frac{1}{3\sqrt{\pi}} \, \delta(T|_{E_n}) - \varepsilon, \quad n = 1, 2, \ldots. \tag{4.13}$$

Since $\lim_n \dim(E_n) = \infty$, from (4.11) we conclude that $\lim k_n = \infty$. This and (4.13) imply

$$\lim_n b_n(T) = \lim_n b_{k_n}(T) \geq \frac{1}{3\sqrt{\pi}} \; \limsup_n \delta(T|_{E_n}) - \varepsilon, \quad n = 1, 2, \ldots. \tag{4.14}$$

From (4.14), since $(E_n)$ is arbitrary, according to Proposition 4.2, we obtain $\lim_n b_n(T) \geq \frac{1}{3\sqrt{\pi}} \overline{\overline{\delta}}(T) - \varepsilon$. Since $\varepsilon$ is also arbitrary, we get $\overline{\overline{\delta}}(T) \leq 3\sqrt{\pi} \, \lim_n b_n(T)$.

(b) follows from (a).

(c) Fix a finite-dimensional vector subspace $M_0 \subset H$ and a number $\varepsilon > 0$.

Let $X \subset H$ be a closed vector subspace such that $T|_X : X \to Y$ is strictly singular. Then, by [52, (1.9.1)], we can find and fix an infinite-dimensional closed vector subspace $H_0 \subset X$ such that $\|T|_{H_0}\| < \varepsilon$. Take now an infinite sequence $(M_n)$ of finite-dimensional subspaces of $H_0$ such that $\dim M_n = n$, $n = 1, 2, \ldots$. According to (3.15), for any fixed $n$ we can write

$$\delta(T|_{M_0+M_n}) \leq \sqrt{\frac{\dim M_0}{\dim M_0 + \dim M_n}}\, \|T|_{M_0}\| + \delta(T|_{M_0})$$
$$\leq \sqrt{\frac{\dim M_0}{\dim M_0 + n}}\, \|T\| + \|T|_{M_n}\| < \sqrt{\frac{\dim M_0}{\dim M_0 + n}}\, \|T\| + \varepsilon.$$

From this we get

$$\alpha_{M_0}(T) \leq \delta(T|_{M_0+M_n}) < \sqrt{\frac{\dim M_0}{\dim M_0 + n}} \|T\| + \varepsilon, \quad n = 1, 2, \ldots .$$

Hence $\alpha_{M_0}(T) \leq \varepsilon$ and, since $M_0$ is arbitrary, $\underline{\underline{\delta}}(T) \leq \varepsilon$. As $\varepsilon$ is also arbitrary, we get $\underline{\underline{\delta}}(T) = 0$. □

The next "AMD-free" statement of independent interest is not a consequence of Theorem 2.18 because it does not require the existence of an unconditional basis.

**Theorem 4.12.** *Let $H$ be a Hilbert space and $Y$ be any Banach space of cotype 2. Then $SSS(H, Y) = K(H, Y)$.*

*Proof.* This follows from Theorem 4.11(b) and Proposition 4.9. □

*Remark* 4.13. (1) Let $2 < p < \infty$ and let $T_{2,p} : \ell_2 \to \ell_p$ be the natural embedding. Then by Proposition 2.10(d) we have that $T_{2,p}$ is a SSS-operator and therefore, by Theorem 4.11, $\overline{\overline{\delta}}(T_{2,p}) = 0$[15]. Since $T_{2,p}$ is not compact, it turns out that Proposition 4.9, as well as Theorem 4.12, may not be true when $Y$ is not of cotype 2. *However, in general, for a Banach space $Y$ the validity even of the inclusion $SS(\ell_2, Y) \subset K(\ell_2, Y)$ may not imply that $Y$ is of cotype* 2 (see Remark 2.19(5)).

(2) Let us say for a moment that a Banach space $Y$ is a *SSSC-space* if $SSS(\ell_2, Y) \subset K(\ell_2, Y)$. By Theorem 4.12 any cotype 2 Banach space is a SSSC-space. Evidently, any Banach space with the Schur property is a SSSC-space too. It would be interesting to find an internal characterization of SSSC-spaces.

## 5. AMD-Numbers and Bounds of the Essential Spectrum

**5.1. Spectrum and spectral radius.** Let $\mathfrak{A}$ be a Banach algebra over $\mathbb{K}$ with unit $e$. The spectrum $\sigma(a)$ of an element $a \in \mathfrak{A}$ is defined as the set of all $\lambda \in \mathbb{K}$ such that the element $a - \lambda e$ is not invertible in $\mathfrak{A}$. For an arbitrary element $a$, $\sigma(a)$ is a compact subset of $\mathbb{K}$ which is non-empty when $\mathbb{K} = \mathbb{C}$ (by Gelfand's theorem), but which may be empty when $\mathbb{K} = \mathbb{R}$.

[15] The equality $\overline{\overline{\delta}}(T_{2,p}) = 0$ can also be proved directly, namely it can be shown that for *any* vector subspace $E \subset \ell_2$ with $\dim E = n$ one has $\delta(T_{2,p}|_E) \leq a_p(\gamma_{\mathbb{K}})(\dim E)^{1/p-1/2}$.

For an element $a \in \mathfrak{A}$ with non-empty spectrum, its *spectral radius* $r(a)$ is defined by the equality

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Notice that $r(a) \leq \|a\|$ for any $a \in \mathfrak{A}$. Clearly, if for a given $a \in \mathfrak{A}$ we have $\sigma(a) \neq \varnothing$ and $\sigma(a) \subset \mathbb{R}$, then $\sigma(a) \subset [\min(\sigma(a)), \max(\sigma(a))]$.

The following assertions are easy to prove:

(Spec1) If $a \in \mathfrak{A}$ is such that $\sigma(a) \neq \varnothing$ and $\lambda \in \mathbb{K}$, then $\sigma(a - \lambda e) = \sigma(a) - \lambda(:= \{\lambda' - \lambda : \lambda' \in \sigma(a)\})$.

(Spec2) If $a \in \mathfrak{A}$ is such that $\sigma(a) \neq \varnothing$ and $a$ is invertible, then $0 \notin \sigma(a)$ and $\sigma(a^{-1}) = (\sigma(a))^{-1}(:= \{\lambda^{-1} : \lambda \in \sigma(a)\})$.

(Spec3) If $a \in \mathfrak{A}$ is such that $\sigma(a) \neq \varnothing$ and $\lambda \notin \sigma(a)$, then $\sigma((a-\lambda e)^{-1}) \neq \varnothing$ and hence $r((a - \lambda e)^{-1})$ is defined.

Taking into account (Spec3) let us say that an element $a \in \mathfrak{A}$ is *normal-like* if $\sigma(a) \neq \varnothing$ and $r((a - \lambda e)^{-1}) = \|(a - \lambda e)^{-1}\| \ \forall \lambda \in \mathbb{K} \setminus \sigma(a)$.

**Proposition 5.1.** *Let $\mathfrak{A}$ be a real or complex Banach algebra with unit, $(a_n)$ be a sequence of elements of $\mathfrak{A}$ such that $\sigma(a_n) \neq \varnothing \ \forall\, n$, $a \in \mathfrak{A}$ and $a_n \to a$ in $\mathfrak{A}$. Then:*

(a) *If $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of scalars such that $\lambda_n \in \sigma(a_n)$ for all $n$, then $(\lambda_n)$ is a bounded sequence and its cluster points belong to $\sigma(a)$. In particular, $\sigma(a) \neq \varnothing$.*

(b) $\limsup_n r(a_n) \leq r(a)$. *Moreover, if $\sigma(a) \subset \mathbb{R}$ and $\sigma(a_n) \subset \mathbb{R}$ for all $n \in \mathbb{N}$, then*

$$\limsup_n (\max(\sigma(a_n))) \leq \max(\sigma(a)),$$

*and*

$$\liminf_n (\min(\sigma(a_n))) \geq \min(\sigma(a)).$$

(c) *If the elements $a_n$, $n = 1, 2, \dots$ are normal-like, then for any $\lambda \in \sigma(a)$ we have $\lim_n d(\lambda, \sigma(a_n)) = 0$.*

(d) *If the assumptions of (c) are satisfied then $r(a) = \lim_n r(a_n)$. Moreover, if $\sigma(a) \subset \mathbb{R}$ and $\sigma(a_n) \subset \mathbb{R}$ for all $n \in \mathbb{N}$, then*

$$\lim_n \max(\sigma(a_n)) = \max(\sigma(a))$$

*and*

$$\lim_n \min \sigma(a_n) = \min \sigma(a).$$

*Proof.* (a) Evidently, $|\lambda_n| \leq \|a_n\| \leq \sup_n \|a_n\| < \infty$ for all $n \in \mathbb{N}$. Hence $(\lambda_n)$ is a bounded sequence. Now it is easy to prove that any cluster point of $(\lambda_n)$ belongs to $\sigma(a)$.

(b) follows easily from (a).

(c) Suppose $\lambda \in \sigma(a)$, but $\lim_n d(\lambda, \sigma(a_n)) \neq 0$. Then there exists a strictly positive number $\varepsilon$ such that $d(\lambda, \sigma(a_n)) \geq \varepsilon$ for infinitely many values of $n$. There is no loss in simplifying the notation and assuming that

$$d(\lambda, \sigma(a_n)) \geq \varepsilon, \quad n = 1, 2, \dots .$$

Hence

$$\lambda \notin \sigma(a_n)), \quad n = 1, 2, \dots .$$

From this, via (Spec2) and according to the definition of the spectral radius, we get

$$r((a_n - \lambda e)^{-1}) \leq \varepsilon^{-1}, \quad n = 1, 2, \dots .$$

The latter relation, together with the supposition that the considered elements are normal-like, gives

$$\|(a_n - \lambda e)^{-1}\| \leq \varepsilon^{-1}, \quad n = 1, 2, \dots . \tag{5.1}$$

Put $b_n = a_n - \lambda e,\ n = 1, 2, \dots$. Using the relation

$$b_n^{-1} - b_m^{-1} = b_m^{-1}(a_m - a_n)b_n^{-1}, \quad , n, m = 1, 2, \dots ,$$

and (5.1), we obtain

$$\|b_n^{-1} - b_m^{-1}\| \leq \|b_m^{-1}\| \cdot \|a_m - a_n\| \cdot \|b_n^{-1}\| \leq \varepsilon^{-2}\|a_m - a_n\|, \quad n, m = 1, 2, \dots ,$$

and since $a_n \to a$, it follows that the squence $(b_n^{-1})$ converges to some element $b$. Then

$$(a - \lambda e)b = \lim_n (a_n - \lambda e) \lim_n b_n^{-1} = \lim_n (a_n - \lambda e)(a_n - \lambda e)^{-1} = e$$

and, similarly, $b(a - \lambda e) = e$. This $a - \lambda e$ is invertible and $\lambda \notin \sigma(a)$. A contradiction.

(d) follows in a standard way from (c) and (b). $\square$

*Remark* 5.2. The reader familiar with the notion of convergence of sets in Kuratowski's sense (see [32, p. 335–340]) can observe easily that Proposition 5.1(a) is a direct formulation of the well-known fact: the set-valued function $\sigma$ is upper semicontinuous on $\mathfrak{A}$. Similarly, Proposition 5.1(c) together with Proposition 5.1(a) implies that this function is continuous on the (closed) set $\mathfrak{A}_0$ consisting of normal-like elements of $\mathfrak{A}$. Our proof is taken from [25] (see the solution of problem 105) where the statement is proved for normal operators in a complex Hilbert space. We see that the proof works for normal-like elements in real algebra too.

**5.2. Essential spectrum.** Let $H$ be an infinite-dimensional real or complex Hilbert space, then $L(H)$ is a Banach algebra with unit. It follows that when $H$ is a complex Hilbert space, any $T \in L(H)$ has a non-empty spectrum. It is well-known that when $T$ is any self-adjoint operator in a real or complex Hilbert space, the fact that $\sigma(T)$ is non-empty can be proved easily without any reference to Gelfand's theorem. Also the following facts have direct and easy proofs.

Let $H$ be a Hilbert space over $\mathbb{K}$, $T \in L(H)$ and $T^* = T$. Then:

- $(Tx|x) \in \mathbb{R}$ $\forall x \in H$ and $\sigma(T) \subset \mathbb{R}$.
- $\sigma(|T|) = \{|\lambda| : \lambda \in \sigma(T)\}$ and $r(T) = \|T\|$.
- $\sup\{(Tx|x) : x \in S_H\} = \max \sigma(T)$ and $\inf\{(Tx|x) : x \in S_H\} = \min \sigma(T)$.

The set $K(H)$ is a closed two-sided ideal of $L(H)$. Let $\pi : L(H) \to L(H)/K(H)$ be the canonical map. Then it is known that $L(H)/K(H)$ too is a Banach algebra with unit $\pi(I)$ and with respect to the quotient norm

$$\|\pi(T)\| = d(T, K(H)), \quad T \in L(H),$$

which is called the *Calkin algebra*. For a given $T \in L(H)$ the *essential spectrum* $\sigma_e(T)$ is by definition the spectrum $\sigma(\pi(T))$ of its canonical image $\pi(T)$ in the Calkin algebra ([16, p. 358]).

Notice that for any $T \in L(H)$ we have $\sigma_e(T) \subset \sigma(T)$. If $H$ is a complex Hilbert space, then for any $T \in L(H)$ the essential spectrum is non-empty (this is so because in any complex Banach algebra with unit any element has a non-empty spectrum). In what follows, we need the fact that any self-adjoint operator in a real Hilbert space also has a non-empty essential spectrum. We shall derive this from the corresponding fact for strictly diagonalizable operators via the following lemma which asserts that such operators are sufficiently many.

**Lemma 5.3.** *Let $H$ be a real or complex infinite-dimensional Hilbert space. The following assertions hold:*

(a) *Let $S \in L(H)$ be any self-adjoint operator. Then there is a sequence $D_n : H \to H$, $n \in \mathbb{N}$ of self-adjoint strictly ortho-diagonalizable operators such that $\lim_n \|D_n - S\| = 0$.*

(b) *If $Y$ is another Hilbert space, then the set of all ortho-diagonalizable operators is norm dense in $L(H, Y)$.*

*Proof.* (a) Fix $S$. According to the spectral representation theorem which holds in the case of real Hilbert space too [63, Theorem 7.17, p.191], there exists a resolution of identity $\mathcal{E}$ on Borel subsets of $\Lambda := \sigma(S)$ such that

$$S = \int_\Lambda \lambda \mathcal{E}(d\lambda).$$

Notice that this integral can be computed as follows: fix a sequence $f_n : \Lambda \to \Lambda$, $n \in \mathbb{N}$, of simple measurable functions such that $\sup\{|f_n(\lambda) - \lambda|, \lambda \in \Lambda\} \to 0$

when $n \to \infty$; put

$$D_n = \int\limits_{\Lambda} f_n(\lambda)\, d\mathcal{E}(\lambda), \quad n \in \mathbb{N}, \tag{5.2}$$

then $D_n : H \to H$, $n \in \mathbb{N}$ are self-adjoint linear operators and $\lim\limits_n \|S - D_n\| = 0$.

The properties of the resolution of identity easily imply that each of the operators $D_n$, $n \in \mathbb{N}$, is strictly ortho-diagonalizable. Therefore $(D_n)$ is the required sequence for $S$.

(b) Fix $T \in L(H, Y)$. Let $H_1$ be the closure of $|T|(H)$, where $|T| = (T^*T)^{\frac{1}{2}}$. There is an isometry $U : H_1 \to Y$ such that $U|T|h = Th \;\forall\, h \in H$. The restriction $S$ of $|T|$ on $H_1$ is a self-adjoint positive linear operator acting in the Hilbert space $H_1$. Since $H = \ker|T| + H_1$, we have $T = USP_1$, where $P_1 : H \to H_1$ is an orthogonal projection. Let now $D_n$, $n \in \mathbb{N}$, be the ortho-diagonalizable operators constructed for $S$ according to (a). Now it is easy to check that the operators $UD_nP_1$, $n \in \mathbb{N}$, are ortho-diagonalizable and $\lim\limits_n \|T - UD_nP_1\| = 0$.[16] $\square$

**Lemma 5.4.** *Let $H$ be a real or complex Hilbert space and let $T \in L(H)$ be a strictly ortho-diagonalizable operator with the diagonal $(\lambda_j)_{j\in J}$. The following assertions are valid:*

(a) *$0 \notin \sigma_e(T)$ if and only if* u-$\liminf_{j\in J} |\lambda_j| > 0$. *Moreover, if $0 \notin \sigma_e(T)$, then $(\pi(T))^{-1} = \pi(B)$, where $B \in L(H)$ is a strictly diagonalizable operator with respect to the same orthonormal basis as $T$.*

(b) *$\sigma_e(T)$ coincides with the set of numbers $\lambda$ which have the form $\lambda = \lim \lambda_{j_n}$ where $(j_n)$ is a sequence of distinct elements of $J$ such that $\lim\limits_n \lambda_{j_n}$ exists. Equivalently, if $\lambda$ is a number, then $\lambda \in \sigma_e(T)$ if and only if* u-$\liminf_{j\in J} |\lambda_j - \lambda| = 0$. *In particular, $\sigma_e(T) \neq \varnothing$.*

(c) *$\sigma_e(|T|) = \{|\lambda| : \lambda \in \sigma_e(T)\}$ and $r(\pi(T)) = \|\pi(T)\|$.*

(d) *If $0 \notin \sigma_e(T)$ then $r((\pi(T))^{-1}) = \|(\pi(T))^{-1}\|$.*

(d′) *The element $\pi(T)$ is normal-like in the Calkin algebra $L(H)/K(H)$.*

(e) *If $A \in L(H)$ is any self-adjoint operator, then $\sigma_e(A) \neq \varnothing$.*

*Proof.* (a) We can suppose that $T$ is a diagonal operator in an orthonormal basis $(e_j)_{j\in J}$ of $H$. Then the first part of (a) can be proved as a similar assertion in [16] (see the proof of the equivalence (a)$\Leftrightarrow$(e) of Theorem 2.3 on pp. 350-351); we note only that in our case of a diagonal operator the proof is easier and it works also in the case of a real Hilbert space. Let now $0 \notin \sigma_e(T)$, then according to the first part we have $\liminf_{j\in J} |\lambda_j| > 0$. This implies that $\inf_{j\notin\Delta} |\lambda_j| > 0$ for some finite subset $\Delta \subset J$. Define now $B \in L(H)$ in the following way: $B(e_j) = 0$ if $j \in \Delta$ and $B(e_j) = \frac{1}{\lambda_j} e_j$ if $j \notin \Delta$. Then $B$ is a diagonal operator and it is easy to see that $\pi(B)$ is the inverse of $\pi(T)$ in the Calkin algebra.

[16] We are indebted to Professor S. Kwapień who communicated to us the idea of deriving this lemma from the spectral theorem. We do not know whether there exists a proof which avoids spectral integrals.

(b) The first part of (b) follows directly from the first part of (a). From the first part of (b) it is clear also that the essential spectrum of $T$ is not empty.

(c) Since $|T|$ is a strictly diagonalizable operator with diagonal $(|\lambda_j|)$, according to (b) we have the first equality from (c). Now again from (b) and from the definition of $\limsup_j |\lambda_j|$ it follows that $r(\pi(T)) = \sup \sigma_e(|T|) = \limsup_j |\lambda_j|$. This and equality (2.6) of Corollary 2.21 imply $r(\pi(T)) = \|\pi(T)\|$.

(d) The inequality $r((\pi(T))^{-1}) \leq \|(\pi(T))^{-1}\|$ is valid for general reasons. The inverse inequality will be proved if we can find $\lambda$ in $\sigma((\pi(T))^{-1})$ such that $|\lambda| = \|(\pi(T))^{-1}\|$. Take $B$ from (a). If we apply now the second equality of (c) to the operator $B$, we can find $\lambda \in \sigma(\pi(B)) = \sigma((\pi(T))^{-1})$ such that $|\lambda| = \|\pi(B)\| = \|(\pi(T))^{-1}\|$ and (d) is proved.

(d$'$) Let $\lambda \in \mathbb{K} \setminus \sigma_e(T)$. Put $T_\lambda = T - \lambda I$. Then $T_\lambda$ is strictly ortho-diagonalizable and $0 \notin \sigma_e(T_\lambda)$. Then, by (d), applied to $T_\lambda$, we get: $r((\pi(T_\lambda))^{-1}) = \|(\pi(T_\lambda))^{-1}\|$. Hence $\pi(T)$ is normal-like.

(e) Via Lemma 5.3(a) we can find a sequence $(T_n)$ of strictly ortho-diagonalizable operators in $H$ such that $\lim\limits_n \|T_n - A\| = 0$. It follows that $\lim\limits_n \|\pi(T_n) - \pi(A)\| = 0$. Now an application of (b) and Proposition 5.1(a) to the Calkin algebra $\mathfrak{A} = L(H)/K(H)$ gives that $\sigma_e(A) := \sigma(\pi(A)) \neq \varnothing$. $\square$

**5.3. Bounds of the essential spectrum.** We begin with an assertion which shows that the mean dilatation numbers in the case of Hilbert spaces depend only on the modulus of operators.

**Lemma 5.5.** *Let $Y$ be a Hilbert space and $T \in L(H,Y)$. Then $\overline{\overline{\overline{\delta}}}(T) = \overline{\overline{\overline{\delta}}}(|T|)$, $\underline{\underline{\delta}}(T) = \underline{\underline{\delta}}(|T|)$. Consequently, $T$ is AMD-regular if and only if $|T|$ is such and, in the case of AMD-regularity, we have $\overline{\delta}(T) = \overline{\delta}(|T|)$.*

*Proof.* This follows easily from the definitions and from the observation that the equality $\|Tx\| = \||T|(x)\|$ holds for all $x \in H$. $\square$

The next assertion, which is the main result of this section, shows that by using mean dilatation numbers it is possible to describe the bounds of the essential spectrum.

**Theorem 5.6.** *Let $H$, $Y$ be infinite-dimensional Hilbert spaces, $T \in L(H,Y)$ be an arbitrary operator. Then*

$$\overline{\overline{\overline{\delta}}}(T) = \max \sigma_e(|T|), \quad \underline{\underline{\delta}}(T) = \min \sigma_e(|T|).$$

*Consequently, $T$ is AMD-regular if and only if $\sigma_e(|T|)$ consists of one point $\lambda_0$, and, in such a case, $\overline{\delta}(T) = \lambda_0$.*

*Proof.* Suppose first that $T$ is an ortho-diagonalizable operator with diagonal $(\lambda_j)$. Then $|T|$ is a strictly ortho-diagonalizable operator with diagonal $(|\lambda_j|)$. Hence by Lemma 5.4, Lemma 2.1 and Corollary 2.21 (through equality (2.6)) we can write the equalities:

$$\max \sigma_e(|T|) = \text{u-}\limsup_j |\lambda_j| = \overline{\overline{\overline{\delta}}}(T), \ \min \sigma_e(|T|) = \text{u-}\liminf_j |\lambda_j| = \underline{\underline{\delta}}(T).$$

Therefore the assertion is true in this case.

Let now $T$ be an arbitrary operator. By Lemma 5.3(b), there is a sequence $(T_n)$ of ortho-diagonalizable operators, $T_n \in L(H,Y)$, such that $\lim_n \|T - T_n\| = 0$. It follows that $\lim_n \||T| - |T_n|\| = 0$. This implies $\lim_n \|\pi(|T|) - \pi(|T_n|)\| = 0$. Notice now that according to Lemma 5.4(d′) the elements $\pi(|T_n|)$, $n = 1, 2, \ldots,$ are normal-like in the Calkin algebra $L(H)/K(H)$. So we can apply Proposition 5.1(d) for the elements $\pi(|T|)$, $\pi(|T_n|)$, $n \in \mathbb{N}$, and write $\max \sigma_e(|T_n|) \to \max \sigma_e(|T|)$ and $\min \sigma_e(|T_n|) \to \min \sigma_e(|T|)$ (let us emphasize that this is the crucial point of the proof, only here the delicate Proposition 5.1(d) is used). Now, using this, Lemma 5.5 and the continuity of functionals $\overline{\overline{\delta}}$ and $\underline{\underline{\delta}}$ (see Proposition 4.1), we obtain:

$$\begin{aligned}\overline{\overline{\delta}}(T) = \overline{\overline{\delta}}(|T|) = \lim_n \overline{\overline{\delta}}(|T_n|) = \\ = \lim_n \max \sigma_e(|T_n|) = \max \sigma_e(|T|).\end{aligned}$$

Similarly,

$$\begin{aligned}\underline{\underline{\delta}}(T) = \underline{\underline{\delta}}(|T|) = \lim_n \underline{\underline{\delta}}(|T_n|) = \\ = \lim_n \min \sigma_e(|T_n|) = \min \sigma_e(|T|).\end{aligned}$$

The assertion about $\overline{\delta}$ is now immediate. □

*Remark* 5.7. The exact bounds of the essential spectrum for an operator in a complex Hilbert space had been studied earlier in several papers, see, e.g., [6], [41], [65]. For the case of a complex Hilbert space $H$ and the description of the essential spectrum by means of Weyl's criterion [57, p.237, Theorem VII.12] together with the observation that for any $T \in L(H)$ one has $\|\pi(T)\| \in \sigma_e(|T|)$ (which follows from the general result concerning the spectrum of an element in a $C^*$-algebra), we get

$$\overline{\overline{\delta}}(T) = \max \sigma_e(|T|).$$

As the reader can see, our proof avoids the usage of the rather delicate fact that the Calkin algebra is a $C^*$-algebra.


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## References


1. D. Aldous and D. H. Fremlin, Colacunary sequences in $L$-spaces. *Studia Math.* **LXXI**(1982), 296–304.
2. Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis, I. *American Mathematical Society Colloquium Publications* 48, *American Mathematical Society, Providence, RI,* 2000.
3. F. Bombal, Strictly singular and strictly cosingular operators on $C(K,E)$. *Math. Nachr.* **143**(1989), 355–364.
4. F. Bombal, Sobre algunas propiedades de espacios de Banach. *Rev. Real Acad. Cienc. Exact. Fis. Natur. Madrid* **84**(1990), 83–116.
5. F. Bombal, Distinguished subsets in vector sequence spaces. *Progress in Functional Analysis. K.D. Biernstedt, J. Bonet, J. Horváth, M. Maestre (Eds.),* 293–306, *Elsevier Science Publishers B.V.,* 1992.
6. R. Bouldin, The essential minimum modulus. *Indiana Univ. Math. J.* **30**(1981), No. 4, 513–517.
7. N. Bourbaki, Éléments de mathématique. Topologie générale. Chapitres 1 à 4. *Hermann, Paris,* 1971.
8. N. Bourbaki, Espaces vectoriels topologiques. *Masson, Paris,* 1981.
9. J. Bourgain and J. Diestel, Limited operators and strict cosingularity. *Math. Nachr.* **119**(1984), 55–58.
10. P. Casazza and N. Nielsen, A Gaussian average property of Banach spaces. *Illinois J. Math.* **41**(1997), No. 4, 559–576.
11. V. Caselles and M. González, Compactness properties of strictly singular operators in Banach lattices. *Semesterbericht Funktionalanalysis,* 175–189, *Tübingen, Sommersemester,* 1987.
12. A.Castejón, E.Corbacho, and V. Tarieladze, MD-numbers and asymptotic MD-numbers of operators. *Georgian Math. J.* **8**(2001), No. 3, 455–468.
13. S.A. Chobanjan and V. I. Tarieladze, Gaussian characterizations of certain Banach spaces. *J. Multivariate Anal.* **7**(1977), 183–203.
14. F. Cobos, A. Manzano, A. Martínez, and P. Matos, On interpolation of strictly singular, strictly cosingular operators and related operator ideals. *Proc. Roy. Soc. Edinburgh Sect. A* **130**(2000), 771–989.
15. F.Cobos and E. Pustylnik, On strictly singular and strictly cosingular embeddings between Banach lattices of functions. *Math. Proc. Cambridge Philos. Soc.* (to appear).
16. J. B. Conway, A course in functional analysis. *Springer-Verlag, New York,* 1990.
17. J. Diestel, Sequences and series in Banach spaces. *Springer-Verlag, New York,* 1984.
18. J. Diestel, H. Jarchow, and A. Tonge, Absolutely summing operators. *Cambridge University Press, Cambridge,* 1995.
19. T. Figiel and N. Tomczak-Jaegermann, Projections onto Hilbertian subspaces of Banach spaces. *Israel J. Math.* **33**(1979), 155–171.
20. A. García del Amo, F. L. Hernández, V. M. Sánchez, and E. M. Semenov, Disjointly strictly singular inclusions between rearrangement invariant spaces. *J. London Math. Soc.* **62**(2000), 239–252.
21. I. C. Gohberg, A. S. Markus, and I. A. Fel'dman, Normally solvable operators and ideals associated with them. (Russian) *Bul. Akad. Štiince RSS Moldoven.* **1960**, No. 10(76), 51–70.

22. S. Goldberg, Unbounded linear operators. Theory and applications. *Dover Publications, Inc. New York,* 1985.
23. S. Goldberg and E. O. Thorp, On some open questions concerning strictly singular operators. *Proc. Amer. Math. Soc.* **14**(1963), No. 2, 334–336.
24. W. T. Govers and B. Maurey, The unconditional basic sequence problem. *J. Amer. Math. Soc.* **6**(1993), 851–874.
25. P. Halmos, A Hilbert space problem book. *Springer-Verlag, New York etc.,* 1982.
26. F. L. Hernández and B. Rodríguez-Salinas, On $\ell_p$-complemented copies in Orlicz spaces. *Israel J. Math.* **68**(1989), 27–55.
27. F. L. Hernández, V. M. Sánchez, and E. M. Semenov, Disjoint strict singularity of inclusions between rearrangement invariant spaces. *Studia Math.* **144**(2001), 209–226.
28. A. Hinrichs and A. Pietsch, Closed ideals and Dvoretzky theorem for operators. (*Unpublished*).
29. H. Jarchow, Locally convex spaces. *B. G. Teubner, Stuttgart,* 1981.
30. T. Kato, Perturbation theory for nullity deficiency and other quantities of linear operators. *J. Anal. Math.* **6**(1958), 273–322.
31. O. V. Kucher and A. M. Plichko, On strict singularity of locally integral operators on Banach function spaces. *Quaestiones Math.* **20**(1997), 549–561.
32. K. Kuratowski, Topology, I. (Translated from the French) *Academic Press, New York-London,* 1966.
33. S. Kwapień, Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Math.* **44**(1972), 583–595.
34. S. Kwapień, A remark on the median and the expectation of convex functions of gaussian vectors. *Probability in Banach spaces,* 9 (*Sandjberg,* 1993), 271–272, *Progr. Probab.,* 35, *Birkhauser Boston, Boston, MA,* 1994.
35. S. Kwapień and B. Szymański, Some remarks on Gaussian measures in Banach spaces. *Probab. Math. Statist.* **1**(1980), 59–65.
36. E. Lacey and R. J. Whitley, Conditions under which all the bounded linear maps are compact. *Math. Ann.* **158**(1965), 1–5.
37. R. Latala and K. Oleszkiewicz, Gaussian measures of dilatations of convex symmetric sets. *Ann. Probab.* **27**(1999), 1922–1938.
38. W. Linde and A. Pietsch, Mappings of Gaussian cylindrical measures in Banach spaces. *Theory Probab. Appl.* **19**(1974), 445–460.
39. W. Linde, S. A. Chobanyan, and V. I. Tarieladze, Characterization of certain classes of Banach Spaces by Properties of Gaussian Measures. *Theory Probab. Appl.* **25**(1980), 159–164.
40. J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. I. *Springer-Verlag, Berlin etc.,* 1977.
41. E. Martín, E.Indurain, A. Plans, and A. Rodés, Two geometric constants for operators acting on a separable Banach space. *Rev. Mat. Univ. Complut. Madrid* **1**(1988), No. 1–3, 23–30.
42. V. D. Milman, Some properties of strictly singular operators. (Russian) *Funktsional. Anal. i Prilozen.* **3**(1969), No. 1, 93–94.
43. V. D. Milman, Spectrum of bounded continuous functions specified on a unit sphere of a Banach space. (Russian) *Funktsional. Anal. i Prilozen.* **3**(1969), No. 2, 67–79.
44. V. D. Milman, Operators of class $C_0$ and $C_0^*$. (Russian) *Teor. Funkciĭ Funktsional. Anal. i Prilozhen.* No. 10,(1970), 15–26.

45. V. D. Milman, Isomorphic symmetrization and geometric inequalities. *Geometric aspects of functional analysis* (1986/87), 107–131, *Lecture Notes in Math.,* 1317, *Springer, Berlin,* 1988.

46. V. D. Milman and G. Schechtman, Asymptotic theory of finite dimensional normed spaces. *Lecture Notes in Mathematics,* 1200, *Springer-Verlag, Berlin,* 1986.

47. V. Milman and R. Wagner, Asymptotic versions of operators and operator ideals. *Convex geometric analysis,* 34, 165–179, *Cambridge Univ. Press, Math. Sci. Res. Inst. Publ.,* 1999.

48. B. S. Mitiagin and A. Pełczyński, Nuclear operators and approximative dimensions. *Proc. Inter. Congress of mathematicians,* 366–372, *Moscow,* 1966.

49. A. Pełczyński Projections in certain Banach spaces. *Studia Math.* **19**(1960), 209–228.

50. A. Pełczyński, On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in C(S)-spaces. *Bull. Acad. Pol. Sci.* **13**(1964), No. 1, 31–36.

51. A. Pełczyński, On strictly singular and strictly cosingular operators. II. Strictly singular and strictly cosingular operators in L($\nu$)-spaces. *Bull. Acad. Pol. Sci.* **13**(1964), No. 1, 37–41.

52. A. Pietsch, Operator ideals. *North-Holland, Amsterdam etc.,* 1980.

53. A. Pietsch, Eigenvalues and $s$-numbers. *Cambridge University Press, Cambridge,* 1987.

54. A. Pietsch and J. Wenzel, Orthonormal systems and Banach space geometry. *Cambridge University Press, Cambridge,* 1998.

55. G. Pisier, The volume of convex bodies and Banach space geometry. *Cambridge University Press, Cambridge,* 1989.

56. A. Plichko, Superstrictly singular and superstrictly cosingular operators. *Preliminary materials,* 2000.

57. M. Reed and B. Simon, Functional analysis. Revised and Enlarged Edition, *Academic Press, San Diego,* 1980.

58. E. Saksman and H-O. Tylli, The Apostol–Fialkow formula for elementary operators on Banach spaces. *J. Funct. Anal.* **161**(1999), 1–26.

59. E. M. Semenov and A. M. Shtenberg, The Orlicz property of symmetric spaces. *Soviet Math. Dokl.* **42**(1991), 679–682.

60. M. Talagrand, Orlicz property and cotype in symmetric sequence spaces. *Israel J. Math.* **87**(1994), 181–192.

61. N. Tomczak-Jaegermann, Banach–Mazur distance and finite dimensional operator ideals. *Longman Scientific Technical, Harlow,* 1989.

62. N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, Probability distributions on Banach spaces. *D. Reidel Publishing Company, Dordrecht,* 1987.

63. J. Weidmann, Linear operators in Hilbert spaces. *Springer-Verlag, New York–Berlin,* 1980.

64. R. J. Whitley, Strictly singular operators and their conjugates. *Trans. Amer. Math. Soc.* **113**(1964), 252–261.

65. J. Zemanek, Geometric characteristics of semi-Fredholm operators and their asymptotic behaviour. *Studia Math.* **80**(1984), 219–234.

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