

CONFORMAL AND QUASICONFORMAL MAPPINGS OF CLOSE MULTIPLY-CONNECTED DOMAINS

Z. SAMSONIA[†] AND L. ZIVZIVADZE

Abstract. Doubly-connected and triply-connected domains close to each other in a certain sense are considered. Some questions connected with conformal and quasiconformal mappings of such domains are studied using integral equations.

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1. CONFORMAL MAPPING OF CLOSE TRIPLY-CONNECTED DOMAINS

Let G be a finite triply-connected domain of a complex plane Z bounded by the simple Lyapunov curves $\Gamma_0, \Gamma_1, \Gamma_2$, one of which Γ_0 envelops the other two and $z = 0 \in \text{int } \Gamma_1$.

Assume first that the boundary $\Gamma = \bigcup_{i=0}^2 \Gamma_i$ belongs to the class C'_α ($\frac{1}{2} < \alpha \leq 1$), while singly-connected domains with boundaries Γ_1 and Γ_0 are star-like with respect to $z = 0$. Let the equations of these curves be given in terms of polar coordinates

$$t = g_1(\varphi) = \rho_1(\varphi) \cdot e^{i\varphi}, \quad t = g_0(\varphi) = \rho_0(\varphi) \cdot e^{i\varphi} \quad (0 \leq \varphi \leq 2\pi)$$

and the finite domain with boundary Γ_2 be star-like with respect to $z_0 \in \text{int } \Gamma_2$. If we assume that the polar axis with a pole in z_0 is parallel to the abscissa axis, then the parametric equation for Γ_2 can be written in the form

$$t = g_2(\varphi) = z_0 + \rho_2(\varphi)e^{i\varphi} \quad (0 \leq \varphi \leq 2\pi).$$

Let us consider the second triply-connected domain \tilde{G} of type G bounded by the curves $\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{\Gamma}_2$ (with the same properties) whose parametric equations are

$$t = \tilde{\rho}_1(\varphi)e^{i\varphi}, \quad t = z_0 + \tilde{\rho}_2(\varphi)e^{i\varphi}, \quad t = \tilde{\rho}_0(\varphi)e^{i\varphi} \quad (0 \leq \varphi \leq 2\pi).$$

We introduce the following notation:

$$d_1 = \rho(\Gamma_1; \Gamma_0), \quad d_2 = \rho(\Gamma_1; \Gamma_2), \quad d_3 = \rho(\Gamma_2; \Gamma_0), \quad d_0 = \min\{d_1; d_2; d_3\}.$$

It is assumed that $\varepsilon \in (0; d_0/2)$.

Definition 1. The domains G and \tilde{G} are called ε -close to each other if the conditions

$$|\rho_i(\varphi) - \tilde{\rho}_i(\varphi)| \leq \varepsilon; \quad \|\rho'_i(\varphi) - \tilde{\rho}'_i(\varphi)\|_{C_\alpha} \leq \varepsilon \quad (i = 0, 1, 2) \quad (1)$$

are fulfilled.

An infinite set of domains ε -close to G are formed for any $\varepsilon \in (0; d/2)$. We denote it by G_ε .

Let us conformally map the domains G and \tilde{G} onto the canonical domains $K(\rho; r; 1)$ and $\tilde{K}(\tilde{\rho}; \tilde{r}; 1)$, respectively, using the assumptions of [1], where $K(\rho; r; 1)$ and $\tilde{K}(\tilde{\rho}; \tilde{r}; 1)$ are annuli with concentric cuts along the arc of the circumferences $|W| = r$ and $|W| = \tilde{r}$, ($\rho < r < 1$, $\tilde{\rho} < \tilde{r} < r < 1$), respectively. Then for the definition of radii we have [1]

$$\begin{aligned} \ln \rho &= \frac{1}{\pi} \int_0^{2\pi} \nu_1(\varphi) \sqrt{\rho_1^2(\varphi) + [\rho'_1(\varphi)]^2} d\varphi, \\ \ln \tilde{\rho} &= \frac{1}{\pi} \int_0^{2\pi} \tilde{\nu}_1(\varphi) \sqrt{\tilde{\rho}_1^2(\varphi) + [\tilde{\rho}'_1(\varphi)]^2} d\varphi, \\ \ln r &= \frac{1}{\pi} \int_0^{2\pi} \nu_2(\varphi) \sqrt{\rho_2^2(\varphi) + [\rho'_2(\varphi)]^2} d\varphi, \\ \ln \tilde{r} &= \frac{1}{\pi} \int_0^{2\pi} \tilde{\nu}_2(\varphi) \sqrt{\tilde{\rho}_2^2(\varphi) + [\tilde{\rho}'_2(\varphi)]^2} d\varphi. \end{aligned} \quad (2)$$

It is assumed here that

$$\nu(t(\varphi)) = \begin{cases} \nu_1(\varphi), & \text{when } t \in \Gamma_1, \\ \nu_2(\varphi), & \text{when } t \in \Gamma_2, \\ \nu_0(\varphi), & \text{when } t \in \Gamma_0. \end{cases}$$

The reasoning for $\tilde{\nu}(\tilde{t}(\varphi))$ is analogous. It is assumed that $\nu(t)$ and $\tilde{\nu}(t)$ are unique solutions of the integral equations

$$\nu(t_0) + \frac{1}{\pi} \int_{\Gamma} K_0(t; t_0) \nu(t) dt = -\ln |t_0|, \quad t_0 \in \Gamma, \quad (3)$$

$$\tilde{\nu}(t_0) + \frac{1}{\pi} \int_{\tilde{\Gamma}} \tilde{K}_0(t; t_0) \tilde{\nu}(t) dt = -\ln |t_0|, \quad t_0 \in \tilde{\Gamma}, \quad (4)$$

where

$$K_0(t; t_0) = \begin{cases} \operatorname{Im} \left(\frac{1}{t - t_0} \cdot \frac{dt}{ds} \right) - 1, & \text{when } t, t_0 \in \Gamma_j \quad (j = 1, 2), \\ \operatorname{Im} \left(\frac{1}{t - t_0} \cdot \frac{dt}{ds} \right), & \text{in all other cases.} \end{cases}$$

$\widetilde{K}_0(t; t_0)$ is defined analogously.

Let us pose the problem: derive an estimate through ε for a difference of the solutions of equations (3) and (4) (in an appropriate norm), and also for the expressions $|\rho - \tilde{\rho}|, |r - \tilde{r}|$.

We can obtain such estimates by using the statements proved below. For this, the integral equations (3) and (4) are represented in the complex form:

$$\nu(t_0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{t'} K_*(t; t_0) \nu(t) dt = -\ln |t_0|, \tag{3_1}$$

$$\tilde{\nu}(t_0) + \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{1}{t'} \widetilde{K}_*(t; t_0) \tilde{\nu}(t) dt = -\ln |t_0|, \tag{4_1}$$

where

$$K_0(t; t_0) = \begin{cases} (t' - t'_0) + (\bar{t}'_0 - \bar{t}') \frac{t - t_0}{\bar{t} - \bar{t}_0} + \left[t'_0 \frac{\bar{t} - \bar{t}_0}{t - t_0} - \bar{t}'_0 \right] \frac{t - t_0}{\bar{t} - \bar{t}_0} \frac{1}{t - t_0} - 2i, \\ \text{when } t, t_0 \in \Gamma_i \ (i = 1, 2), \\ (t' - t'_0) + (\bar{t}'_0 - \bar{t}') \frac{t - t_0}{\bar{t} - \bar{t}_0} + \left[t'_0 \frac{\bar{t} - \bar{t}_0}{t - t_0} - \bar{t}'_0 \right] \frac{t - t_0}{\bar{t} - \bar{t}_0} \frac{1}{t - t_0} \\ \text{in all other cases.} \end{cases} \tag{5}$$

Here $t' = g'_j(\varphi), t'_0 = g'_j(\varphi_0)$ and j take values 0, 1, 2 depending on the fact to which contour Γ_j ($j = 0, 1, 2$) the point t or t_0 belongs. $\widetilde{K}^*(t; t_0)$, too, is constructed analogously to (5). Clearly, in that case $t = \tilde{g}_j(\varphi), t_0 = \tilde{g}_j(\varphi_0), t' = \tilde{g}'_j(\varphi), t'_0 = \tilde{g}'_j(\varphi_0)$ ($j = 0, 1, 2$).

Let us represent the integral equations (3₁), (4₁) in the operator form:

$$A\nu = (I + H)\nu = f_0, \tag{3'_1}$$

$$\tilde{A}\tilde{\nu} = (I + \tilde{H})\tilde{\nu} = \tilde{f}_0. \tag{4'_1}$$

It is assumed that (analogously to $\nu[t(\varphi)]$), $\tilde{\nu}(t), f_0(t), \tilde{f}_0(t)$ are column-matrices, I is the unit matrix of third order, and

$$H\nu = \begin{pmatrix} H_{11}\nu_1 + H_{12}\nu_2 + H_{10}\nu_0 \\ H_{21}\nu_1 + H_{22}\nu_2 + H_{20}\nu_0 \\ H_{01}\nu_1 + H_{02}\nu_2 + H_{00}\nu_0 \end{pmatrix},$$

$$\tilde{H}\tilde{\nu} = \begin{pmatrix} \tilde{H}_{11}\tilde{\nu}_1 + \tilde{H}_{12}\tilde{\nu}_2 + \tilde{H}_{10}\tilde{\nu}_0 \\ \tilde{H}_{21}\tilde{\nu}_1 + \tilde{H}_{22}\tilde{\nu}_2 + \tilde{H}_{20}\tilde{\nu}_0 \\ \tilde{H}_{01}\tilde{\nu}_1 + \tilde{H}_{02}\tilde{\nu}_2 + \tilde{H}_{00}\tilde{\nu}_0 \end{pmatrix}.$$

Here H_{ij}, \tilde{H}_{ij} ($i, j = 0, 1, 2$) are the concrete integral operators. Before representing them explicitly, let us make some additional observations.

Observe that

$$\left. \frac{dt}{t-t_0} \right|_{\substack{t \in \Gamma_j \\ t_0 \in \Gamma_k}} = \frac{g'_j(\varphi)d\varphi}{g_j(\varphi) - g_k(\varphi_0)} = M_{jk}(\varphi; \varphi_0) \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) d\varphi,$$

where the function

$$M_{jk}(\varphi; \varphi_0) = \frac{g'_j(\varphi)}{2ie^{i\varphi}} \cdot \frac{e^{i\varphi} - e^{i\varphi_0}}{g_j(\varphi) - g_k(\varphi_0)} \quad (6)$$

satisfies the Hölder condition with index α with respect to its arguments, i.e., belongs to the class $C_\alpha(\Gamma_j \times \Gamma_k)$.

The function

$$R_{jk}(t_0; t) = \frac{1}{t'} \left\{ (t' - t'_0) + (t'_0 - t') \frac{t - t_0}{\bar{t} - \bar{t}_0} + \left[t'_0 \frac{\bar{t} - \bar{t}_0}{t - t_0} - t'_0 \right] \frac{t - t_0}{\bar{t} - \bar{t}_0} \right\}$$

($t \in \Gamma_j, t_0 \in \Gamma_k, j, k = 0, 1, 2$) figuring in (5) also belongs to the class $C_\alpha(\Gamma_j \times \Gamma_k)$ and $R_{kk}(t_0; t_0) = 0$.

We have in terms of the parameter φ

$$\begin{aligned} S_{jk}(\varphi; \varphi_0) &= R_{jk}[g_j(\varphi); g_k(\varphi_0)] = \frac{1}{g'_j(\varphi)} \left\{ g'_j(\varphi) - g'_k(\varphi_0) \right. \\ &\quad + (\bar{g}'_k(\varphi_0) - \bar{g}'_j(\varphi)) \frac{g_j(\varphi) - g_k(\varphi_0)}{\bar{g}_j(\varphi) - \bar{g}_k(\varphi_0)} \\ &\quad \left. + \left[g'_k(\varphi_0) \frac{\bar{g}_j(\varphi) - \bar{g}_k(\varphi_0)}{g_j(\varphi) - g_k(\varphi_0)} - \bar{g}'_k(\varphi_0) \right] \frac{g_j(\varphi) - g_k(\varphi_0)}{\bar{g}_j(\varphi) - \bar{g}_k(\varphi_0)} \right\}, \quad (5') \end{aligned}$$

$S_{jk}(\varphi; \varphi_0) \in C_\alpha(\Gamma_j \times \Gamma_k)$, $S_{kk}(\varphi_0; \varphi_0) = 0$ ($0 \leq \varphi \leq 2\pi$).

$\tilde{M}_{jk}(\varphi, \varphi_0)$ and $\tilde{S}_{jk}(\varphi; \varphi_0)$ are defined analogously.

Having introduced the notation, we can write that

$$\begin{aligned} & [H_{11}\nu_1 + H_{12}\nu_2 + H_{10}\nu_0](\varphi_0) \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{11}(\varphi_0; \varphi)M_{11}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta - 2i \right\} \nu_1(\varphi) d\varphi \\ &\quad + \frac{1}{2\pi i} \int_0^{2\pi} \frac{S_{12}(\varphi_0; \varphi)M_{12}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \nu_2(\varphi) d\varphi \\ &\quad + \frac{1}{2\pi i} \int_0^{2\pi} \frac{S_{10}(\varphi_0; \varphi)M_{10}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \nu_0(\varphi) d\varphi, \\ & [H_{21}\nu_1 + H_{22}\nu_2 + H_{20}\nu_0](\varphi_0) \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{21}(\varphi_0; \varphi)M_{21}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \right\} \nu_1(\varphi) d\varphi \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{22}(\varphi_0; \varphi) M_{22}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta - 2i \right\} \nu_2(\varphi) d\varphi \\
 & + \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{20}(\varphi_0; \varphi) M_{20}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \right\} \nu_0(\varphi) d\varphi, \\
 & \qquad [H_{01}\nu_1 + H_{02}\nu_2 + H_{00}\nu_0](\varphi_0) \\
 & = \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{01}(\varphi_0; \varphi) M_{01}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \right\} \nu_1(\varphi) d\varphi \\
 & + \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{02}(\varphi_0; \varphi) M_{02}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \right\} \nu_2(\varphi) d\varphi \\
 & + \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \frac{S_{00}(\varphi_0; \varphi) M_{00}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \cdot \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \right\} \nu_0(\varphi) d\varphi,
 \end{aligned}$$

where β is any number satisfying the condition $\frac{1}{2} < \beta \leq \alpha$.

From the above formulas we obtain

$$\begin{aligned}
 (H_{11} - \widetilde{H}_{11})\nu_1 & = \frac{1}{\pi i} \int_0^{2\pi} \frac{S_{11}(\varphi_0; \varphi) M_{11}(\varphi_0; \varphi) - \widetilde{S}_{11}(\varphi_0; \varphi) \widetilde{M}_{11}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \\
 & \quad \times \frac{1}{2} \left(\operatorname{ctg} \frac{\varphi - \varphi_0}{2} + i \right) |e^{i\varphi} - e^{i\varphi_0}|^\beta \nu_1(\varphi) d\varphi \\
 & = \frac{1}{\pi i} \int_0^{2\pi} \frac{S_{11}(\varphi_0; \varphi) M_{11}(\varphi_0; \varphi) - \widetilde{S}_{11}(\varphi_0; \varphi) \widetilde{M}_{11}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta} \\
 & \quad \times \frac{ie^{i\varphi}}{e^{i\varphi} - e^{i\varphi_0}} |e^{i\varphi} - e^{i\varphi_0}|^\beta \nu_1(\varphi) d\varphi \\
 & = \frac{1}{\pi i} \int_0^{2\pi} [K_{11}(\varphi_0; \varphi) - \widetilde{K}_{11}(\varphi_0; \varphi)] \nu_1(\varphi) d\varphi. \tag{7}
 \end{aligned}$$

In (7)

$$\begin{aligned}
 K_{11}(\varphi_0; \varphi) & = K_{11}^{(1)}(\varphi_0; \varphi) K_{11}^{(2)}(\varphi_0; \varphi), \\
 \widetilde{K}_{11}(\varphi_0; \varphi) & = \widetilde{K}_{11}^{(1)}(\varphi_0; \varphi) \widetilde{K}_{11}^{(2)}(\varphi_0; \varphi),
 \end{aligned} \tag{7'}$$

where

$$\begin{aligned}
 K_{11}^{(1)}(\varphi_0; \varphi) & = \frac{M_{11}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta}; & K_{11}^{(2)}(\varphi_0; \varphi) & = \frac{iS_{11}(\varphi_0; \varphi)e^{i\varphi}}{|e^{i\varphi} - e^{i\varphi_0}|^{1-\beta} \cdot e^{i \arg(e^{i\varphi} - e^{i\varphi_0})}}; \\
 \widetilde{K}_{11}^{(1)}(\varphi_0; \varphi) & = \frac{\widetilde{M}_{11}(\varphi_0; \varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta}; & \widetilde{K}_{11}^{(2)}(\varphi_0; \varphi) & = \frac{i\widetilde{S}_{11}(\varphi_0; \varphi)e^{i\varphi}}{|e^{i\varphi} - e^{i\varphi_0}|^{1-\beta} \cdot e^{i \arg(e^{i\varphi} - e^{i\varphi_0})}}.
 \end{aligned} \tag{8}$$

According to [2], if $\varphi(t) \in C\mu(\Gamma)$, $0 < \mu \leq 1$, then the function of two variables $(t, t_0 \in \Gamma)$

$$\psi(t; t_0) = \frac{\varphi(t) - \varphi(t_0)}{|t - t_0|^\lambda},$$

$0 \leq \lambda < \mu \leq 1$, satisfies, on Γ , the Hölder condition with index $\mu - \lambda$. Moreover, the estimate

$$\|\psi(t_0; t)\|_{C_{\mu-\lambda}} \leq A^*(1 + \lambda) \tag{9}$$

holds true, where $A^* \geq \frac{|\varphi(t) - \varphi(t_0)|}{|t - t_0|^\lambda}$ (see §§5, 6 of [2]). Taking into account the structure of the functions $M_{11}(\varphi; \varphi_0)$, $S_{11}(\varphi; \varphi_0)$, $\widetilde{M}_{11}(\varphi; \varphi_0)$, $\widetilde{S}_{11}(\varphi; \varphi_0)$, this result implies that the functions $K_{11}(\varphi; \varphi_0)$ and $\widetilde{K}_{11}(\varphi; \varphi_0)$ are continuous in the Hölder sense with respect to φ_0 and φ with index δ , $\delta = \min\{\alpha - \beta; \alpha + \beta - 1\}$ and

$$\begin{aligned} \|K_{11}(\varphi; \varphi_0) - \widetilde{K}_{11}(\varphi; \varphi_0)\|_{C_\delta} &= \|K_{11}^{(1)} K_{11}^{(2)} - \widetilde{K}_{11}^{(1)} \widetilde{K}_{11}^{(2)}\|_{C_\delta} \\ &\leq \|K_{11}^{(2)}\|_{C_\delta} \cdot \|K_{11}^{(1)} - \widetilde{K}_{11}^{(1)}\|_{C_\delta} + \|\widetilde{K}_{11}^{(1)}\|_{C_\delta} \cdot \|K_{11}^{(2)} - \widetilde{K}_{11}^{(2)}\|_{C_\delta}. \end{aligned} \tag{10}$$

Let us define the order of smallness with respect to ε in (10). Preliminarily, we will prove the validity of the following propositions.

For small values of ε the following inequalities are fulfilled:

- I. $\| [g'_1(\varphi) - g'(\varphi_0)] - [\widetilde{g}'_1(\varphi) - \widetilde{g}'(\varphi_0)] \|_{C_\alpha} \leq 2\varepsilon,$
 $\| [\overline{g}'_1(\varphi) - \overline{g}'(\varphi_0)] - [\widetilde{\overline{g}}'_1(\varphi) - \widetilde{\overline{g}}'(\varphi_0)] \|_{C_\alpha} \leq 2\varepsilon,$
- II. $\left\| \frac{e^{i\varphi} - e^{i\varphi_0}}{g_1(\varphi) - g_1(\varphi_0)} \cdot \frac{g'_1(\varphi)}{2ie^{i\varphi}} - \frac{e^{i\varphi} - e^{i\varphi_0}}{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)} \cdot \frac{\widetilde{g}'_1(\varphi)}{2ie^{i\varphi}} \right\|_{C_\alpha} \leq A_1 \cdot \varepsilon,$
- III. $\left\| \frac{g_1(\varphi) - g_1(\varphi_0)}{\overline{g}_1(\varphi) - \overline{g}_1(\varphi_0)} - \frac{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)}{\widetilde{\overline{g}}_1(\varphi) - \widetilde{\overline{g}}_1(\varphi_0)} \right\|_{C_\alpha} \leq A_2 \cdot \varepsilon,$
- IV. $|\widetilde{g}'_1(\varphi)| \geq \frac{\min_{[0, 2\pi]} |g'_1(\varphi)|}{2}$ for $\varepsilon \leq \frac{\min_{[0, 2\pi]} |g'_1(\varphi)|}{2},$
- V. $\left\| \left(g'_1(\varphi_0) \cdot \frac{\overline{g}_1(\varphi) - \overline{g}_1(\varphi_0)}{g_1(\varphi) - g_1(\varphi_0)} - g'_1(\varphi_0) \right) \right.$
 $\left. \left(-\widetilde{g}'_1(\varphi_0) \cdot \frac{\widetilde{\overline{g}}_1(\varphi) - \widetilde{\overline{g}}_1(\varphi_0)}{\widetilde{g}_1(\varphi) - \widetilde{g}_1(\varphi_0)} - \widetilde{\overline{g}}'_1(\varphi_0) \right) \right\|_{C_\alpha} \leq A_3\varepsilon$

and all constants contained in the estimates do not depend on \widetilde{G} .

Proof. Inequality I immediately follows from the definition of domain closeness, i.e., from (1).

Further we have

$$|g'_1(\varphi)| - |\widetilde{g}'_1(\varphi)| \leq \|g'_1(\varphi) - \widetilde{g}'_1(\varphi)\|_C \leq \|g'_1(\varphi) - \widetilde{g}'_1(\varphi)\|_{C_\alpha} < \varepsilon$$

and if it is assumed that $\varepsilon \leq \frac{1}{2} \min_{[0,2\pi]} |g'_1(\varphi)|$, then the validity of inequality IV is proved.

Before proving inequality II, note the following: for $|s - s_0| \leq \frac{l_1}{2}$, where l_1 is the length of Γ_1 , and s and s_0 are the arc abscissas of the points $t = g_1(\varphi)$ and $t_0 = g_1(\varphi_0)$, we have

$$\frac{|t - t_0|^2}{|\varphi - \varphi_0|^2} = \frac{|g_1(\varphi) - g_1(\varphi_0)|^2}{|s - s_0|^2} \cdot \frac{|s - s_0|^2}{|\varphi - \varphi_0|^2} \geq k_1^2 \cdot \rho_1^2, \tag{11}$$

where $\rho_1 = \min_{[0,2\pi]} |g'_1(\varphi)| = \rho(0; \Gamma_1) > 0$, and k_1 ($0 < k_1 < 1$) is the constant defined by giving the contour Γ_1 .

On the other hand,

$$\begin{aligned} [g_1(\varphi) - g_1(\varphi_0)] - [\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)] &= [g_1(\varphi) - \tilde{g}_1(\varphi)] - [g_1(\varphi_0) - \tilde{g}_1(\varphi_0)] \\ &= \operatorname{Re}[g_1(\varphi) - \tilde{g}_1(\varphi)] - \operatorname{Re}[g_1(\varphi_0) - \tilde{g}_1(\varphi_0)] \\ &\quad + i(\operatorname{Im}[g_1(\varphi) - \tilde{g}_1(\varphi)] - \operatorname{Im}[g_1(\varphi_0) - \tilde{g}_1(\varphi_0)]) \\ &= (\varphi - \varphi_0) \operatorname{Re}[g'_1(\xi) - \tilde{g}'_1(\xi)] + i(\varphi - \varphi_0) \operatorname{Im}[g'_1(\eta) - \tilde{g}'_1(\eta)], \end{aligned}$$

where the numbers ξ and η lie between φ and φ_0 .

By virtue of (1) we can write that

$$|[\tilde{g}(\varphi) - \tilde{g}(\varphi_0)] - [g_1(\varphi) - g_1(\varphi_0)]| \leq 4\varepsilon|\varphi - \varphi_0|.$$

Moreover, since

$$\begin{aligned} &|g_1(\varphi) - g_1(\varphi_0)| - |\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)| \\ &\leq |[g_1(\varphi) - g_1(\varphi_0)] - [\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)]| \leq 4\varepsilon|\varphi - \varphi_0| \end{aligned}$$

we have

$$|\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)| \geq |g_1(\varphi) - g_1(\varphi_0)| - 4\varepsilon|\varphi - \varphi_0|.$$

Hence, assuming that $\varepsilon \leq \frac{k_1\rho_1}{8}$, by (11) we obtain

$$\frac{|\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)|}{|\varphi - \varphi_0|} > \frac{k_1\rho_1}{2}. \tag{12}$$

Now we return to proving inequality II. Since

$$\begin{aligned} &\frac{e^{i\varphi} - e^{i\varphi_0}}{g_1(\varphi) - g_1(\varphi_0)} \cdot \frac{g'_1(\varphi)}{2ie^{i\varphi}} - \frac{e^{i\varphi} - e^{i\varphi_0}}{\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)} \cdot \frac{\tilde{g}'_1(\varphi)}{2ie^{i\varphi}} \\ &= \frac{e^{i\varphi} - e^{i\varphi_0}}{2ie^{i\varphi}} \cdot \frac{g'_1(\varphi)[\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)] - \tilde{g}'_1(\varphi)[g_1(\varphi) - g_1(\varphi_0)]}{[g_1(\varphi) - g_1(\varphi_0)][\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)]} = \frac{e^{i\varphi} - e^{i\varphi_0}}{2ie^{i\varphi}} \\ &\quad \times \frac{(\varphi - \varphi_0) \left[g'_1(\varphi) \int_0^1 \tilde{g}'_1[\varphi_0 + u(\varphi - \varphi_0)] du - \tilde{g}'_1(\varphi) \int_0^1 g'_1[\varphi_0 + u(\varphi - \varphi_0)] du \right]}{[g_1(\varphi) - g_1(\varphi_0)][\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)]}, \end{aligned}$$

by virtue of I, IV, (11) and (12) this expression obviously implies that inequality II is valid.

Just in the same way, after carrying out analogous transformations, from

$$\frac{g_1(\varphi) - g_1(\varphi_0)}{\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)} - \frac{\tilde{g}_1(\varphi) - \tilde{g}_1(\varphi_0)}{\tilde{\tilde{g}}_1(\varphi) - \tilde{\tilde{g}}_1(\varphi_0)} = \frac{(\varphi - \varphi_0)^2}{[\bar{g}_1(\varphi) - \bar{g}_1(\varphi_0)][\tilde{\tilde{g}}_1(\varphi) - \tilde{\tilde{g}}_1(\varphi_0)]} \times \left[\int_0^1 g_1'[\varphi_0 + u(\varphi - \varphi_0)]du \int_0^1 \tilde{\tilde{g}}_1[\varphi_0 + u(\varphi - \varphi_0)]du - \int_0^1 \tilde{\tilde{g}}_1'[\varphi_0 + u(\varphi + \varphi_0)]du \int_0^1 \bar{g}_1[\varphi_0 + u(\varphi - \varphi_0)]du \right]$$

follows the validity of inequality III.

Inequality V immediately follows from inequalities I, II and IV.

The validity of I–V is proved. \square

Now taking into account formulas (6) and (5') for $M_{11}(\varphi, \varphi_0)$, $S_{11}(\varphi, \varphi_0)$ and analogous formula for $\tilde{M}_{11}(\varphi, \varphi_0)$, $\tilde{S}_{11}(\varphi, \varphi_0)$, we can state by virtue of the above inequalities and $S_{11}(\varphi_0, \varphi_0) = \tilde{S}_{11}(\varphi_0, \varphi_0)$ that

$$\begin{aligned} \|S_{11}(\varphi, \varphi_0) e^{i \arg(e^i \varphi - e^i \varphi_0)} - \tilde{S}_{11}(\varphi, \varphi_0) e^{i \arg(e^i \varphi - e^i \varphi_0)}\|_{c_\alpha} &\leq B_1 \cdot \varepsilon, \\ \|M_{11}(\varphi, \varphi_0) - \tilde{M}_{11}(\varphi, \varphi_0)\|_{c_\alpha} &\leq B_2 \cdot \varepsilon, \end{aligned} \tag{13}$$

where the constants B_1 and B_2 depend only on the domain G .

By virtue of (9) and (13), inequality (10) immediately gives rise to the following estimates of its individual terms:

$$\begin{aligned} \|K_{11}^{(1)}(\varphi, \varphi_0) - \tilde{K}_{11}^{(1)}(\varphi, \varphi_0)\|_{c_\delta} &\leq N_1 \cdot \varepsilon, \\ \|K_{11}^{(2)}(\varphi, \varphi_0) - \tilde{K}_{11}^{(2)}(\varphi, \varphi_0)\|_{c_\delta} &\leq N_2 \cdot \varepsilon, \\ \|\tilde{K}_{11}^{(1)}(\varphi, \varphi_0)\|_{c_\delta} &\leq N_3, \quad \|\tilde{K}_{11}^{(2)}(\varphi, \varphi_0)\|_{c_\delta} \leq N_4, \end{aligned} \tag{14}$$

($\delta = \min\{\alpha - \beta; \alpha + \beta - 1\}$), where all constants are expressed in terms of the initial domain G , i.e., for small ε the estimates are uniform with respect to domains $\tilde{G} \in G_\varepsilon$.

Therefore for the kernels represented by formulas (7') we have

$$\|K_{11}(\varphi; \varphi_0) - \tilde{K}_{11}(\varphi; \varphi_0)\|_{c_\delta} \leq N_1(\beta) \cdot \varepsilon. \tag{15}$$

The estimates

$$\begin{aligned} \|K_{22}(\varphi; \varphi_0) - \tilde{K}_{22}(\varphi; \varphi_0)\|_{c_\delta} &\leq N_2(\beta) \cdot \varepsilon, \\ \|K_{00}(\varphi; \varphi_0) - \tilde{K}_{00}(\varphi; \varphi_0)\|_{c_\delta} &\leq N_0(\beta) \cdot \varepsilon \end{aligned} \tag{16}$$

are established analogously.

If the points $t(\varphi)$, $t(\varphi_0)$ and the corresponding points $\tilde{t}(\varphi)$, $\tilde{t}(\varphi_0)$ lie on different curves Γ_i , Γ_j and $\tilde{\Gamma}_i$, $\tilde{\Gamma}_j$ ($i \neq j$), respectively, then estimates for the values $\|K_{ij} - \tilde{K}_{ij}\|_{c_\delta}$ ($i \neq j$, $i, j = 0, 1, 2$) are established immediately and have order

$O(\varepsilon)$. This can be verified at once if, for the kernels of equations (3₁), (4₁) representable by formulas of form (5), the difference

$$K_*[g_i(\varphi), g_j(\varphi_0)] - \widetilde{K}_*[\widetilde{g}_i(\varphi); \widetilde{g}_j(\varphi_0)]$$

is reduced to the common denominator and the latter is estimated from below by the number $d_0 = \min\{d_1; d_2; d_3\}$ assuming that $0 < \varepsilon < \frac{d_0}{2}$, while the difference of their numerators is estimated using inequality (1).

We have thus shown that the following theorem holds for small values of the parameter ε .

Theorem 1. *If the domains G and $\widetilde{G} \in G_\varepsilon$ ($0 < \varepsilon \leq \varepsilon_0$) belong to the class G'_α ($\frac{1}{2} < \alpha \leq 1$), then the inequality*

$$\|K_*(\varphi; \varphi_0) - \widetilde{K}_*(\varphi; \varphi_0)\|_{c_\delta} < A_0(G; \beta) \cdot \varepsilon \tag{17}$$

holds, where $K_*(\varphi; \varphi_0)$, $\widetilde{K}_*(\varphi; \varphi_0)$ are the kernels of the integral equations (3₁), (4₁), respectively, $\delta = \min\{\alpha - \beta; \alpha + \beta - 1\}$ and $\frac{1}{2} < \beta \leq \alpha$. The constant $A_0(G; \beta)$ and small ε_0 are completely defined by giving the initial domain G

$$\varepsilon_0 = \frac{1}{2} \min \{d_0; k\rho/4; g(\varphi)\},$$

where k is the constant defined by giving the contour Γ , $\rho = \min\{\rho(0, \Gamma_j) : j = 0, 1, 2\}$, $g(\varphi) = \min\{|g'_j(\varphi)| : j = 0, 1, 2\}$.

The proven theorem makes it possible to obtain an estimate for the difference $\nu(\varphi) - \widetilde{\nu}(\varphi)$ in an adequate norm. It is of order $O(\varepsilon)$, but it can be obtained in a stronger form if we use the result from [3]. This technique implies estimating, through ε , the difference of the corresponding integral operators.

Let us estimate the difference $(H - \widetilde{H})\nu$. Having in mind the structure of this difference, it suffices to confine the investigation to the case $(H_{11} - \widetilde{H}_{11})\nu_1$.

From (7) we have

$$\begin{aligned} & (H_{11} - \widetilde{H}_{11}) \nu_1 \\ &= \frac{1}{\pi i} \int_0^{2\pi} \widetilde{K}_{11}^*(\varphi; \varphi_0) i e^{i\varphi} \cdot \exp(-i \arg(e^{i\varphi} - e^{i\varphi_0})) \frac{\nu_1(\varphi)}{|e^{i\varphi} - e^{i\varphi_0}|^{1-\beta}}, \end{aligned} \tag{18}$$

where

$$\widetilde{K}_{11}^*(\varphi, \varphi_0) = \frac{S_{11}(\varphi; \varphi_0)M_{11}(\varphi; \varphi_0) - \widetilde{S}_{11}(\varphi; \varphi_0)\widetilde{M}_{11}(\varphi; \varphi_0)}{|e^{i\varphi} - e^{i\varphi_0}|^\beta}. \tag{19}$$

As has already been noted, the function $\widetilde{K}_{11}^*(\varphi; \varphi_0)$ is continuous in the Hölder sense with index $\alpha - \beta$. Taking into account (9) and (19) as well as inequalities I–V, we see that the inequality

$$\|\widetilde{K}_{11}^*(\varphi; \varphi_0)\|_{c_{\alpha-\beta}} \leq C_1(\beta) \varepsilon \tag{20}$$

is valid for small values of the parameter ε .

Let us use the following result from [2]. A function of form (18)

$$\omega(t(\varphi_0)) = \frac{1}{\pi} \int_0^{2\pi} \widetilde{K}_{11}^*(\varphi; \varphi_0) r(\varphi; \varphi_0) \frac{\nu_1(\varphi) d\varphi}{|e^{i\varphi} - e^{i\varphi_0}|^{1-\beta}},$$

where $r(\varphi; \varphi_0) = ie^{i\varphi} \exp(-i \arg(e^{i\varphi} - e^{i\varphi_0}))$ belongs to the class C_δ for any bounded function $\nu_1(\varphi)$ and $\delta = \alpha - \beta$ (see §51 of [2]).

Now, by virtue of this result, (18) and (20) give rise to the estimate

$$\|(H_{11} - \widetilde{H}_{11}) \nu_1\|_{C_{\alpha-\beta}} \leq C'(\beta) \|\nu_1\|_{C_{\alpha-\beta}} \cdot \varepsilon, \tag{21}$$

where $C'(\beta)$ and $C_1(\beta)$ from (20) are the absolute constants, while $\nu(\varphi)$ is any function of the class $C_{\alpha-\beta}$.

The proof that the values $\|(H_{ii} - \widetilde{H}_{ii})\nu_i\|_{C_{\alpha-\beta}}$ for $i = 0, 2$ have order $O(\varepsilon)$ repeats the proof of inequality (21).

The estimate for the values $\|(H_{ij} - \widetilde{H}_{ij})\nu_j\|_{C_{\alpha-\beta}}$ when $i \neq j$, i.e., when the points $t(\varphi)$ and $t_0(\varphi_0)$ belong to different curves Γ_1 and Γ_2 , while the corresponding points $\tilde{t}(\varphi)$ and $\tilde{t}_0(\varphi_0)$ belong to the contours $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, is obtained immediately if it is assumed, for instance, $\varepsilon < \frac{d_0}{2}$. The constants in all such estimates are expressed only in terms of the curves Γ_1 and Γ_2 .

Hence the estimate

$$\|(H - \widetilde{H}) \nu\|_{C_{\alpha-\beta}} \leq C_0(\beta) \cdot \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon, \tag{22}$$

where $C_0(\beta)$ is the absolute constant, is valid.

Assume now that $\nu(\varphi)$ and $\tilde{\nu}(\varphi)$ are solutions of equations (3₁) and (4₁), respectively. From (3'₁) and (4'₁) we have

$$\begin{aligned} \nu - \tilde{\nu} &= \tilde{A}^{-1}(\tilde{A} - A) \nu - \tilde{A}^{-1}(\tilde{f}_0 - f_0) \\ &= \tilde{A}^{-1}(\widetilde{H} - H) \nu - \tilde{A}^{-1}(\tilde{f}_0 - f_0). \end{aligned} \tag{23}$$

But by (22)

$$\begin{aligned} \|A - \tilde{A}\|_{C_{\alpha-\beta}} &= \sup_{\|\nu\|_{C_{\alpha-\beta}}=1} \|(\tilde{A} - A) \nu\|_{C_{\alpha-\beta}} \\ &\leq \sup_{\|\nu\|_{C_{\alpha-\beta}}=1} C_0(\beta) \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon = C_0(\beta) \cdot \varepsilon \end{aligned} \tag{24}$$

and

$$\|(\tilde{A} - A) A^{-1}\|_{C_{\alpha-\beta}} \leq \|A - \tilde{A}\|_{C_{\alpha-\beta}} \cdot \|A^{-1}\|_{C_{\alpha-\beta}} \leq C_0(\beta) \|A^{-1}\|_{C_{\alpha-\beta}} \cdot \varepsilon$$

and if it is assumed that $\varepsilon < 1/(C_0(\beta) \cdot \|A^{-1}\|_{C_{\alpha-\beta}})$, then the norm of the inverse operator \tilde{A}^{-1} [4] is uniformly (with respect to $\tilde{\Gamma}$) bounded in the space $C_{\alpha-\beta}$,

$$\|\tilde{A}^{-1}\|_{C_{\alpha-\beta}} \leq \frac{\|A^{-1}\|_{C_{\alpha-\beta}}}{1 - \|(\widetilde{H} - H)A^{-1}\|_{C_{\alpha-\beta}}}.$$

Using further the obvious estimate $\|f_0 - \tilde{f}_0\|$ and (24), from (23) it follows that the inequality

$$\|\nu(\varphi) - \tilde{\nu}(\varphi)\|_{C_{\alpha-\beta}} < B_0(\beta) \cdot \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon$$

is valid for all $\varepsilon \leq \varepsilon_1$, where

$$\varepsilon_1 = \left\{ \frac{d_0}{2}; \frac{g(\varphi)}{2}; \frac{1}{C_0(\beta)\|A^{-1}\|_{C_{\alpha-\beta}}} \right\},$$

$B_0(\beta)$ is the constant depending only on G and β (β is any positive number smaller than α). Thus we have proved

Theorem 2. *If the boundaries of the domains G and $\tilde{G} \in G_\varepsilon$, $0 < \varepsilon \leq \varepsilon_1$, belong to the class C'_α ($0 < \alpha < 1$), then the inequality*

$$\|\nu(\varphi) - \tilde{\nu}(\varphi)\|_{C_{\alpha-\beta}} < B_0(\beta) \cdot \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon \tag{25}$$

is valid, where $\nu(\varphi)$ and $\tilde{\nu}(\varphi)$ are unique solutions of the integral equations (3₁), (4₁), respectively. The constant $B_0(\beta)$ and small ε_1 are defined by giving the initial domain G ; β is any positive number smaller than α .

The proven theorem solves the problem we have posed. As for estimating the differences $\|r - \tilde{r}\|$ and $\|\rho - \tilde{\rho}\|$, such an estimate is immediately implied by Theorem 2 and formulas (2).

Corollary 1. *If the triply-connected domains G and \tilde{G} are ε -close to each other ($0 < \varepsilon \leq \varepsilon_1$), then the inequalities*

$$\|\rho - \tilde{\rho}\| < Q_1 \cdot \varepsilon; \quad \|r - \tilde{r}\| < Q_2 \cdot \varepsilon \tag{26}$$

are valid, where $\rho, r, \tilde{\rho}, \tilde{r}$ are the radii defining the canonical domains $K(\rho; r; 1)$ and $\tilde{K}(\tilde{\rho}; \tilde{r}; 1)$, respectively, while the constant Q_1 and Q_2 depend only on the domain G .

2. CONFORMAL MAPPING OF CLOSE DOUBLY-CONNECTED DOMAINS

In the complex plane Z let us consider the finite doubly-connected domain G whose boundary Γ consists of the simple closed Lyapunov curves Γ_0 and Γ_1 , one of which Γ_0 envelops the other, and $z = 0 \in \text{int } \Gamma_1$.

Assume that the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of the given domain belongs to the class C'_α ($0 < \alpha < 1$) and is given parametrically by the equations

$$t = g_1(\tau), \quad t = g_0(\tau) \quad (0 \leq \tau \leq 2\pi; \quad g_i(0) = g_i(2\pi), \quad i = 0, 1).$$

Let $d_0 = \rho(\Gamma_1; \Gamma_0)$ and assume that $0 < \varepsilon < d_0/2$.

Consider another doubly-connected domain \tilde{G} of type G .

Definition 2. The domains G and \tilde{G} whose boundary $\tilde{\Gamma}$ consists of $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ whose parametric equations are

$$t = \tilde{g}_1(\tau), \quad t = \tilde{g}_0(\tau) \quad (0 \leq \tau \leq 2\pi; \tilde{g}_i(0) = \tilde{g}_i(2\pi), \quad i = 0, 1)$$

are called ε -close to each other if the conditions

$$|g_i(\tau) - \tilde{g}_i(\tau)| \leq \varepsilon, \quad \|g'_i(\tau) - \tilde{g}'_i(\tau)\|_{C_\alpha} \leq \varepsilon \quad (i = 0, 1) \tag{1'}$$

are fulfilled.

As has already been noted, G_ε denotes a set of domains ε -close to G for any $0 < \varepsilon < d_0/2$.

Let us map conformally (under the assumptions of [5]) the close domains G and \tilde{G} onto the canonical domains $K(\rho; 1)$ and $\tilde{K}(\tilde{\rho}; 1)$, where $K(\rho; 1)$ and $\tilde{K}(\tilde{\rho}; 1)$ are respectively annuli with $\rho < |w| < 1$ and $\tilde{\rho} < |w| < 1$, while the radii ρ and $\tilde{\rho}$ are defined by the formulas

$$\begin{aligned} \ln \rho &= \frac{1}{\pi} \int_0^{2\pi} \nu_1(\tau) |g'_1(\tau)| d\tau, \\ \ln \tilde{\rho} &= \frac{1}{\pi} \int_0^{2\pi} \tilde{\nu}_1(\tau) |\tilde{g}'_1(\tau)| d\tau, \end{aligned}$$

where

$$\nu(t) = \begin{cases} \nu_1(t), & \text{when } t \in \Gamma_1, \\ \nu_0(t), & \text{when } t \in \Gamma_0. \end{cases}$$

Apply an analogous treatment to $\tilde{\nu}(t)$ too. Note that $\nu(t)$ and $\tilde{\nu}(t)$ are solutions of integral equations of form (3₁) and (4₁), respectively, derived for the doubly-connected domains G and \tilde{G} .

Using the methods from Section 1 one can similarly obtain an estimate for the norm $\|\nu - \tilde{\nu}\|_{C_{\alpha-\beta}}$ of difference of solutions of integral equations. Namely, we have

Theorem 3. *If the boundaries of doubly-connected domains G and $\tilde{G} \in G_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, belong to the class C'_α ($\frac{1}{2} < \alpha < 1$), then the inequalities*

$$\begin{aligned} \|\nu(\tau) - \tilde{\nu}(\tau)\|_{C_{\alpha-\beta}} &< B'_0(\beta) \cdot \|\nu\|_{C_{\alpha-\beta}} \cdot \varepsilon, \\ |\ln \rho - \ln \tilde{\rho}| &< Q_0 \cdot \varepsilon \end{aligned}$$

are valid, where $0 < \beta < \alpha$ and the constants $B'_0(\beta)$, Q_0 and ε_0 are defined by giving the initial domain G .

These estimates allow us to construct, with the aid of the function $\nu(t)$ defined by giving the initial domain G , an approximation to the function $w = \tilde{f}_\nu(z)$ ($\tilde{f}_\nu(\tilde{z}_1) = 1$, $\tilde{z}_1 > 0$, $\tilde{z}_1 \in \tilde{\Gamma}_0$) which maps conformally an arbitrary doubly-connected domain $\tilde{G} \in G_\varepsilon$ ($0 < \varepsilon \leq \varepsilon_0$) onto the canonical domain $K(\tilde{\rho}; 1)$.

By virtue of inequalities (25) and (26), we can regard the function (see [3])

$$w = \tilde{f}_\nu(z) = z \cdot \exp \left(\frac{1}{\pi i} \int_{\tilde{\Gamma}} \frac{\nu(t) dt}{t - z} + ic \right) \tag{27}$$

($\tilde{f}_\nu(z_1) = 1, z_1 > 0, z_1 \in \tilde{\Gamma}_0$), where $\nu(t)$ is a solution of equation (3₁), as an approximation to the function $w = \tilde{f}_\nu(z)$. By the proof of the Plemelj–Privalov theorem [2] (§18) the use of (25) and (26) leads to

Theorem 4. *The function $w = \tilde{f}_\nu(z)$ ($\tilde{f}_\nu(z_1) = 1, z_1 > 0, z_1 \in \tilde{\Gamma}_0$) given in the doubly-connected domain \tilde{G} by formula (27), where $\tilde{G} \in G_\varepsilon$ ($0 < \varepsilon \leq \varepsilon_1^*$), admits, in \tilde{G} , an estimate*

$$|\tilde{f}_\nu(z) - \tilde{f}_\nu(z)| < P \cdot \varepsilon,$$

where P depends only on the domain G .

Note that it is assumed here that the given boundary points z_1 and \tilde{z}_1 correspond to one and the same value of the parameter τ (say, $\tau = 0$).

3. TO THE QUASICONFORMAL MAPPING OF CLOSE DOUBLY-CONNECTED DOMAINS

Let us consider the problem of quasiconformal mapping of close domains. As a construction tool we take the method of integral equations [5], which stipulates the knowledge of the concrete global homeomorphism of the Beltrami equation

$$\begin{aligned} W_{\bar{z}} &= q(z) \cdot W_z, \\ |q(z)| &\leq q_0 < 1, \end{aligned} \tag{28}$$

constructed by I. N. Vekua’s scheme [6]. This homeomorphism figures in the kernels of integral equations whose solutions are used to construct the wanted functions.

Assume that the coefficient $q(z)$ of the Beltrami equation is given in some doubly-connected domain G_0 containing the initial domain G and all domains $\tilde{G} \in G_\varepsilon$ which are ε -close to G (in the sense of (1')). As G_0 we can take, for instance, an annulus $\frac{\rho_0}{2} < |z| < R_0$, where $\rho_0 = \rho(0; \Gamma_1), R_0 = \max_{t_0 \in \Gamma_0} \rho(0; t_0)$.

Assume further that the boundary of the domain G belongs to the class C'_α ($0 < \alpha < 1$), and $q(z) \in C'_\gamma(\tilde{G}_0)$, ($0 < \gamma < 1$), and the so-called Vekua basic homeomorphism $\tilde{W}(z)$ of equation (28) is constructed with the coefficient

$$\tilde{q}(z) = \begin{cases} q(z), & \text{when } z \in \tilde{G}_0, \\ 0, & \text{when } z \text{ lies outside } G_0. \end{cases}$$

In that case $\tilde{q}(z)$ belongs to any Lebesgue class $L_p(E)$ (where E is the entire complex plane) and, according to [7], $\tilde{W}_{\bar{z}}$ and \tilde{W}_z satisfy the Hölder condition with index γ_0 and $0 < \gamma_0 < \min\{\alpha; \gamma\}$. In what follows this global homeomorphism of equation (28) is denoted by $\tilde{W}_{G_0}(z)$.

We use the technique of exit in the plane of this homeomorphism. Then the integral equations (3₁) and (4₁) take the form

$$\begin{aligned} \mu(\xi_0) + \frac{1}{2\pi i} \int_{\widetilde{W}_{G_0}(\Gamma)} \frac{1}{\xi'} K_*(\xi; \xi_0) \mu(\xi) d\xi \\ = -\ln |\xi_0 - \widetilde{W}_{G_0}(0)|, \quad \xi_0 \in \widetilde{W}_{G_0}(\Gamma), \end{aligned} \tag{29}$$

$$\begin{aligned} \tilde{\mu}(\xi_0) + \frac{1}{2\pi i} \int_{\widetilde{W}_{G_0}(\tilde{\Gamma})} \frac{1}{\xi'} \widetilde{K}_*(\xi; \xi_0) \tilde{\mu}(\xi) d\xi \\ = -\ln |\xi_0 - \widetilde{W}_{G_0}(0)|, \quad \xi_0 \in \widetilde{W}_{G_0}(\tilde{\Gamma}), \end{aligned} \tag{30}$$

where $\xi = \widetilde{W}_{G_0}[g_k(\tau)]$, $\xi = \widetilde{W}_{G_0}[\tilde{g}_k(\tau)]$ ($k = 0, 1$) are the parametric equations of the curves $\widetilde{W}_{G_0}(\Gamma_k)$, $\widetilde{W}_{G_0}(\tilde{\Gamma}_k)$, respectively. $K_*(\xi; \xi_0) = K_*[\widetilde{W}_{G_0}[g_k(\tau); g_l(\tau_0)]]$ ($k, l = 0, 1$). In an analogous manner we define $\widetilde{K}_*(\xi; \xi_0)$. Moreover, $[\widetilde{W}_{G_0}(z)]_{\bar{z}} \in C_{\gamma_0}(\bar{G}_0)$ [7] and $\mu[\xi(\tau)]$, $\tilde{\mu}[\xi(\tau)] \in C_{\gamma_0}[0; 2\pi]$. We can also write that $\mu(\xi_0) = \nu[\widetilde{W}_{G_0}^{-1}(g(\tau_0))]$.

By virtue of (1') we have

$$\begin{aligned} |\widetilde{W}_{G_0}[g_j(\tau)] - \widetilde{W}_{G_0}[\tilde{g}_j(\tau)]| &\leq C_j(\widetilde{W}_{G_0}; G_0) |g_j(\tau) - \tilde{g}_j(\tau)| \\ &< C_j(\widetilde{W}_{G_2}; G_0) \cdot \varepsilon \quad (j = 0, 1). \end{aligned} \tag{31}$$

For convenience, denote $\widetilde{W}_{G_0}(z) = \widetilde{W}(z)$.

We have

$$\begin{aligned} (\widetilde{W}[g_j(\tau)])'_\tau &= \widetilde{W}_t[g(\tau)](g'_j(\tau))_\tau + \widetilde{W}_{\bar{t}}[g_j(\tau)](\tilde{g}_j(\tau))'_\tau, \\ (\widetilde{W}[\tilde{g}_j(\tau)])'_\tau &= \widetilde{W}_t[\tilde{g}(\tau)](\tilde{g}'_j(\tau))_\tau + \widetilde{W}_{\bar{t}}\tilde{g}_j(\tau)'_\tau \end{aligned}$$

and

$$(\widetilde{W}[g_j])'_\tau \in C_{\gamma_0}[0, 2\pi], \quad (\widetilde{W}[\tilde{g}_j(\tau)])'_\tau \in C_{\gamma_0}[0; 2\pi]; \quad (j = 0, 1).$$

Compose the difference

$$\begin{aligned} (\widetilde{W}[g_j(\tau)])'_\tau - (\widetilde{W}[\tilde{g}_j(\tau)])'_\tau &= \widetilde{W}_t(g_j)(g'_j)_\tau - \widetilde{W}_t(g_j)(\tilde{g}'_j)_\tau - \widetilde{W}_{\bar{t}}(\tilde{g}_j)(\tilde{g}_j)'_\tau \\ &\quad + \widetilde{W}_t(g_j)(\tilde{g}'_j)_\tau + \widetilde{W}_{\bar{t}}(g_j)(\tilde{g}_j)'_\tau - \widetilde{W}_{\bar{t}}(g_j)(\tilde{g}_j)'_\tau \\ &\quad - \widetilde{W}_{\bar{t}}(\tilde{g}_j)(\tilde{g}_j)'_\tau + \widetilde{W}_{\bar{t}}(g_j)(\tilde{g}_j)'_\tau \quad (j = 0, 1). \end{aligned}$$

Using (1') in these expressions and taking into account the inequalities

$$\begin{aligned} \|g'_j(\tau) - \tilde{g}'_j(\tau)\|_{C_{\gamma_0/2}} &\leq \text{const}_{j1} \cdot \|g'_j(\tau) - \tilde{g}_j'(\tau)\|_{C_\alpha}, \\ \|\tilde{g}'_j(\tau) - \tilde{g}_j'(\tau)\|_{C_{\gamma_0/2}} &\leq \text{const}_{j2} \cdot \|\tilde{g}'_j(\tau) - \tilde{g}_j'(\tau)\|_{C_\alpha}, \end{aligned}$$

we obtain

$$\begin{aligned} \|\widetilde{W}_t(g_j) - \widetilde{W}_t(\tilde{g}_j)\|_{C_{\gamma_0/2}} &\leq C_{2j}^*(\widetilde{W}_{G_0}; G_0) |g_j(\tau) - \tilde{g}_j(\tau)|^{\gamma_0/2} \\ &\leq C_{3j}^* \varepsilon^{\gamma_0/2} \quad (j = 0, 1). \end{aligned} \tag{32}$$

We see that the closeness conditions (1') of the contours Γ_j and $\tilde{\Gamma}_j$ in the plane of the homeomorphism $\tilde{W}_{G_0}(z)$ for the corresponding curves $\tilde{W}_{G_0}(\Gamma_j)$ and $\tilde{W}_{G_0}(\tilde{\Gamma}_j)$ are replaced by conditions (31) and (32).

Now, by virtue of the estimate established in Theorem 3, we come to the validity of the following proposition.

Theorem 5. *If the doubly-connected domains $G \subset G_0$ and $\tilde{G} \subset G_0$ whose boundaries belong to the class C'_α ($0 < \alpha < 1$) are ε -close to each other in the sense of (1'), then the estimate*

$$\|\mu(\xi) - \tilde{\mu}(\xi)\|_{C_{\frac{\gamma_0}{2}-\beta}} < Q^*(\tilde{W}_{G_0}; G; \beta) \cdot \|\mu\|_{C_{\frac{\gamma_0}{2}-\beta}} \cdot \varepsilon^{\frac{\gamma_0}{2}} \quad (33)$$

holds for all $\varepsilon \in [0; \varepsilon^*]$; here $\mu(\xi)$ and $\tilde{\mu}(\xi)$ are unique solutions of the integral equations (29) and (30), respectively, β is any positive number smaller than $\gamma_0/2$, the constant Q^* and small ε^* are completely defined by giving the initial domain G and the homeomorphism $\tilde{W}_{G_0}(z)$.

Note that in Theorem 5 the order of smallness for ε can be obtained arbitrarily close to γ_0 [3], for instance, $O(\varepsilon^{\gamma_0-\eta})$, where $\eta < \gamma_0$ is any positive number, but in that case we can estimate only "small" norms of the value $\|\mu(\xi) - \tilde{\mu}(\xi)\|_{C_\eta}$ (the Hölder index η decreases). In the considered situation the choice $\eta = \frac{\gamma_0}{2}$ seems optimal to us.

In conclusion, also note that, analogously to the conformal case, the estimate in terms of ε for the difference of modules $\ln \rho(\tilde{q}) - \ln \tilde{\rho}(\tilde{q})$ calculated by [8] can be established with the aid of estimate (33).

REFERENCES

1. L. ZIVZIVADZE, On conformal and quasiconformal transformations of triply connected fields. *Bull. Georgian Acad. Sci.* **163**(2001), No. 2, 226–229.
2. N. I. MUSKHELISHVILI, Singular integral equations. Boundary value problems of the theory of functions and some of their applications in mathematical physics. (Russian) 3rd ed. *Nauka, Moscow*, 1968; English translation from 1st Russian ed. (1946): *P. Noordhoff, Groningen*, 1953, corrected reprint *Dover Publications, Inc.*, N. Y., 1992.
3. Z. SAMSONIA and I. SAMKHARADZE, On quasiconformal mappings corresponding to the Beltrami equation. (Russian) *Ukrain. Mat. Zh.* **51**(1999), No. 10, 1391–1397.
4. V. TRENIGIN, Functional analysis. (Russian) *Nauka, Moscow*, 1980.
5. D. KVESELAVA, On the application of integral equations in the theory of conformal mappings. (Russian) *Trudy Vychisl. Centra Akad. Nauk Gruzin. SSR* **2**(1961), 3–15.
6. I. VEKUA, Generalized analytic functions. (Russian) *Nauka, Moscow*, 1988; English translation from the 1st Russian edition: *Pergamon Press, Oxford-London-New York-Paris*, 1962; *Addison-Wesley, Reading, Mass.-London*.
7. G. MANDZHAVIDZE, A boundary value problem of linear conjugation with displacements and its connection to the theory of generalized analytic functions. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **33**(1967), 82–87.
8. D. KVESELAVA and Z. SAMSONIA, On the quasiconformal mapping of domains. *Metric Problems of the Function Theory (Russian)*, 53–65, *Naukova Dumka, Kiev*, 1980.

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Author's address:

L. Zivzivadze
A. Tsereteli Kutaisi State University
55, Queen Tamar St., Kutaisi 384000
Georgia