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PERIODIC POINTS AND CHAOTIC-LIKE DYNAMICS OF PLANAR MAPS ASSOCIATED TO NONLINEAR HILL'S EQUATIONS WITH INDEFINITE WEIGHT

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Abstract. We prove some results about the existence of fixed points, periodic points and chaotic-like dynamics for a class of planar maps which satisfy a suitable property of "arc expansion" type. We also outline some applications to the nonlinear Hill's equations with indefinite weight.

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1. Introduction

The study of the existence of oscillatory and periodic solutions of the nonlinear scalar second-order differential equation

$$\ddot{x} + q(t)g(x) = 0 \tag{1.1}$$

has motivated a great deal of research activities during the last fifty years.

Here and henceforth, we suppose that $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and denote by

$$G(s) := \int_0^s g(\xi)\, d\xi$$

a primitive of g. The "weight" function $q : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous. Further (mild) conditions on $q(t)$ will be added later, if necessary. In the sequel, we will also confine our discussion to the situation in which $q(t)$ changes its sign.

In [69], Waltman initiated the study of the oscillatory solutions of the nonlinear Hill's equation

$$\ddot{x} + q(t)x^{2n+1} = 0, \quad n \geq 1, \tag{1.2}$$

when the coefficient $q(t)$ is not necessarily of definite sign. Contributions to this problem were also given by Kiguradze [36], Wong [76], Bobisud [10], Onose [48], Butler [12], [13], Kwong and Wong [39] and others. The interest in investigating the oscillatory (or nonoscillatory) solutions of equation (1.2) or equation (1.1)

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for a sign indefinite weight is witnessed by a broad literature (see, e.g., [77], [78] and the references therein).

In the case of a periodic $q(\cdot)$, Butler in [14] first obtained the existence of infinitely many periodic solutions to equation (1.1) for a function $g(x)$ having a superlinear growth at infinity, thus including the case of equation (1.2). Recent developments along Butler's work have been obtained in [49], [50], [51].

In the past ten years, starting with Lassoued [40], the study of boundary value problems for superlinear equations with an indefinite weight has received much attention in the literature. Besides the case of partial differential equations that we don't discuss here (see, e.g., [1], [7], [8]), general results have been obtained for the periodic solutions of Hamiltonian systems and, in particular, second order vector differential equations of the form $\ddot{u} + \nabla_u V(t,u) = 0$, using critical point theory (see [5], [6], [22], [25], [26], [43]). Other applications of variational methods to the periodic problem for the equation

$$\ddot{x} + q(t)|x|^{p-1}x = 0 \tag{1.3}$$

are contained in [41] and [57].

In [68], Terracini and Verzini showed that, for a sign changing weight, equation (1.2) (as well as its "forced" extension) may possess solutions which oscillate in a quite "wild" sense. In fact, in [68] the authors gave evidence of the possibility of a chaotic behavior for the solutions of (1.2), by proving that there is a double-sided sequence of positive integers m_i^* $(i \in \mathbb{Z})$, corresponding to a suitably labeled sequence of intervals J_i^+ in which $q > 0$, such that, for every $m_i \geq m_i^*$, there are solutions having m_i zeros in the i-th interval of positivity J_i^+. These solutions vanish once in the intervals where $q < 0$ (see [68] for the precise statement).

In [52], we considered some boundary value problems (including the Sturm-Liouville boundary conditions) for equation (1.1) with $q(t)$ a sign changing function defined in a interval and satisfying suitable assumptions in order to have, like in [14], the continuability of the solutions across the sub-intervals where $q(t) \geq 0$. Therein, under the hypothesis that

$$(g_0) \qquad g(s)s > 0 \quad \text{for} \quad s \neq 0$$

and assuming also that g satisfies a condition of superlinear growth at infinity, we proved the existence of solutions having an arbitrarily large number of zeros in the intervals where $q(t) > 0$ and having precisely one zero or no zeros at all (according to any sequence of 1's and 0's fixed in advance) in the intervals where $q(t) < 0$. Further results in this direction and for the superlinear indefinite case have been obtained in [16], [50], [51]. In particular, a detailed investigation concerning the existence of chaotic-like oscillatory solutions of (1.1) is contained in the recent article [16] by Capietto, Dambrosio and Papini.

With this respect, the aim of this paper is not that of reproducing the argument which has been already used in [16], yet we would like first to focus our attention to the properties of the planar transformations which are associated to the solutions of equation (1.1) and then to prove the existence of a chaotic-like dynamics for a broad class of planar maps which emulate the Poincaré map of (1.1).

The key properties of the solutions of (1.1) which were used in the proof of the main results in [52] are described in the following lemmas taken from [52, Lemma 3, Lemma 4], that we recall now for the reader's convenience.

For the purposes of the present work it is not necessary to repeat here the precise assumptions of superlinear growth at infinity which were considered in [52]. For simplicity in the exposition, we restrict ourselves to the special case of

$(g_{\rm mon})$ $\qquad g$ is monotone nondecreasing in a neighborhood of $\pm\infty$

and

$$(g_\infty) \qquad \lim_{s\to\pm\infty}\frac{g(s)}{s}=+\infty, \qquad \left|\int^{\pm\infty}|G(s)|^{-\frac{1}{2}}\,ds\right|<\infty.$$

Clearly, all the conditions (g_0), $(g_{\rm mon})$ and (g_∞) are satisfied for

$$g(x)=|x|^{p-1}x, \quad p>1,$$

which is the case of equation (1.3). However, we point out that in our results the oddness or the homogeneity of $g(x)$ are not required. More general assumptions for $g(x)$ in terms of time-mappings, like in [23], are given in [52].

Following the notation in [52] we denote by A_i, for $i=1,2,3,4$, the open quadrants of the plane, counted in the counterclockwise sense starting from $A_1=\{(x,y):x>0,y>0\}$. We also denote by $z(\cdot;t_0,z_0)$ the solution of

$$\dot x=y, \quad \dot y=-q(t)g(x) \tag{1.4}$$

satisfying the initial condition $z(t_0)=(x(t_0),y(t_0))=z_0$. We observe that, due to the fact that $q<0$ in some intervals, we have that $z(\cdot;t_0,z_0)$ may be not globally defined for some $z_0\in\mathbb{R}^2$ (see [11], [15]). On the other hand, we assume (like in [14], [52]) the continuability of the solutions across the sub-intervals where $q\geq 0$.

Lemma 1 ([52, Lemma 3])**.** *Let $a<b<c$, be such that*

$$q\geq 0 \ \text{ and } \ q\not\equiv 0 \ \text{ on } \ [a,b], \qquad q\leq 0 \ \text{ and } \ q\not\equiv 0 \ \text{ on } \ [b,c].$$

Then there is a constant R^* (depending only on g and $q|_{[b,c]}$) such that the following holds:

For each $R>0$, there is $n^*=n^*_R>0$ such that, for each $n>n^*$, and for each continuous curve $\gamma:[\alpha,\beta[\,\to\bar A_1$ (respectively, $\gamma:[\alpha,\beta[\,\to\bar A_3$), with

$$|\gamma(\alpha)|\leq R \quad \text{and} \quad |\gamma(s)|\to\infty, \ \text{as} \ s\to\beta^-,$$

there is an interval $[\alpha_n,\beta_n]\subset\,]\alpha,\beta[$ such that for each $s\in\,]\alpha_n,\beta_n]$ we have:

- *$z(t;a,\gamma(s))$ is defined for all $t\in[a,c]$,*
- *$x(\cdot;a,\gamma(s))$ has exactly n zeros in $]a,b[$, no zeros in $[b,c]$ and exactly one change of sign of the derivative in $]b,c[$.*

Moreover, setting

$$\gamma_n(s) := z(c; a, \gamma(s)), \quad \forall\, s \in \,]\alpha_n, \beta_n],$$

we have that

$$|\gamma_n(\beta_n)| \leq R^* \quad \textit{and} \;\; |\gamma_n(s)| \to \infty, \;\textit{as}\;\; s \to \alpha_n^+,$$

and γ_n *lies in* $\bar{A}_1$ *or in* $\bar{A}_3$ *according to the fact that* n *is even or odd (respectively,* γ_n *lies in* $\bar{A}_3$ *or in* $\bar{A}_1$ *according to the fact that* n *is even or odd).*

Lemma 2 ([52, Lemma 4]**).** *Let* $a < b < c$, *be such that*

$$q \geq 0 \;\;\textit{and}\;\; q \not\equiv 0 \;\;\textit{on}\;\; [a, b], \quad q \leq 0 \;\;\textit{and}\, q \not\equiv 0 \;\;\textit{on}\; ; [b, c].$$

Then there is a constant R^* *(depending only on* g *and* $q|_{[b,c]}$*) such that the following holds:*

For each $R > 0$, *there is* $n^* = n^*_R > 0$ *such that, for each* $n > n^*$, *and for each continuous curve* $\gamma : [\alpha, \beta[\, \to \bar{A}_1$ *(respectively,* $\gamma : [\alpha, \beta[\, \to \bar{A}_3$*), with*

$$|\gamma(\alpha)| \leq R \quad \textit{and} \;\; |\gamma(s)| \to \infty, \;\textit{as}\;\; s \to \beta^-,$$

there is an interval $[\alpha_n, \beta_n] \subset \,]\alpha, \beta[$ *such that for each* $s \in [\alpha_n, \beta_n[$ *we have:*

- $z(t; a, \gamma(s))$ *is defined for all* $t \in [a, c]$,
- $x(\cdot; a, \gamma(s))$ *has exactly* n *zeros in* $]a, b[$, *exactly one zero in* $]b, c[$ *and no zeros of the derivative in* $[b, c]$.

Moreover, setting

$$\gamma_n(s) := z(c; a, \gamma(s)), \quad \forall\, s \in [\alpha_n, \beta_n[\,,$$

we have that

$$|\gamma_n(\alpha_n)| \leq R^* \quad \textit{and} \;\; |\gamma_n(s)| \to \infty, \;\textit{as}\;\; s \to \beta_n^-$$

and γ_n *lies in* $\bar{A}_3$ *or in* $\bar{A}_1$ *according to the fact that* n *is even or odd (respectively,* γ_n *lies in* $\bar{A}_1$ *or in* $\bar{A}_3$ *according to the fact that* n *is even or odd).*

Let us define, for $0 < r_1 < r_2$, the closed annulus

$$\mathcal{A}[r_1, r_2] := \{z \in \mathbb{R}^2 : \; r_1 \leq |z| \leq r_2\}$$

and, for $r > 0$, the circumference $C_r := \{z \in \mathbb{R}^2 : \; |z| = r\}$. Now, from Lemma 1 and Lemma 2 we can obtain the following.

Proposition 1. *Assume* (g_0), (g_{mon}) *and* (g_∞). *Let* $a < b < c$, *be such that*

$$q \geq 0 \;\;\textit{and}\;\; q \not\equiv 0 \;\;\textit{on}\;\; [a, b], \quad q \leq 0 \;\;\textit{and}\;\; q \not\equiv 0 \;\;\textit{on}\;\; [b, c]$$

and suppose that the solutions of (1.1) *are continuable across* $[a, b]$. *Then there is a constant* R^* *(depending only on* g *and* $q|_{[b,c]}$*) such that for each* $R \geq R^*$, *there is* $n^* = n^*_R > 0$ *such that for each* $j > n^*$ *there is* $R_j > R$ *in order to satisfy the following arc expansion property:*

Let us fix $i = 1, 3$ *and* $\delta = 0, 1$. *Given any path* σ *contained in* $\bar{A}_i \cap \mathcal{A}[R, R_j]$ *and such that*

$$\sigma \cap C_R \neq \varnothing, \quad \sigma \cap C_{R_j} \neq \varnothing,$$

there is a sub-path σ' *such that, for each* $z_0 \in \sigma'$:

$z(\cdot; a, z_0)$ is defined for all $t \in [a, c]$,

$x(b)x(c)\dot{x}(b)\dot{x}(c) \neq 0,$

$x(\cdot; a, z_0)$ has exactly j zeros in $]a, b[$, exactly δ zeros in $]b, c[$ and exactly $1 - \delta$ changes of sign of the derivative in $]b, c[$.

Moreover, for the map $\phi : z_0 \mapsto z(c; a, z_0)$, we have that:

$$\phi(\sigma') \cap C_R \neq \varnothing, \quad \phi(\sigma') \cap C_{R_j} \neq \varnothing$$

and

$$\phi(\sigma') \subset \bar{A}_\ell \cap \mathcal{A}[R, R_j],$$

with $\ell = i$ or $\ell = i + 2$ (mod. 4) according to the fact that $j + \delta$ is even or odd.

Proof (sketched). The proof of Proposition 1 is a straightforward consequence of Lemma 1 and Lemma 2 using the following two facts.

First, by our assumptions, we have the continuability of the solutions across the intervals of positivity of $q(t)$. From this and the fact that g is lipschitzean around zero, it follows that solutions with an initial point $z(a) \neq 0$ in a compact set have a number of zeros in $[a, b]$ which is (uniformly) upper bounded. Indeed, for the number of zeros $n(x; a, b)$ of $x(\cdot)$ in $[a, b[$, we have that $\lfloor 2\mathrm{rot}(z; a, b) \rfloor \leq n(x; a, b) \leq 2\mathrm{rot}(z; a, b) + 1$ where

$$\mathrm{rot}(z; a, b) = \frac{1}{2\pi} \int_a^b \frac{y(t)^2 + q(t)g(x(t))x(t)}{y(t)^2 + x(t)^2}\, dt.$$

Second, thanks to the assumption of superlinear growth at infinity for $g(x)$, we know (see, e.g., [14], [29], [33], [44], [65]) that in the interval of positivity of $q(t)$ solutions having at most j zeros are uniformly bounded (by a constant depending on j) in the C^1-norm on $[a, b]$.

By virtue of these facts, we can "cut" the paths γ and γ_n of Lemma 1 and Lemma 2 in order to reduce the corresponding results to compact sets, provided that the number of the zeros of the solutions is sufficiently large but bounded.

As to the precise dynamics of the trajectories we have that the following four situations are possible for a solution $z(\cdot)$ with $z(a)$ belonging to a suitable sub-path σ' of σ in $\bar{A}_i \cap \mathcal{A}[R, R_j]$:

if j is even and $\delta = 0$, we have that $x(\cdot)$ has exactly j-zeros in $]a, b[$, with $z(b) \in A_{i-1}$ (mod. 4), $x(t) \neq 0$ for all $t \in [b, c]$ and $\dot{x}$ changes sign exactly once in $]b, c[$, with $z(c) \in A_i$;

if j is even and $\delta = 1$, we have that $z(\cdot)$ has exactly j-zeros in $]a, b[$, with $z(b) \in A_{i-1}$ (mod. 4), $\dot{x}(t) \neq 0$ for all $t \in [b, c]$ and x vanishes exactly once in $]b, c[$, with $z(c) \in A_{i+2}$ (mod. 4);

if j is odd and $\delta = 0$, we have that $z(\cdot)$ has exactly j-zeros in $]a, b[$, with $z(b) \in A_{i+1}$, $x(t) \neq 0$ for all $t \in [b, c]$ and $\dot{x}$ changes sign exactly once in $]b, c[$, with $z(c) \in A_{i+2}$ (mod. 4);
if j is odd and $\delta = 1$, we have that $z(\cdot)$ has exactly j-zeros in $]a, b[$, with $z(b) \in A_{i+1}$, $\dot{x}(t) \neq 0$ for all $t \in [b, c]$ and x vanishes exactly once in $]b, c[$, with $z(c) \in A_i$.

With respect to Lemma 1 and Lemma 2 we have added here also the information that $z(c)$ is not on the coordinate axes. This would follow by reworking along the argument in [52], or by a corresponding remark in [51].

Note that from $\dot{x} = y$ and by an elementary analysis of the direction of the vector field in system (1.4), it follows that all the zeros of $x(\cdot)$ are simple and that (roughly) the transition from a quadrant to another is of the type $A_2 \to A_1$, $A_4 \to A_3$ and, moreover, $A_1 \to A_4$, $A_3 \to A_2$ for $t \in]a, b[$, $A_4 \to A_1$, $A_2 \to A_3$ for $t \in]b, c[$. This way, we have a complete control of the position of $z(c)$ according to the number of the zeros of the solution. In particular, from the above four cases, we have also that $\phi(z_0) \in A_i$ or $\phi(z_0) \in A_{i+2}$ (mod. 4) according to the fact that $j + \delta$ is even or odd. Hence the conclusion follows. □

Now, our aim will be that of showing that the arc expansion property described in Proposition 1 implies the existence of fixed points and of periodic points of arbitrary order for the Poincaré map associated to (1.1) as well as of a chaotic-like dynamics of coin tossing type [37]. [1] This goal will be achieved by means of some theorems about the dynamics of a planar map which entails some properties of the Poincaré map associated to the nonlinear Hill's equations with indefinite weight. We stress that, even if in the present article we have made our main reference to the case of a restoring field $g(x)$ having superlinear growth at infinity (like in [16] and in [52]), in principle (and due to the more general setting of planar maps instead of that of a special class of differential equations) our results can be applied to a wide family of nonlinear scalar ordinary differential equations, including some cases of asymptotically linear systems (provided that a suitable gap between zero and infinity is assumed). More details and further results in this direction will be given in [53].

At the end of this introductory section, we should also mention two recent articles by Kennedy and Yorke [35] and Kennedy, Koçak and Yorke [34] in which the authors develop a general topological framework in order to deduce the presence of chaotic dynamics for a broad variety of different situations. In particular, in [34], the authors define the concept of a "family of expanders" and it seems that it should be possible to show that some of the results following from the arc expansion property of Proposition 1 (and that we are going to develop in the next sections) are likely to be obtainable as a consequence of the results [34] or [35]. On the other hand, in our setting, we are able to prove the existence

[1] Recall that, according to [37], a discrete dynamical system (X, ψ) is chaotic in the sense of coin tossing, if there are two disjoint compact sets X_0 and X_1 of X such that, given any two-sided sequence $(\dots, s_{-2}, s_{-1}, s_0, s_1, s_2, \dots)$, with $s_i \in \{0, 1\}$, there is an orbit $(x_n)_n$ of ψ with $x_n \in X_{s_n}$ for each $n \in \mathbb{Z}$.

of fixed points and of periodic points, via an elementary fixed point theorem for planar maps (see Theorem 1 and Theorem 2[case (w_2)]). It is therefore our hope that the approach proposed in the present paper (even if limited to the study of a two-dimensional map) may be of some independent interest as well. In the last section, we try to describe the presence of an underlying general topological structure for our setting, in order to make more transparent the connections between our results for the nonlinear Hill equation and some arguments which belongs to the recent research on "topological horseshoes".

Throughout the paper, we recall that by an *arc* (respectively, a *path*) we mean the homeomorphic image (respectively, the image through a continuous map) of a compact nondegenerate interval of the real line. A *continuum* is a compact connected set. In the sequel, all the arcs, paths and continua that will be considered are subsets of the plane $\mathbb{R}^2$. It will be also useful to have in mind that a plane continuum is not necessarily a path and that each path joining two different points contains an arc joining the same two points [24], [79]. Moreover, if $\Gamma \subset \mathbb{R}^2$ is a continuum, then, for any two points $P, Q \in \Gamma$, with $P \neq Q$ and any $\varepsilon > 0$, there is a path (or, if we like, an arc) $\sigma = \sigma^{\varepsilon}_{(P,Q)}$ joining P with Q and contained in the ε-neighborhood $B(\Gamma, \varepsilon)$ of Γ. This property clearly allows to suitably approximate a continuum with arcs.

As a last definition, we call a *two-dimensional cell* $\mathcal{R} \subset \mathbb{R}^2$ the homeomorphic image of the square $Q = [-1, 1]^2$.

Finally, and in order to avoid misunderstanding, we notice that we denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of positive integers and when we speak of a map ψ we implicitly assume that it is a continuous one (unless when the contrary is explicitly mentioned).

2. Fixed Points and Dynamics of Planar Maps

We start this section with some results on the existence of fixed points for a (continuous) map $\psi : \mathbb{R}^2 \supset \mathrm{dom}(\psi) := D_\psi \to \mathbb{R}^2$.

Let $v = (v_1, v_2)$ and $w = (w_1, w_2)$ be two unit vectors of the plane with $v \neq w$. We denote by $\ell_v = \{tv : t > 0\}$ and $\ell_w = \{tw : t > 0\}$ the (open) half-lines from the origin through v and w. The 0-*pointed angle* $\widehat{v0w}$ is the union of ℓ_v, ℓ_w with all the half-lines ℓ_u such that the triple (v, u, w) is positively oriented. Given an angle $\widehat{v0w}$ we denote by $\nu_v = (v_2, -v_1)$ and $\nu_w = (-w_2, w_1)$ the outer normals to $\widehat{v0w}$ at the points of ℓ_v and ℓ_w (clearly, these points lie on the boundary of $\widehat{v0w}$).

Note that in the above definition, $0 \notin \widehat{v0w}$. Clearly, the closure of $\widehat{v0w}$ is $\widehat{v0w} \cup \{0\}$, that will be denoted by $\overline{v0w}$.

Lemma 3. *Let $\widehat{v0w}$ be a pointed angle at the origin. Assume that there is a connected set $\Gamma \subset D_\psi \cap \widehat{v0w}$, with $\Gamma \cap \ell_v \neq \varnothing$ and $\Gamma \cap \ell_w \neq \varnothing$ and such that*

$$(\psi(z)|\nu_v) \leq 0, \ \forall\, z \in \Gamma \cap \ell_v \quad \text{and} \quad (\psi(z)|\nu_w) \leq 0, \ \forall\, z \in \Gamma \cap \ell_w \tag{2.5}$$

(*or*

$$(\psi(z)|\nu_v) \geq 0,\ \forall\, z \in \Gamma \cap \ell_v \quad \text{ and } \ (\psi(z)|\nu_w) \geq 0,\ \forall\, z \in \Gamma \cap \ell_w\,, \tag{2.6}$$

respectively.) *Then, there is at least one point* $\tilde{z} \in \Gamma$ *such that* $\psi(\tilde{z}) = \lambda\tilde{z}$ *for some* $\lambda \in \mathbb{R}$.

The interested reader is invited to draw a picture in order to visualize the geometrical meaning of conditions (2.5), (2.6).

Proof. Assume (2.5). Let $\psi = (\psi_1, \psi_2)$ and consider the map $h:\ z = (z_1, z_2) \mapsto z_1\psi_2(z) - z_2\psi_1(z)$, for $z \in \Gamma$. If $z \in \Gamma \cap \ell_v$, we have that $z = tv$ for some $t > 0$. Hence, $h(z) = t(v_1\psi_2(tv) - v_2\psi_1(tv)) = -t(\psi(tv)|\nu_v) \geq 0$. With analogous computation, we can see that $h(z) \leq 0$ for $z \in \Gamma \cap \ell_w$. By the continuity of ψ and the connectedness of Γ, we conclude that there is $\tilde{z} \in \Gamma$ such that $h(\tilde{z}) = 0$ (Bolzano theorem). Since $\tilde{z} \neq 0$ (because $0 \notin \Gamma$), this implies that there is a $\lambda \in \mathbb{R}$ such that $\psi(\tilde{z}) = \lambda\tilde{z}$ and the proof is complete. □

Corollary 1. *Let* $\widehat{v0w}$ *be a pointed angle at the origin such that its closure* $\overline{v0w}$ *is convex. Assume that there is a connected set* $\Gamma \subset D_\psi \cap \widehat{v0w}$, *with* $\Gamma \cap \ell_v \neq \varnothing$ *and* $\Gamma \cap \ell_w \neq \varnothing$ *and such that*

$$\psi(\Gamma) \subset \widehat{v0w}. \tag{2.7}$$

Suppose further that

$$\begin{aligned} \psi(z) &\neq \mu v, \qquad \forall\, z \in \Gamma \cap \ell_v\,,\ \forall \mu < 0,\\ \psi(z) &\neq \mu w, \qquad \forall\, z \in \Gamma \cap \ell_w\,,\ \forall \mu < 0, \end{aligned} \tag{2.8}$$

is satisfied. Then, there is at least one point $\tilde{z} \in \Gamma$ *such that* $\psi(\tilde{z}) = \lambda\tilde{z}$ *for some* $\lambda > 0$.

Proof. Since $\overline{v0w}$ is closed and convex, the assumption that $\psi(\Gamma) \subset \widehat{v0w}$, implies that $(\psi(z)|\nu_v) \leq 0$, for all $z \in \Gamma \cap \ell_v$ and $(\psi(z)|\nu_w) \leq 0$, for all $z \in \Gamma \cap \ell_w$. Hence, we can apply Lemma 3 and find $\tilde{z} \in \Gamma$ and $\lambda \in \mathbb{R}$ such that $\psi(\tilde{z}) = \lambda\tilde{z}$. As a consequence of $\psi(\Gamma) \subset \widehat{v0w}$, we find that $\lambda\tilde{z} \in \widehat{v0w}$ with $\tilde{z} \in \widehat{v0w}$ and hence $\lambda \neq 0$ for $0 \notin \widehat{v0w}$. If, by contradiction, $\lambda < 0$, we have that both $\tilde{z}$ and $-\tilde{z}$ belong to $\widehat{v0w}$ and this is possible only if $\tilde{z} \in \ell_v \cup \ell_w$, but then, we contradict (2.8). □

Remark 1. In the frame of Corollary 1, we notice that condition (2.8) is always fulfilled when $w \neq -v$ (that is, $\widehat{v0w} \cup \{0\}$ is a cone [21, Ch.6]) or when $\psi(\Gamma) \subset \operatorname{int}(\widehat{v0w})$.

It will be also useful to introduce the following definition.

Let $\widehat{v0w}$ be a pointed angle at the origin. For r_1 and r_2 with $0 < r_1 < r_2$, we define the (closed) set

$$[\widehat{v0w}]_{r_1}^{r_2} := \Big\{ z \in \widehat{v0w} :\ r_1 \leq |z| \leq r_2 \Big\} = \widehat{v0w} \cap \mathcal{A}[r_1, r_2].$$

We say that $[\widehat{v0w}]_{r_1}^{r_2}$ is a *conical shell* if it is determined by a pointed angle $\widehat{v0w}$ at the origin such that $\overline{v0w}$ is convex and also $w \neq -v$. This definition corresponds to that given in [21, p. 239].

Corollary 2. *Let $\mathcal{W} = [\widehat{v0w}]_{r_1}^{r_2}$ be a conical shell and suppose that there is a connected set $\Gamma \subset D_\psi \cap \mathcal{W}$ with $\Gamma \cap \ell_v \neq \varnothing$ and $\Gamma \cap \ell_w \neq \varnothing$ and such that $\psi(\Gamma) \subset \mathcal{W}$ and $|\psi(z)| = |z|$ for all $z \in \Gamma$. Then, the map ψ has at least one fixed point $\tilde{z} \in \Gamma$.*

Proof. By Corollary 1 and Remark 1 we have that there is $\tilde{z} \in \Gamma$ such that $\psi(\tilde{z}) = \lambda\tilde{z}$ for some $\lambda > 0$. But, evidently, $\lambda = 1$ as $|\psi(z)| = |z|$ on Γ. □

For the next steps, we recall a slight variant of a result from plane topology previously used also in [14], [20] (more or less explicitly). Details can be found in [54].

Lemma 4. *Let $\mathcal{R} \subset \mathbb{R}^2$ be a two-dimensional cell and let $h : Q = [-1,1]^2 \to \mathcal{R}$, be a surjective homeomorphism. Let us define $H^\pm := h([-1,1] \times \{\pm 1\})$ and $V^\pm := h(\{\pm 1\} \times [-1,1])$. Let $\mathcal{S} \subset \mathcal{R}$ be a closed set such that*

$$\sigma \cap \mathcal{S} \neq \varnothing,$$

for each path σ contained in $\mathcal{R}$ and joining H^- with H^+. Then, $\mathcal{S}$ contains a closed connected set $\mathcal{C}$ joining V^- with V^+.

Using the property that any path joining two distinct points contains an arc joining the same two points (as recalled at the end of the Introduction), it is easy to see that it is completely equivalent if in the assumption of Lemma 4 we suppose that σ is an arc.

Example 1. Let us consider the set $\mathcal{S} = \mathcal{S}_0 \cap Q$, where

$$\mathcal{S}_0 = \Big\{(x,y) : \Big(x \in \mathbb{Q}, y \in \mathbb{Q}, -\frac{1}{2} \leq y \leq 0\Big) \vee \Big(x \in \mathbb{R}\backslash\mathbb{Q}, y \in \mathbb{R}\backslash\mathbb{Q}, 0 < y < \frac{1}{2}\Big)\Big\}.$$

The set $\mathcal{S}$ is not closed, not connected and does not contain any connected subset joining V^- with V^+. Though, any path from H^- to H^+ has nonempty intersection with $\mathcal{S}$.

For the subsequent results, we also need a property of the domain (or a part of the domain) on which we want to have the map ψ defined.

Let us consider a set $\mathcal{D} \subset D_\psi$. We say that the (continuous) map is *proper on* $\mathcal{D}$ if $\psi^{-1}(\mathcal{K}) \cap \mathcal{D}$ is compact for each compact set $\mathcal{K} \subset \mathbb{R}^2$. Equivalently, we have that whenever there is a sequence $z_n \in \mathcal{D}$ and a point $w \in \mathbb{R}^2$ such that $\psi(z_n) \to w$, then, there is a subsequence z_{i_n} of z_n and a point $z^* \in \mathcal{D}$ such that $z_{i_n} \to z^*$.

In this situation, we will also say that the pair $(\mathcal{D}, \psi)$ is proper.

To be used in the next steps, we note that a composition of proper maps is proper and that if $(\mathcal{D}, \psi)$ is proper, then ψ is proper on $\mathcal{D} \cap C$, for each closed set $C \subset \mathbb{R}^2$ (see the Appendix for more details).

Given a map ψ and a set $\mathcal{D} \subset D_\psi$, we also define $(\mathcal{D}, \psi)$ *proper on compact sets* if, for each K compact, ψ is proper on $\mathcal{D} \cap K$.

Example 2. Let $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Carathéodory function and assume the uniqueness of the solutions for the Cauchy problems

$$\begin{cases} \dot{z} = f(t,z) \\ z(t_0) = z_0 \end{cases} \tag{2.9}$$

with $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^2$.

If we denote by $z(\cdot; t_0, z_0)$ the noncontinuable solution of (2.9), defined on its maximal interval of existence $I_{(t_0,z_0)}$, we have that for each $t_0, t \in \mathbb{R}$, the map $\phi_{t_0}^t : z_0 \mapsto z(t; t_0, z_0)$ is continuous and is defined in a open (possibly empty) set $D(t_0, t) \subset \mathbb{R}^2$ (by the Peano's theorem for the Carathéodory equations [28] $D(t_0, t) \neq \varnothing$ if $|t - t_0|$ is sufficiently small). We also have that $\phi_{t_1}^{t_2} \circ \phi_{t_0}^{t_1} = \phi_{t_0}^{t_2}$.

We look now for conditions to be satisfied by the differential equation in order that the pair $(D(t_0, t), \phi_{t_0}^t)$ is proper on compact sets. In fact, in general, the map $\phi_{t_0}^t$ may fail to be proper even when it is restricted to the compact sets. An example of this fact is shown, for instance, by the the planar system $\dot{x} = 1,\ \dot{y} = -xy^2$.

Let $t_0 < t_1$ and let K be a compact set. Suppose that (2.9) has solutions defined on $[t_0, t_1]$ for each $z_0 \in K$. Then it is easy to prove that $\phi_{t_0}^{t_1}$ is proper on $K \subset D_{t_0}^{t_1}$. A more general result is the following.

Claim 1. Assume that there is a continuous function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|z(s; t_0, z_0)| \leq \eta\Big(\max\{|z_0|, |z(t; t_0, z_0)|\}\Big),\ \forall\, t_0 \leq s \leq t,\ \forall\, z_0 \in D(t_0, t). \tag{2.10}$$

Then, for any compact set $K \subset \mathbb{R}^2$, the map $\phi_{t_0}^t$ is proper on $D(t_0, t) \cap K$.

Proof. Let $t > t_0$ be fixed and suppose that $D(t_0, t) \neq \varnothing$ (otherwise the claim is vacuously true). Let $\mathcal{E} = K \cap D(t_0, t)$, with K a compact subset of $\mathbb{R}^2$.

Let $z_0^n \in \mathcal{E}$ be a bounded sequence of initial points and, without loss of generality, assume that $z_0^n \to z_0 \in K$ and $w_n := z(t; t_0, z_0^n) \to w_0 \in \mathbb{R}^2$. Then, we have that $|z_0^n|, |w_n| \leq M$, for all n and for some $M > 0$, so that, by (2.10),

$$|z(s; t_0, z_0^n)| \leq \eta(M) := M_1, \quad \forall\, s \in [t_0, t],\ n \in \mathbb{N}.$$

By the Carathéodory assumption, there is $\rho \in L^1([t_0, t], \mathbb{R}^+)$ such that

$$|f(s, z(s; t_0, z_0^n))| \leq \rho(s), \qquad \text{for a.e. } s \in [t_0, t],\ \forall\, n \in \mathbb{N}.$$

Using the Ascoli–Arzelá theorem and the Lebesgue dominated convergence theorem in the integral relation

$$z(s; t_0, z_0^n) = z_0^n + \int_{t_0}^{t} f(\xi, z(\xi; t_0, z_0^n))\, d\xi, \quad \forall\, s \in [t_0, t],$$

(cf.[28, p.29]) we can prove that there is a continuous function $\hat{z} : [t_0, t] \to \mathbb{R}^2$ such that

$$\hat{z}(s) = z_0 + \int_{t_0}^{t} f(\xi, \hat{z}(\xi))\, d\xi, \quad \forall\, s \in [t_0, t].$$

From this, it follows that $z_0 \in D(t_0, t)$, $\hat{z}(s) = z(s; t_0, z_0)$ and $w_0 = \hat{z}(t)$. Hence the claim is proved. $\square$

Example 3. Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitzean function satisfying (g_0) and let $q : \mathbb{R} \to \mathbb{R}$ be a locally summable function.

- *Let* $[a, b]$ *be an interval such that* $q \geq 0$ *on* $[a, b]$ *and* q *is continuous and of bounded variation on* $[a, b]$. *Suppose that the set* $\{t \in [a, b] : q(t) > 0\}$ *is the union of a finite number of intervals. If, further,* $q(\cdot)$ *is monotone in a left neighborhood of any point* $s \in]a, b]$ *where* $q(s) = 0$, *then, for each* $t_0 \in [a, b[$ *any solution of* (1.1) *is continuable to* $[t_0, b]$. *Respectively, if* $q(\cdot)$ *is monotone in a right neighborhood of any point* $s \in [a, b[$ *where* $q(s) = 0$, *then, for each* $t_0 \in]a, b]$ *any solution of* (1.1) *is continuable to* $[a, t_0]$.

Proof. The result follows from [19], [14] (see also [50] for more details).

For completeness, we give a sketch of the proof in the simpler case when $q(\cdot)$ is absolutely continuous. This assumption can be relaxed to that of $q(\cdot)$ of bounded variation on $[a, b]$ arguing like in [19].

Let $[t_0, t_1] \subset [a, b]$ be such that $q(t) > 0$ on $]t_0, t_1[$ and suppose that $q(t_1) = 0$ and q is monotone nonincreasing in a left neighborhood of t_1. By Peano's theorem, we have that the solution $x(\cdot)$ is defined on $[t_0, t_2]$ for some $t_2 \in]t_0, t_1[$. Consider now $V(t, x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + q(t)G(x) + 1$, where $G(x) = \int_0^x g(\xi)\, d\xi$. Differentiation of $v(t) := V(t, x(t), \dot{x}(t))$ yields to

$$\dot{v}(t) = x\dot{x} + \dot{q}(t)G(x) \leq \zeta(t)v(t),$$

for a.e. $t \in [t_2, t_1]$ where $x(\cdot)$ is defined and where ζ is a suitably chosen function, summable on $[t_2, t_1]$ and such that $\zeta(t)q(t) \geq \dot{q}(t)$ for a.e. $t \in [t_2, t_1]$. Hence, $v(t) \leq v(t_2) \exp \int_{t_2}^{t_1} \zeta(s)\, ds := N$ holds for all $t \in [t_2, t_1]$. From this a priori bound and the fact that $V(t, x, \dot{x}) \to +\infty$ uniformly for $t \in [a, b]$ as $|x| + |\dot{x}| \to +\infty$, we obtain the continuability of the solution up to $t = t_1$. By a finite number of steps, we can then obtain the continuability on $[t_0, b]$. $\square$

- *Let* $[b, c]$ *be an interval such that* $q \leq 0$ *on* $[b, c]$. *Then, property* (2.10) *holds for each* $[t_0, t] \subset [b, c]$.

Proof. Let $x(\cdot)$ be a solution of (1.1). Multiplying the equation by $x(s)$ gives $\ddot{x}(s)x(s) \geq 0$ for a.e. $t \in [t_0, t]$, that is, for $\ell(t) = \frac{1}{2}x(t)^2$, we have

$$\frac{d^2}{dt^2}\ell(s) \geq 0, \quad \text{for a.e. } s \in [t_0, t].$$

Hence, it follows immediately that

$$|x(s)| \leq \max\Big\{|x(t_0)|, |x(t)|\Big\} \leq \max\Big\{|x(t_0)| + |\dot{x}(t_0)|, |x(t)| + |\dot{x}(t)|\Big\} := L_1$$

holds for all $s \in [t_0, t]$. Now, we easily conclude, using the equation.

In fact, $|\ddot{x}(s)| \le |q(s)| \max_{|\xi| \le L_1}\{|g(\xi)|\} := \mu(t)$, for a.e. $s \in [t_0, t]$ and then $|\dot{x}(s)| \le L_1 + \int_{t_0}^{t} \mu(\xi)\, d\xi := L_2$. From the proof it is clear that $L_1 + L_2$ is a continuous function of $(\,|z(t_0)|, |\dot{z}(t_0)|\,)$. □

As a consequence of the above results, we can give the following corollary.

- *Let $q(\cdot)$ be a continuous and piece-wise monotone function defined in an interval I. Suppose that $I = [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{k-1}, t_k]$ and either $q \ge 0$ or $q \le 0$, on each of the sub-intervals $I_j = [t_{j-1}, t_j]$. Then, for any compact set $K \subset \mathbb{R}^2$, and for each $t_0, t \in I$, the map $\phi_{t_0}^{t}$ is proper on $D(t_0, t) \cap K$.*

Proof. It is sufficient to observe that the composition of maps fits with the property of properness on compact sets that we have defined. □

Now we come back to the general setting considered at the beginning and prove the following fixed point theorem.

Theorem 1. *Suppose that $(\mathcal{D}, \psi)$ is proper on compact sets and consider a conical shell $\mathcal{W} = [\widehat{v0w}]_{r_1}^{r_2}$. Assume that any path σ contained in $\mathcal{W}$ and intersecting C_{r_1} and C_{r_2}, contains a sub-path $\sigma' \subset \mathcal{D}$ such that*

$$\psi(\sigma') \subset \widehat{v0w}\,, \quad \text{and} \;\; \psi(\sigma') \cap C_{r_1} \neq \varnothing, \;\; \psi(\sigma') \cap C_{r_2} \neq \varnothing.$$

Then, the map ψ has at least one fixed point $\tilde{z} \in \mathcal{D} \cap \mathcal{W}$.

Remark 2. In the above theorem we don't assume ψ to be defined on the whole set $\mathcal{W}$.

Proof. We consider the set

$$\mathcal{S} = \Big\{ z \in \mathcal{D} \cap \mathcal{W} \,:\, \psi(z) \in \mathcal{W},\; |\psi(z)| = |z| \Big\}.$$

Since $\mathcal{W}$ is compact, by the properness of $(\mathcal{D}, \psi)$ on compact sets, we have that $\mathcal{S}$ is compact.

Consider now an arbitrary path σ contained in $\mathcal{W}$ and intersecting C_{r_1} and C_{r_2}. By the assumption, σ contains a path σ', with $\sigma' \subset \mathcal{D}$ such that $\psi(\sigma')$ intersects C_{r_1} and C_{r_2}. This implies that there are $z_1, z_2 \in \sigma'$ such that $|\psi(z_1)| = r_1 \le |z_1|$ and $|\psi(z_2)| = r_2 \ge |z_2|$, and therefore, by the continuity of ψ and the connectedness of σ', we can conclude that σ' intersects $\mathcal{S}$. Thus we have proved that the closed set $\mathcal{S}$ intersects any path σ contained in $\mathcal{W}$ and intersecting C_{r_1} and C_{r_2}.

At this point, using the elementary fact that $\mathcal{W}$ is homeomorphic to a rectangle, Lemma 4 guarantees the existence of a continuum $\mathcal{C} \subset \mathcal{S}$ with $\mathcal{C}$ intersecting both ℓ_v and ℓ_w. By the assumptions, we also have that ψ is defined on $\mathcal{C}$ and $\psi(\mathcal{C}) \subset \mathcal{W}$ with $|\psi(z)| = |z|$ on $\mathcal{C}$. Then, Corollary 2 implies the existence of a fixed point for ψ in $\mathcal{C}$. □

We consider now two conical shells

$$\mathcal{W}_0 := [\widehat{v_0 0 w_0}]_{r_1^0}^{r_2^0}\,, \quad \mathcal{W}_1 := [\widehat{v_1 0 w_1}]_{r_1^1}^{r_2^1}\,,$$

which we assume to be *disjoint*.

In the next result, we denote by ψ^j the j-th iterate of ψ, with the convention that $\psi^1 = \psi$.

Theorem 2. *Suppose that $(\mathcal{D}, \psi)$ is proper on compact sets. For $\mathcal{W}_0$ and $\mathcal{W}_1$ as above, let us suppose that the following property hold:*

(a_0) *any path σ contained in $\mathcal{W}_0$ and intersecting $C_{r_1^0}$ and $C_{r_2^0}$, contains a sub-path $\sigma'_{(0,0)} \subset \mathcal{D}$ such that*

$$\psi(\sigma'_{(0,0)}) \subset \widehat{v_0 0 w_0}\,, \quad \psi(\sigma'_{(0,0)}) \cap C_{r_1^0} \neq \varnothing, \quad \psi(\sigma'_{(0,0)}) \cap C_{r_2^0} \neq \varnothing,$$

as well as a sub-path $\sigma'_{(0,1)} \subset \mathcal{D}$ such that

$$\psi(\sigma'_{(0,1)}) \subset \widehat{v_1 0 w_1}\,, \quad \psi(\sigma'_{(0,1)}) \cap C_{r_1^1} \neq \varnothing, \quad \psi(\sigma'_{(0,1)}) \cap C_{r_2^1} \neq \varnothing;$$

any path σ contained in $\mathcal{W}_1$ and intersecting $C_{r_1^1}$ and $C_{r_2^1}$, contains a sub-path $\sigma'_{(1,0)} \subset \mathcal{D}$ such that

$$\psi(\sigma'_{(1,0)}) \subset \widehat{v_0 0 w_0}\,, \quad \psi(\sigma'_{(1,0)}) \cap C_{r_1^0} \neq \varnothing, \quad \psi(\sigma'_{(1,0)}) \cap C_{r_2^0} \neq \varnothing,$$

as well as a sub-path $\sigma'_{(1,1)} \subset \mathcal{D}$ such that

$$\psi(\sigma'_{(1,1)}) \subset \widehat{v_1 0 w_1}\,, \quad \psi(\sigma'_{(1,1)}) \cap C_{r_1^1} \neq \varnothing, \quad \psi(\sigma'_{(1,1)}) \cap C_{r_2^1} \neq \varnothing.$$

Then, the following conclusions hold:

(w_1) *for each $i = 0, 1$, the map ψ has at least one fixed point $\tilde z_i \in \mathcal{D} \cap \mathcal{W}_i$;*

(w_2) *for each $i = 0, 1$ and for any finite sequence $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$, with $k \geq 1$ and $\delta_j \in \{0, 1\}$ (for all $j = 1, \dots, k$), there are points $\tilde z_{(i,\ )} \in \mathcal{D} \cap \mathcal{W}_i$ which are fixed points of ψ^{k+1} and satisfy*

$$\psi^j(\tilde z_{(i,\ )}) \in \mathcal{W}_{\delta_j}\,, \quad \forall\, j = 1, \dots, k;$$

(w_3) *for each $i = 0, 1$ and for any sequence $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots)$, with $\delta_j \in \{0, 1\}$ ($\forall\, j \in \mathbb{N}$), there is a continuum $\Gamma_i \subset \mathcal{W}_i$, with $\Gamma_i \cap \ell_{v_i} \neq \varnothing$ and $\Gamma_i \cap \ell_{w_i} \neq \varnothing$ such that for each $z \in \Gamma_i$, it follows that*

$$\psi^j(z) \in \mathcal{W}_{\delta_j}\,, \quad \forall\, j \in \mathbb{N};$$

(w_4) *for any doubly-infinite sequence $\boldsymbol{\delta} = (\dots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \dots)$, with $\delta_j \in \{0, 1\}$ ($\forall\, j \in \mathbb{N}$), there is a double-sided sequence of points*

$$\tilde w \ = (\dots, \tilde w_{-2}, \tilde w_{-1}, \tilde w_0, \tilde w_1, \tilde w_2, \dots),$$

with $\tilde w_j \in \mathcal{D} \cap \mathcal{W}_{\delta_j}$ and such that

$$\psi(\tilde w_j) = \tilde w_{j+1}\,, \quad \forall\, j \in \mathbb{Z}.$$

Proof. Clearly, (w_1) is a straightforward consequence of Theorem 1, so that we can pass to the proof of the other properties.

Let us fix a finite sequence $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$ for some $k \geq 1$ and with $\delta_i \in \{0, 1\}$ for all $i = 1, \dots, k$. Let $\gamma : [0, 1] \to \mathcal{W}_i$ be a continuous map such that $\gamma(0) \in C_{r_1^i}$

and $\gamma(1) \in C_{r_2^i}$. By assumption (a_0), we have that there is a closed interval $I_1 \subset [0,1]$ such that ψ is defined in $\gamma(I_1)$, $\psi(\gamma(t)) \in \mathcal{W}_{\delta_1}$, for all $t \in I_1$ and

$$\psi(\gamma(t_1')) \in C_{r_1^{\delta_1}}\,, \quad \psi(\gamma(t_1'')) \in C_{r_2^{\delta_1}}\,, \ \text{ for } \ \{t_1', t_1''\} = \partial I_1\,.$$

We apply now the same argument to the continuous map $\gamma_1 : I_1 \to \mathcal{W}_{\delta_1}$ defined by $\gamma_1(t) := \psi(\gamma(t))$ and find a closed interval $I_2 \subset I_1$, such that ψ is defined in $\gamma_1(I_2)$, $\psi(\gamma_1(t)) \in \mathcal{W}_{\delta_2}$, for all $t \in I_2$ and

$$\psi(\gamma_1(t_2')) \in C_{r_1^{\delta_2}}\,, \quad \psi(\gamma(t_2'')) \in C_{r_2^{\delta_2}}\,, \ \text{ for } \ \{t_2', t_2''\} = \partial I_2\,.$$

Continuing this way for a finite number of steps, we have that for each $j = 1, \dots, k, k+1$ there is a closed interval I_j, with

$$[0,1] \supset I_1 \supset \cdots \supset I_j \supset I_{j+1} \supset \cdots \supset I_k \supset I_{k+1}\,,$$

such that

$$\psi^j(\gamma(I_{k+1})) \subset \mathcal{W}_{\delta_j} \quad \text{for } \ j = 1, \dots, k$$

and

$$\psi^{k+1}(\gamma(t_{k+1}')) \in C_{r_1^i}\,, \quad \psi(\gamma(t_{k+1}'')) \in C_{r_2^i}\,, \ \text{ for } \ \{t_{k+1}', t_{k+1}''\} = \partial I_{k+1}\,.$$

We can consider now the set

$$\mathcal{S}_i^k = \Big\{ z \in \mathcal{D} \cap \mathcal{W}_i \ : \ \psi^j(z) \in \mathcal{W}_{\delta_j}, \ \forall\, j = 1 \dots, k, \ |\psi^{k+1}(z)| = |z| \Big\}$$

for $i = 0, 1$. By the properness of $(\mathcal{D}, \psi)$ on the compact sets, we have that the set $\mathcal{S}_i^k$ is a compact subset of $\mathcal{W}_i$ and, arguing like in the proof of Theorem 1 (using Lemma 4), we have that $\mathcal{S}_i^k$ contains a continuum $\mathcal{C}_i^k$ which intersects both ℓ_{v_i} and ℓ_{w_i}. Then, Corollary 2 ensures the existence of a point $\tilde{z} \in \mathcal{W}_i$ which is a fixed point for ψ^{k+1} and such that $\psi^j(\tilde{z}) \in \mathcal{W}_{\delta_j}$. Thus (w_2) is proved.

To prove (w_3), let us fix $i \in \{0,1\}$ and choose a sequence $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots)$ with $\delta_j \in \{0,1\}$. Repeating the first part of the proof of (w_2), we have that for any continuous map $\gamma : [0,1] \to \mathcal{W}_i$ such that $\gamma(0) \in C_{r_1^i}$ and $\gamma(1) \in C_{r_2^i}$, and for each $j \in \mathbb{N}$, there is a compact interval $I_j \subset [0,1]$ such that

$$[0,1] \supset I_1 \supset I_2 \supset \cdots \supset I_j \supset I_{j+1} \supset \dots,$$

moreover, $\psi^j(\gamma(t)) \subset \mathcal{W}_{\delta_j}$ for all $t \in I_j$ and

$$\psi^j(\gamma(t_j')) \in C_{r_1^{\delta_j}}\,, \quad \psi(\gamma(t_j'')) \in C_{r_2^{\delta_j}}\,, \ \text{ for } \ \{t_j', t_j''\} = \partial I_j\,.$$

We can consider now the set

$$\mathcal{S}_i = \Big\{ z \in \mathcal{D} \cap \mathcal{W}_i \ : \ \psi^j(z) \in \mathcal{W}_{\delta_j}, \ \forall\, j \in \mathbb{N} \Big\}$$

for $i = 0, 1$ which is compact by the assumption of properness of $(\mathcal{D}, \psi)$. If we take $s \in \cap_{j=1}^{\infty} I_j$, we have that $\gamma(s) \in \mathcal{S}_i$ and thus we have proved that $\sigma \cap \mathcal{S}_i \neq \varnothing$ for each path $\sigma \subset \mathcal{W}_i$ with σ intersecting both $C_{r_1^i}$ and $C_{r_2^i}$. Hence, Lemma 4 ensures the existence of a continuum $\mathcal{C}_i \subset \mathcal{S}_i$ such that $\mathcal{C}_i \cap \ell_{v_i} \neq \varnothing$ and $\mathcal{C}_i \cap \ell_{w_i} \neq \varnothing$. From the definition of $\mathcal{S}_i$ we have that, for each $z \in \mathcal{C}_i$, $\psi^j(z) \in \mathcal{W}_{\delta_j}$ for all $j \in \mathbb{N}$. Therefore, (w_3) is proved.

To prove (w_4), let us fix a double sequence $\boldsymbol{\delta} = (\dots, \delta_{-1}, \delta_0, \delta_1, \dots)$ with $\delta_j \in \{0,1\}$ for each $j \in \mathbb{Z}$. By (w_3) we have that for each $n = 1, 2, \dots,$ there is a continuum $\Gamma_n \subset \mathcal{W}_{\delta_{-n}}$ such that $\psi^j(\Gamma_n) \subset \mathcal{W}_{\delta_{j-n}}$. We take a point $y_{j,n} \in \psi^{j+n}(\Gamma_n) \subset \mathcal{W}_{\delta_j}$, for each $j \geq -n$, in order to form the infinite matrix

$$\begin{array}{ccccccccc} & & & y_{-1,1} & y_{0,1} & y_{1,1} & \cdots & y_{j,1} & \cdots \\ & & y_{-2,2} & y_{-1,2} & y_{0,2} & y_{1,2} & \cdots & y_{j,2} & \cdots \\ & \ddots & & & & & & & \\ y_{-n,n} & \cdots & y_{-2,2} & y_{-1,n} & y_{0,n} & y_{1,n} & \cdots & y_{j,n} & \cdots \\ & \ddots & & & & & & & \end{array}$$

where, for each n and j, we have that $\psi(y_{j,n}) = y_{j+1,n}$. At this point, a standard compactness argument allows to pass to the limit on each "column" along a common subsequence of indexes in order to find, for each $j \in \mathbb{Z}$ a point $\tilde{w}_j \in \mathcal{W}_{\delta_j}$ and the continuity of ψ implies also that $\psi(\tilde{w}_j) = \tilde{w}_{j+1}$, $\forall\, j \in \mathbb{Z}$. $\square$

3. Tagged Maps

In order to apply the results of the preceding section to the Poincaré operator associated to a second order ordinary differential equation with a changing sign weight, it may be convenient to take advantage also of the information about the rotation number that we obtain from the analysis of the solutions in the time-intervals in which the weight is positive. With this respect, we are led to attach a "tag" to the points of the domain of ψ. Such a tag, in the applications, will be somehow linked to the rotation number associated to the solution departing from the given initial point. Accordingly, we define any continuous map $\eta : D_\psi \setminus \{0\} \to \mathbb{R}$, as a *tag* associated to ψ. The pair (ψ, η) will be called a *tagged map*. The choice of excluding the origin from the domain of η is purely conventional and it is made here only in order to remember the fact that the number of turns around the origin is defined only for those solutions not passing through it.

Now, a "tagged" version of Theorem 1 reads as follows.

Theorem 3. *Let $\mathcal{D} \subset D_\psi$ and let (ψ, η) be a tagged map with $(\mathcal{D}, \psi)$ proper on the compact sets. Let $\mathcal{W} = [\widehat{v0w}]_{r_1}^{r_2}$ be a conical shell. Suppose that there is a (non-empty) set of indexes $\mathcal{N} \subset \mathbb{N}$ such that, for each $k \in \mathcal{N}$, there is a compact interval $\Lambda^k \subset \mathbb{R}$, with $\Lambda^k \cap \Lambda^j = \varnothing$ for $k \neq j$ and the following property holds:*

(e_1) *for each $k \in \mathcal{N}$ and any path σ contained in $\mathcal{W}$ and intersecting C_{r_1} and C_{r_2}, there is a sub-path $\sigma^k \subset \mathcal{D}$ such that*

$$\psi(\sigma^k) \subset \widehat{v0w}\,, \quad \eta(\sigma^k) \subset \Lambda^k\,,$$

and

$$\psi(\sigma^k) \cap C_{r_1} \neq \varnothing, \quad \psi(\sigma^k) \cap C_{r_2} \neq \varnothing.$$

Then, for each $k \in \mathcal{N}$, the map ψ has at least one fixed point $\tilde{z}_k \in \mathcal{D} \cap \mathcal{W}$ with $\eta(\tilde{z}_k) \in \Lambda^k$. Hence, $\tilde{z}_k \neq \tilde{z}_j$ for $k \neq j$ and therefore, ψ has at least $\#(\mathcal{N})$ distinct fixed points in $\mathcal{W} \cap \mathcal{D}$.

Proof. Let us fix $k \in \mathcal{N}$ and consider the set

$$\mathcal{S}_k = \Big\{ z \in \mathcal{D} \cap \mathcal{W} \,:\, |\psi(z)| = |z|\,,\ \eta(z) \in \Lambda^k \Big\}.$$

Take a sequence $(z_n)_n \in \mathcal{S}_k$, with $(z_n)_n \to \bar{z}$ as $n \to \infty$. Since $|\psi(z_n)| \leq r_2$ for all n, by the properness of $(\mathcal{D}, \psi)$, we have that $\bar{z} \in \mathcal{D}$. Moreover, $\bar{z} \neq 0$, as $\bar{z} \in \mathcal{W}$. From the continuity of ψ and η and the fact that Λ^k is closed, it follows that $\bar{z} \in \mathcal{S}_k$ and thus we have proved that $\mathcal{S}_k$ is closed.

Consider now an arbitrary path σ contained in $\mathcal{W}$ and intersecting C_{r_1} and C_{r_2}. By (e_1) and repeating the same argument in the proof Theorem 1, we find that the sub-path σ^k, of σ intersects $\mathcal{S}_k$. At this point, Lemma 4 guarantees the existence of a continuum $\mathcal{C}_k \subset \mathcal{S}_k$ with $\mathcal{C}_k$ intersecting both ℓ_v and ℓ_w. By the assumptions, we also have that ψ is defined on $\mathcal{C}_k$, $\psi(\mathcal{C}_k) \subset \widehat{v0w}$ and $\eta(\mathcal{C}_k) \subset \Lambda^k$.

Hence, Corollary 2 implies the existence of a fixed point $\tilde{z}_k$ for ψ in $\mathcal{C}_k$. □

It may be interesting to observe that Theorem 3 could be directly obtained from Theorem 1, just applying Theorem 1 on ψ restricted to the domain $\mathcal{D} \cap \eta^{-1}(\Lambda_k)$.

We can also write the following generalized version of Theorem 2.

Theorem 4. *Let $\mathcal{D} \subset D_\psi$ and let (ψ, η) be a tagged map with $(\mathcal{D}, \psi)$ proper on the compact sets. For $\mathcal{W}_0$ and $\mathcal{W}_1$ as in Theorem 2, let us suppose that the following properties hold:*

- *there is a (non-empty) set of indexes $\mathcal{N}_{(0,0)} \subset \mathbb{N}$ such that, for each $k \in \mathcal{N}_{(0,0)}$ there is a compact interval $\Lambda^k = \Lambda^k_{(0,0)} \subset \mathbb{R}$, with $\Lambda^k \cap \Lambda^j = \varnothing$ for $k \neq j$, and such that any path σ contained in $\mathcal{W}_0$ and intersecting $C_{r_1^0}$ and $C_{r_2^0}$, contains a sub-path $\sigma^k_{(0,0)} \subset \mathcal{D}$ such that*

$$\psi(\sigma^k_{(0,0)}) \subset \widehat{v_0 0 w_0}\,, \quad \eta(\sigma^k_{(0,0)}) \subset \Lambda^k$$

  and

$$\psi(\sigma^k_{(0,0)}) \cap C_{r_1^0} \neq \varnothing, \quad \psi(\sigma^k_{(0,0)}) \cap C_{r_2^0} \neq \varnothing;$$

- *there is a (non-empty) set of indexes $\mathcal{N}_{(0,1)} \subset \mathbb{N}$ such that, for each $k \in \mathcal{N}_{(0,1)}$ there is a compact interval $\Lambda^k = \Lambda^k_{(0,1)} \subset \mathbb{R}$, with $\Lambda^k \cap \Lambda^j = \varnothing$ for $k \neq j$, and such that any path σ contained in $\mathcal{W}_0$ and intersecting $C_{r_1^0}$ and $C_{r_2^0}$, contains a sub-path $\sigma^k_{(0,1)} \subset \mathcal{D}$ such that*

$$\psi(\sigma^k_{(0,1)}) \subset \widehat{v_1 0 w_1}\,, \quad \eta(\sigma^k_{(0,1)}) \subset \Lambda^k$$

  and

$$\psi(\sigma^k_{(0,1)}) \cap C_{r_1^1} \neq \varnothing, \quad \psi(\sigma^k_{(0,1)}) \cap C_{r_2^1} \neq \varnothing;$$

- *there is a (non-empty) set of indexes* $\mathcal{N}_{(1,0)} \subset \mathbb{N}$ *such that, for each* $k \in \mathcal{N}_{(1,0)}$ *there is a compact interval* $\Lambda^k = \Lambda^k_{(1,0)} \subset \mathbb{R}$, *with* $\Lambda^k \cap \Lambda^j = \varnothing$ *for* $k \neq j$, *and such that any path* σ *contained in* $\mathcal{W}_1$ *and intersecting* $C_{r_1^1}$ *and* $C_{r_2^1}$, *contains a sub-path* $\sigma^k_{(1,0)} \subset \mathcal{D}$ *such that*

$$\psi(\sigma^k_{(1,0)}) \subset \widehat{v_0 0 w_0}\,, \quad \eta(\sigma^k_{(1,0)}) \subset \Lambda^k$$

  and

$$\psi(\sigma^k_{(1,0)}) \cap C_{r_1^0} \neq \varnothing, \quad \psi(\sigma^k_{(1,0)}) \cap C_{r_2^0} \neq \varnothing;$$

- *there is a (non-empty) set of indexes* $\mathcal{N}_{(1,1)} \subset \mathbb{N}$ *such that, for each* $k \in \mathcal{N}_{(1,1)}$ *there is a compact interval* $\Lambda^k = \Lambda^k_{(1,1)} \subset \mathbb{R}$, *with* $\Lambda^k \cap \Lambda^j = \varnothing$ *for* $k \neq j$, *and such that any path* σ *contained in* $\mathcal{W}_1$ *and intersecting* $C_{r_1^1}$ *and* $C_{r_2^1}$, *contains a sub-path* $\sigma^k_{(1,1)} \subset \mathcal{D}$ *such that*

$$\psi(\sigma^k_{(1,1)}) \subset \widehat{v_1 0 w_1}\,, \quad \eta(\sigma^k_{(1,1)}) \subset \Lambda^k$$

  and

$$\psi(\sigma^k_{(1,1)}) \cap C_{r_1^1} \neq \varnothing, \quad \psi(\sigma^k_{(1,1)}) \cap C_{r_2^1} \neq \varnothing.$$

Then, the following conclusions hold:

- *for each* $i = 0, 1$ *and* $k_i \in \mathcal{N}_{(i,i)}$, *the map* ψ *has at least one fixed point* $\tilde{z}_{k_i} \in \mathcal{D} \cap \mathcal{W}_i$ *with* $\eta(\tilde{z}_{k_i}) \in \Lambda^{k_i}_{(i,i)}$. *In particular,* $\tilde{z}_{k_i} \neq \tilde{z}_{m_j}$ *for* $k_i \neq m_j$, *where* $i, j \in \{0,1\}$ *and* $k_i \in \mathcal{N}_{(i,i)}$, $m_j \in \mathcal{N}_{(j,j)}$. *In particular, for* $i = 0, 1$, *the map* ψ *has at least* $\#(\mathcal{N}_{(i,i)})$ *distinct fixed points in* $\mathcal{W}_i$;
- *for each* $i = 0, 1$, *for any finite sequence* $\boldsymbol{\delta} = (\delta_1, \dots, \delta_m)$, *with* $m \geq 1$ *and* $\delta_j \in \{0,1\}$ *(for each* $j = 1, \dots, m$*) and for any finite sequence* $\kappa = (k_1, \dots, k_m, k_{m+1})$, *with*

$$k_1 \in \mathcal{N}_{(i,\delta_1)}\,, k_2 \in \mathcal{N}_{(\delta_1,\delta_2)}\,, \dots, k_m \in \mathcal{N}_{(\delta_{m-1},\delta_m)}\,, k_{m+1} \in \mathcal{N}_{(\delta_m,i)}\,,$$

  there are points $\tilde{z}_{(i,\ ,\kappa)} \in \mathcal{D} \cap \mathcal{W}_i$ *which are fixed points of* ψ^{m+1} *and satisfy*

$$\psi^j(\tilde{z}_{(i,\ ,\kappa)}) \in \mathcal{W}_{\delta_j}\,, \quad \eta(\psi^j(\tilde{z}_{(i,\ ,\kappa)})) \in \Lambda^{k_j}_{(\delta_{j-1},\delta_j)}\,, \quad \forall\, j = 1, \dots, m+1$$

  (*we use here the convention that* $\delta_0 = \delta_{m+1} = i$);
- *for each* $i = 0, 1$, *for any sequence* $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots)$, *with* $\delta_j \in \{0,1\}$ $(\forall\, j \in \mathbb{N})$ *and for any sequence* $\kappa = (k_1, k_2, \dots)$, *with*

$$k_1 \in \mathcal{N}_{(i,\delta_1)}\,, k_2 \in \mathcal{N}_{(\delta_1,\delta_2)}\,, \dots, k_m \in \mathcal{N}_{(\delta_{m-1},\delta_m)}\,, \dots\,,$$

  there is a continuum $\Gamma_i^{\ ,\kappa} \subset \mathcal{W}_i$, *with* $\Gamma_i^{\ ,\kappa} \cap \ell_{v_i} \neq \varnothing$ *and* $\Gamma_i^{\ ,\kappa} \cap \ell_{w_i} \neq \varnothing$ *such that for each* $z \in \Gamma_i^{\ ,\kappa}$, *it follows that*

$$\psi^j(z) \in \mathcal{W}_{\delta_j}\,, \quad \eta(\psi^j(z)) \in \Lambda^{k_j}_{(\delta_{j-1},\delta_j)}\,, \quad \forall\, j \in \mathbb{N};$$

- *for any doubly-infinite sequence* $\boldsymbol{\delta} = (\dots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \dots)$, *with* $\delta_j \in \{0,1\}$ $(\forall\, j \in \mathbb{Z})$ *and for any* $\kappa = (\dots, k_{-2}, k_{-1}, k_0, k_1, k_2, \dots)$, *with*

$$k_m \in \mathcal{N}_{(\delta_{m-1},\delta_m)}\,, \quad (\forall\, m \in \mathbb{Z}),$$

there is a double-sided sequence of points

$$\tilde{w}_{(\ ,\kappa)} = (\dots, \tilde{w}_{-2}, \tilde{w}_{-1}, \tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \dots),$$

with $\tilde{w}_j \in \mathcal{D} \cap \mathcal{W}_{\delta_j}$ *such that*

$$\psi(\tilde{w}_j) = \tilde{w}_{j+1}\,, \quad \eta(\psi^j(\tilde{w}_j)) \in \Lambda^{k_j}_{(\delta_{j-1},\delta_j)}\,, \quad \forall\, j \in \mathbb{Z}.$$

The proof is omitted, as it is just a repetition of that of Theorem 2.

An application of Theorem 4 to the nonlinear Hill's equation is the following, where we give a new proof to some of the results contained in [16], [51]. We warn that we don't claim that all the solutions of equation (1.1) are oscillatory under the mild sign conditions on $q(t)$ considered here. What we prove is that there exist solutions with such a complicated oscillatory behavior like those depicted in the next Theorem 5.

Theorem 5. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a locally Lipschitz continuous function satisfying* (g_0), (g_{mon}) *and* (g_∞). *Let* $q : \mathbb{R} \to \mathbb{R}$ *be a continuous,* T*-periodic and piece-wise monotone function. Suppose that there is a finite sequence of points* $t_0 < t_1 < \dots < t_{2k+1} < t_{2k+2} = t_0 + T$ *such that for* $j = 0, \dots, k$,

$$q \ge 0 \ \text{ and } \ q \not\equiv 0 \ \text{ on } \ [t_{2j}, t_{2j+1}], \quad q \le 0 \ \text{ and } \ q \not\equiv 0 \ \text{ on } \ [t_{2j+1}, t_{2j+2}].$$

Then, there are positive integers m_0^*, m_1^* $\dots$, m_k^*, *such that the following conclusion hold:*

- *For each* $n \ge 1$ *and for each finite sequence of integers*

$$(m_0^1, \dots, m_k^1, m_0^2, \dots, m_k^2, \dots, m_0^n, \dots, m_k^n)$$

with $m_j^s \ge m_j^*$ *and for each finite sequence*

$$(\delta_0^1, \dots, \delta_k^1, \delta_0^2, \dots, \delta_k^2, \dots, \delta_0^n, \dots, \delta_k^n),$$

with $\delta_j^s \in \{0,1\}$, *(for each* $s = 1, \dots n$ *and* $j = 0, \dots, k$*) and such that*

$$\sum_{\substack{s=1,\dots,n\\ j=0,\dots k}} m_j^s + \delta_j^s = \textit{even},$$

there are at least two nontrivial nT*-periodic solutions* $x(\cdot)$ *of equation* (1.1) *such that* x *has exactly* m_j^s *zeros in the interval* $]t_{2j}+(s-1)T, t_{2j+1}+(s-1)T[$ *and exactly* δ_j^s *zeros and* $1-\delta_j^s$ *changes of sign of* $\dot{x}$ *in the interval* $]t_{2j+1}+(s-1)T, t_{2j+2}+(s-1)T[$. *These two solutions are such that* $x(t_0) \ge 0, \dot{x}(t_0) \ge 0$ *and* $x(t_0) \le 0, \dot{x}(t_0) \le 0$, *respectively.*

- *For each* $N > \max\{m_i^* : i = 0, \dots, k\}$ *and for each sequence of integers*

$$(m_0, m_1, \dots, m_j, \dots)$$

with $m_j \in [m_i^*, N]$ *for* $j \equiv i$ *(mod k) and for each sequence*

$$(\delta_0, \delta_1, \dots, \delta_j, \dots)$$

with $\delta_j \in \{0, 1\}$, *there are two closed connected sets* $\mathcal{C}^+ \subset \bar{A}_1 \setminus \{0\}$ *and* $\mathcal{C}^- \subset \bar{A}_3 \setminus \{0\}$, *with* $\mathcal{C}^+$ *and* $\mathcal{C}^-$ *intersecting both the* x *and the* y *axes such that for each* $(a, b) \in \mathcal{C}^+ \cup \mathcal{C}^-$, *there is at least one solution* $x(\cdot)$ *of equation* (1.1) *with* $x(t_0) = a$, $\dot{x}(t_0) = b$, *such that, if* $j = sk + i$, *then* x *has exactly* m_j *zeros in the interval* $]t_{2i} + (s-1)T, t_{2i+1} + (s-1)T[$ *and exactly* δ_j *zeros and* $1 - \delta_j$ *changes of sign of* $\dot{x}$ *in the interval* $]t_{2i+1} + (s-1)T, t_{2i+2} + (s-1)T[$;
- *For each* $N > \max\{m_i^* : i = 0, \dots, k\}$ *and for each double-sided sequence of integers*

$$(\dots, m_{-1}, m_0, m_1, \dots, m_j, \dots)$$

with $m_j \in [m_i^*, N]$ *for* $j \equiv i$ *(mod k) and for each sequence*

$$(\dots, \delta_{-1}, \delta_0, \delta_1, \dots, \delta_j, \dots)$$

with $\delta_j \in \{0, 1\}$, *there is at least one solution* $x(\cdot)$ *of equation* (1.1) *such that* x *has exactly* m_j *zeros in the interval* $]t_{2i} + (s-1)T, t_{2i+1} + (s-1)T[$ *and exactly* δ_j *zeros and* $1 - \delta_j$ *changes of sign of* $\dot{x}$ *in the interval* $]t_{2i+1} + (s-1)T, t_{2i+2} + (s-1)T[$, *when* $j = sk + i$.

Proof. We give a proof in the simpler case in which $k = 0$. We also assume $t_0 = 0$ (there is no loss of generality in this latter assumption), so that we can split the interval $[0, T]$ as the union of two intervals $[0, \tau]$ and $[\tau, T]$ such that

$$q \geq 0 \ \text{ and } \ q \not\equiv 0 \ \text{ on } \ [0, \tau], \quad q \leq 0 \ \text{ and } \ q \not\equiv 0 \ \text{ on } \ [\tau, T].$$

Now we take as ψ the Poincaré map ϕ_0^T associated to the planar system (1.4). We note that, due to the superlinear growth of g at infinity, we have that some solutions of (1.1) blow up in the intervals where $q \leq 0$ (cf. [11]), so that D_ψ is a proper open subset of $\mathbb{R}^2$ which may possess a complicated structure [15]. In any case, thanks to Example 3, we have that ψ is proper on the compact sets. Then we define

$$\mathcal{W}_0 = \bar{A}_1 \cap \mathcal{A}[r_1, r_2], \quad \mathcal{W}_1 = \bar{A}_3 \cap \mathcal{A}[r_1, r_2],$$

where r_1 and r_2 are chosen as follows: from Proposition 1, we first fix $r_1 > R^*$. This determines a number $n_{r_1}^*$ and hence we can define $m_0^* = n_{r_1}^* + 1$. At this point, we can choose any number $N > m_0^*$ and take $r_2 > \max\{R_j : j \in [m_0^*, N]\}$ for the R_j's as in Proposition 1. After these definitions, and using the arc expansion property ensured by Proposition 1, we can easily enter into the frame of Theorem 4 and arrive to the conclusion.

For the case $k \geq 1$, one should use a more general version of Theorem 4, using a decomposition of ψ of the form

$$\psi = \phi_{t_{2k}}^{t_{2k+2}} \circ \cdots \circ \phi_{t_0}^{t_2}$$

and repeating a similar argument like that of the proof of Theorem 4, but taking into account the the whole information about the oscillation of the solutions in the single intervals $[t_{2j}, t_{2j+2}]$. For a different proof of these results, see [51] for the periodic problem and [16] for the "chaotic solutions". □

Remark 3. The upper bound N for the numbers of zeros in the intervals of positivity of the function $q(t)$ is assumed in Theorem 5 only for the purpose to have a direct application of Theorem 4. Using more carefully the properties given in Lemma 1 and Lemma 2 (instead of putting a bound for the oscillations like in Proposition 1) one could obtain a more general result in which the request of upper bound N can be avoided. Results in this direction have been obtained by Terracini and Verzini in [68] and by Capietto, Dambrosio and Papini in [16].

We also point out that the same result of Theorem 5 can be obtained for the damped nonlinear Hill's equation

$$\ddot{x} + c\dot{x} + q(t)g(x) = 0, \tag{3.11}$$

with $c \in \mathbb{R}$ and $q(t)$ and $g(x)$ satisfying the same conditions of Theorem 5 (see also [16] and [51]). In fact, Lemma 1 and Lemma 2 and hence Proposition 1 are true for equation (3.11), too.

4. Appendix 1: Looking for an Underlying Geometric Structure

Here, we show how the results of the previous sections can be generalized to the case of arbitrary two-dimensional cells. In doing this, we try to make more explicit the geometric structure which allows to obtain results like Theorem 1 and Theorem 2.

Results about chaotic dynamical systems are nowadays available in various excellent books and articles. Probably, the most famous model for the chaotic dynamics is given by the celebrated Smale Horseshoe [58], [59]. General results about the horseshoe, as well applications to various examples of dynamical systems can be found, for instance, in [27], [47], [70], [55]. In some recent years, various authors have developed topological approaches in order to prove the existence of chaotic dynamics either for some specific examples of differential equations ([30], [32], [67]) or for dynamical systems satisfying suitable assumptions in order to apply some kind of index theory [17], [18], [45], [60], [61], [62], [63], [64], [66], [71], [73], [74], [75], [80]. Other approaches based on variational methods have been applied to differential equations as well (see, e.g., [2], [3], [56] for results in that direction).

The aim of this section is just to show how the arguments described in Theorem 1 and Theorem 2 can be easily extended to the more general setting of two-dimensional cells (the case of one-dimensional cells is much more simple and is only sketched in Remark 6). In some recent articles, [34], [35], a general theory for expanding maps yielding to chaotic dynamics has been developed. The settings of [34] and [35] appear to be more general than ours. On the other hand, in our case, as a by-product of the arc expansion property, it is possible to obtain a fixed point theorem which, in turns, allows to prove, in an elementary manner, the existence of periodic orbits (and periodic solutions for the nonlinear Hill's equation) too.

First of all, we introduce some definitions.

Definition 1. We define an *oriented cell* $\widetilde{\mathcal{R}}$ as a pair $(\mathcal{R}, \mathcal{R}^-)$, where $\mathcal{R} \subset \mathbb{R}^2$ is a two-dimensional cell (i.e., a subset of the plane homeomorphic to the unit square $Q = [-1,1]^2$) and $\mathcal{R}^- \subset \partial\mathcal{R}$ is the union of two disjoint compact arcs, too. The two components of $\mathcal{R}^-$ will be denoted by $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$ and conventionally called the left and the right sides of $\mathcal{R}$. The order in which we make the choice of naming $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$ is not important in what follows.

Since $\partial\mathcal{R}$ is a simple closed curve, we have that $\mathcal{R}^+ := \overline{\partial\mathcal{R} \setminus \mathcal{R}^-}$ is the union of two disjoint compact arcs, too. The two components of $\mathcal{R}^+$ will be denoted by $\mathcal{R}_b^+$ and $\mathcal{R}_t^+$ (the base and the top of $\mathcal{R}$); again, the order is not important.

To consider the unit square as an oriented cell, a natural choice will be that of $Q_l^- = \{-1\} \times [-1,1]$, $Q_r^- = \{1\} \times [-1,1]$, $Q_b^+ = [-1,1] \times \{-1\}$ and $Q_t^+ = [-1,1] \times \{+1\}$.

As a consequence of the Shoenflies theorem [31] or [46], it is not difficult to see that, given an oriented cell $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$, there is a homeomorphism h of the plane onto itself such that $h(Q) = \mathcal{R}$ and $h(Q_l^-) = \mathcal{R}_l^-$, $h(Q_r^-) = \mathcal{R}_r^-$, $h(Q_b^+ \cup Q_t^+) = \mathcal{R}^+$.

Let $\psi : \mathbb{R}^2 \supset D_\psi \to \mathbb{R}^2$ be a (continuous) map that we consider to be defined on a set D_ψ. Let $\mathcal{D}$ be a subset of D_ψ and let also $\mathcal{K} \subset \mathbb{R}^2$ be compact set.

Definition 2. We say that $(\mathcal{D}, \psi)$ is *proper on* $\mathcal{K}$ if ψ is continuous on $\mathcal{D}$ and, for each sequence $z_n \in \mathcal{D} \cap \mathcal{K}$ such that $\psi(z_n)$ is bounded, there is a subsequence z_{j_n} converging to a point of $\mathcal{D}$.

This, clearly, is equivalent to the requirement that if there is any $w \in (\partial\mathcal{D} \setminus \mathcal{D}) \cap \mathcal{K}$, then $\lim_{\substack{z \to w \\ z \in \mathcal{D} \cap \mathcal{K}}} |\psi(z)| = \infty$.

Remark 4. *a*) In the above definition $\mathcal{D}$ is not necessarily the whole domain of ψ. *b*) If $(\mathcal{D}, \psi)$ is proper on $\mathcal{K}$, then $(\mathcal{D}, \psi)$ is proper on any closed subset of $\mathcal{K}$, that is $(\mathcal{D} \cap C, \psi)$ is proper on $\mathcal{K}$, for each closed set $C \subset \mathbb{R}^2$. In particular, if $\mathcal{D} \cap \mathcal{K}$ is closed (for instance, if $\mathcal{K} \subset \mathcal{D}$), then, any continuous map ψ is such that $(\mathcal{D}, \psi)$ is proper on $\mathcal{K}$. *c*) If $(\mathcal{D}_1, \psi_1)$ is proper on $\mathcal{K}_1$ and $(\mathcal{D}_2, \psi_2)$ is proper on $\mathcal{K}_2$ then, $(\mathcal{D}_{1,2}, \psi_2 \circ \psi_1)$ is proper on $\mathcal{K}_1$ for

$$\mathcal{D}_{1,2} := \mathcal{D}_1 \cap \psi_1^{-1}(\mathcal{D}_2 \cap \mathcal{K}_2).$$

The set $\mathcal{D}_{1,2}$ will be considered as the natural domain for the composition of two proper maps. By induction, for the composition of $\psi_1, \ldots, \psi_n$, with $(\mathcal{D}_i, \psi_i)$ proper on $\mathcal{K}_i$ for $i = 1, \ldots, n$, the corresponding "natural domain" will be the set

$$\mathcal{D}_{1,\dots,n} = \Big\{ z \in \mathcal{D}_1 \, : \, \psi_1(z) \in \mathcal{D}_{2,\dots,n} \cap \mathcal{K}_2 \Big\}.$$

Definition 3. Let $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ be two oriented cells and let $\psi : \mathbb{R}^2 \supset D_\psi \to \mathbb{R}^2$ be a map. Consider also a set $\mathcal{D} \subset D_\psi$. We say that $(\mathcal{D}, \psi)$ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ and write

$$(\mathcal{D}, \psi) : \widetilde{\mathcal{A}} \multimap \widetilde{\mathcal{B}}$$

if

- *$(\mathcal{D}, \psi)$ is proper on $\mathcal{A}$,*
- *for any path $\Gamma \subset \mathcal{A}$ such that $\Gamma \cap \mathcal{A}_l^- \neq \varnothing$ and $\Gamma \cap \mathcal{A}_r^- \neq \varnothing$, there is a path $\Gamma' \subset \Gamma \cap \mathcal{D}$ such that*

$$\psi(\Gamma') \subset \mathcal{B}, \quad \psi(\Gamma') \cap \mathcal{B}_l^- \neq \varnothing, \quad \psi(\Gamma') \cap \mathcal{B}_r^- \neq \varnothing.$$

Lemma 5. *Suppose that $(\mathcal{D}_1, \psi_1) : \widetilde{\mathcal{A}} \multimap \widetilde{\mathcal{B}}$ and $(\mathcal{D}_2, \psi_2) : \widetilde{\mathcal{B}} \multimap \widetilde{\mathcal{C}}$. Then $(\mathcal{D}_{1,2}, \psi_2 \circ \psi_1) : \widetilde{\mathcal{A}} \multimap \widetilde{\mathcal{C}}$.*

Proof. The result easily follows from the definition. □

Theorem 6. *Suppose that there is an oriented cell $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$, a map $\psi : \mathbb{R}^2 \supset D_\psi \to \mathbb{R}^2$ and a set $\mathcal{D} \subset \mathbb{R}^2$, such that $(\mathcal{D}, \psi) : \widetilde{\mathcal{R}} \multimap \widetilde{\mathcal{R}}$. Then, there is at least one $z^* \in \mathcal{R} \cap \mathcal{D}$ such that $\psi(z^*) = z^*$.*

Proof. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism such that $h(Q) = \mathcal{R}$, $h(Q_l^-) = \mathcal{R}_l^-$ and $h(Q_r^-) = \mathcal{R}_r^-$ and consider the map

$$\phi(x) := h^{-1}(\psi(h(x)), \quad x \in \mathcal{E} := h^{-1}(\mathcal{D}).$$

By the assumptions on $(\mathcal{D}, \psi)$ and observing also that h and h^{-1} take bounded sets to bounded sets, is is straightforward to check that $(\mathcal{E}, \phi) : (Q, Q^-) \multimap (Q, Q^-)$.

For $\phi = (\phi_1, \phi_2)$ and $x = (x_1, x_2)$, let us consider the set

$$\mathcal{S} = \Big\{ x \in \mathcal{E} \cap Q : \ \ \phi_1(x) = x_1 \, , \ |\phi_2(x)| \leq 1 \Big\}.$$

The properness of $(\mathcal{E}, \phi)$ on Q implies that $\mathcal{S}$ is closed.

Let $\gamma : [0, 1] \to Q$ be a continuous map such that $\gamma(0) \in Q_l^-$ and $\gamma(1) \in Q_r^-$. By the "stretching" hypothesis, there are $t_1, t_2 \in [0, 1]$ such that $\phi(\gamma(t_1)) \in Q_l^-$ and $\phi(\gamma(t_2)) \in Q_r^-$. Moreover, if we denote by $I \subset [0, 1]$ the interval determined by t_1 and t_2, we can also assume that $\gamma(I) \subset \mathcal{E}$ and $\phi(\gamma(I)) \subset Q$.

Hence, $\phi_1(\gamma(t_1)) \leq \gamma_1(t_1)$ and $\phi_1(\gamma(t_2)) \geq \gamma_1(t_2)$ and therefore there is some $t_0 \in I$ such that $\phi_1(\gamma(t_0)) = \gamma_1(t_0)$. As $\phi(\gamma(t_0)) \in Q$, we also have that $|\phi_2(\gamma(t_0))| \leq 1$.

In this manner, we have proved that any path contained in Q and joining Q_l^- with Q_r^- intersects the set $\mathcal{S}$. By Lemma 4, it follows that $\mathcal{S}$ contains a continuum $\mathcal{C}$ joining Q_b^+ with Q_t^+. From the definition of $\mathcal{S}$ it also follows that

$$\phi_2(\mathcal{C}) \subset [-1, 1]$$

and hence $x_2 - \phi_2(x_1, x_2) \leq 0$ for $(x_1, x_2) \in \mathcal{C} \cap Q_b^+$, as well as $x_2 - \phi_2(x_1, x_2) \geq 0$ for $(x_1, x_2) \in \mathcal{C} \cap Q_t^+$. Then, the Bolzano theorem guarantees the existence of a $w^* = (w_1^*, w_2^*) \in \mathcal{C}$ such that $\phi_2(w^*) = w_2^*$. From the definition of $\mathcal{S}$ we can conclude that

$$\phi(w^*) = w^* .$$

Clearly, $z^* := h(w^*) \in \mathcal{D} \cap \mathcal{R}$ is a fixed point of ψ. This concludes the proof. □

Theorem 7. *Suppose that there are two oriented cells* $\widetilde{\mathcal{R}}_0 = (\mathcal{R}_0, \mathcal{R}_0^-)$, *and* $\widetilde{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-)$, *with* $\mathcal{R}_0 \cap \mathcal{R}_1 = \varnothing$. *Let* $\psi : \mathbb{R}^2 \supset D_\psi \to \mathbb{R}^2$ *be a continuous map and assume that there is a set* $\mathcal{D} \subset \mathbb{R}^2$, *such that*

$$(\mathcal{D}, \psi) : \widetilde{\mathcal{R}}_i \mathrel{\triangleleft\!\!\rightsquigarrow} \widetilde{\mathcal{R}}_j$$

for each $i, j \in \{0, 1\}$. *Then, the following conclusions hold:*

- (r_1) *for each* $i = 0, 1$, *the map* ψ *has at least one fixed point* $\tilde{z}_i \in \mathcal{D} \cap \mathcal{R}_i$;
- (r_2) *for each* $i = 0, 1$ *and for any finite sequence* $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$, *with* $k \geq 1$ *and* $\delta_j \in \{0, 1\}$ *(for all* $j = 1, \dots, k$*), there are points* $\tilde{z}_{(i,\ )} \in \mathcal{D} \cap \mathcal{R}_i$ *which are fixed points of* ψ^{k+1} *and satisfy*

$$\psi^j(\tilde{z}_{(i,\ )}) \in \mathcal{R}_{\delta_j}, \quad \forall\, j = 1, \dots, k;$$

- (r_3) *for each* $i = 0, 1$ *and for any sequence* $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots)$, *with* $\delta_j \in \{0, 1\}$ $(\forall\, j \in \mathbb{N})$, *there is a continuum* $\Gamma_i \subset \mathcal{R}_i$, *with* Γ_i *intersecting both the components of* $\mathcal{R}_i^+$, *such that for each* $z \in \Gamma_i$, *it follows that*

$$\psi^j(z) \in \mathcal{R}_{\delta_j}, \quad \forall\, j \in \mathbb{N};$$

- (r_4) *for any doubly-infinite sequence* $\boldsymbol{\delta} = (\dots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \dots)$, *with* $\delta_j \in \{0, 1\}$ $(\forall\, j \in \mathbb{N})$, *there is a double-sided sequence of points*

$$\tilde{z}\ \ = (\dots, \tilde{z}_{-2}, \tilde{z}_{-1}, \tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \dots),$$

with $\tilde{z}_j \in \mathcal{D} \cap \mathcal{R}_{\delta_j}$ *such that*

$$\psi(\tilde{z}_j) = \tilde{z}_{j+1}, \quad \forall\, j \in \mathbb{Z}.$$

Proof. The proof of (r_1) and (r_2) is a direct consequence of Theorem 6. For the proof of (r_3) and (r_4) we have only to repeat that of Theorem 2 with $\mathcal{R}_i$ playing the same role of $\mathcal{W}_i$. □

Remark 5. We notice that the hypothesis that $\mathcal{R}_0 \cap \mathcal{R}_1 = \varnothing$ is assumed here only in order to avoid the possibility of "trivial" cases for the sequence $(\tilde{z})_j$ as well as to have different fixed points for the map ψ and its iterates. The condition of non-intersection for the cells can be avoided if we have some control on the sequence of points which enter in $\mathcal{R}_0 \cap \mathcal{R}_1$ after some step. Sometimes, this will be not a difficult task, for instance, when $\mathcal{R}_0 \cap \mathcal{R}_1$ is a sufficiently small set (e.g., a singleton).

The same remark clearly applies to Theorem 2 and Theorem 4 with respect to $\mathcal{W}_0$ and $\mathcal{W}_1$.

Remark 6. The argument we have used in the proof of Theorem 2 can be applied in order to show the well known fact that an expanding self-map of the unit interval produces a very complicated dynamics through its iterates [42].

Let $f : [0,1] \to [0,+\infty)$ be a continuous function such that $f(0) = f(1) = 0$ and suppose that $f(c) \geq 1$ for some $c \in]0,1[$. Let us set $\mathcal{W}_1 = [0,a]$ and $\mathcal{W}_2 = [b,1]$, where $a = \min\{t \in [0,1] : f(t) = 1\}$ and $b = \max\{t \in [0,1] : f(t) = 1\}$. Then it is easy to see that an expansion arc property holds and all the consequences of Theorem 7 hold in this one-dimensional case. (see [34], [38] for the same situation). To be more precise, if $a < b$ (which is surely true when $f(c) > 1$), we enter in the setting of disjoint cells. Otherwise, if $f(c) = 1$ and $a = c = b$, we could apply Theorem 7 with Remark 5 and produce any sequence of symbols "left-right" (where "left" means "in the interval $[0,a]$" and "right" means "in the interval $[b,1]$") as we know that the only sequences which have a term $z_i = c$, must be identically equal to zero since then.

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