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LINEAR DYNAMICAL SYSTEMS: AN AXIOMATIC APPROACH

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Abstract. Linear dynamical systems are introduced in a general axiomatic way, and their development is carried out in great simplicity. The approach is closely related both with the classical transfer function approach and with the Willems behavioral approach.

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1. Introduction

The main goal of this paper is to describe in an intrinsic way a continuous-time linear dynamical system, which usually is defined as the solution space of a set of linear constant coefficient differential equations. This natural problem was posed in Willems [21], and the reader is refered to that paper for the discussion. In the discrete-time case, as is well-known, there is a very elegant characterization of linear dynamical systems (see Willems [20],[21]).

Let k be $\mathbb{R}$ or $\mathbb{C}$, and let s be an indeterminate. Let $\mathcal{H}$ denote the space of infinitely often differentiable functions of the nonnegative variable and O the ring of proper rational functions of the indeterminate s. There is a canonical k-linear map $L : O \to \mathcal{H}$ which we call the (inverse) Laplace transform and which is defined by the formula

$$L(g)(t) = b_0 + b_1\frac{t}{1!} + b_2\frac{t^2}{2!} + \cdots, \quad t \geq 0,$$

where b_i are the coefficients of expansion of g at infinity. Clearly, the image of L consists just of exponential functions. The crucial fact to be used in this paper is that $\mathcal{H}$ possesses a natural structure of a module over O. This can be introduced by the formula

$$g\xi = (L(g) * \xi)'.$$

(Differentiation is needed in order to normalize the multiplication rule, that is, in order to have $1\xi = \xi$.) Note that the element s^{-1} acts by integration: if $\xi \in \mathcal{H}$, then $s^{-1}\xi$ is the function $t \mapsto \int_0^t \xi$. Hence the fundamental Newton-Leibniz formula may be expressed as follows: $\xi = s^{-1}\xi' + L\xi(0)$. It is remarkable that the module $\mathcal{H}$ has no torsion. The reader recognizes that this is a weak

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form of the Titchmarsh theorem (see [13]), which can be easily shown. Indeed, multiplication by s^{-1} is an injective operator, and this already implies the statement since every element of the ring O is a power of s^{-1} multiplied by an invertible element. It makes sense therefore to consider the fraction space of $\mathcal{H}$. We call this the Mikusinski space and denote by $\mathcal{M}$; its elements are regarded as generalized functions. By definition, $\mathcal{M}$ is a linear space over $k(s)$. It is worthwhile to remark that the operators s and d/dt are distinct. (First of all, s acts on all generalized functions, while d/dt acts on regular functions only; next, by the Newton–Leibniz formula, we have $s\xi = d\xi/dt + sL\xi(0)$ for a regular function ξ.) We recall that in the literature on systems and control (see Oberst [14], for example) these two operators are equal.

Let now q be a signal number, and suppose we are given a linear dynamical system $\mathcal{B}$ with signal number q. Consider its any "minimal" representation

$$R(d/dt)\xi = 0, \quad \xi \in \mathcal{H}^q,$$

where $R(s) = R_0 + R_1 s + \dots + R_n s^n$ is a polynomial matrix, say, with rank p. Using the formula $d\xi/dt = s\xi - sL\xi(0)$, the above differential equation can be written as

$$R(s)\xi = [s \dots s^n]Tp(R)L\bar{\xi}(0), \quad \xi \in \mathcal{H}^q,$$

where

$$Tp(R) = \begin{bmatrix} R_1 & R_2 & \dots & R_n \\ R_2 & R_3 & & 0 \\ \vdots & & & \vdots \\ R_n & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad \bar{\xi}(0) = \begin{bmatrix} \xi(0) \\ \xi^{(1)}(0) \\ \vdots \\ \xi^{(n-1)}(0) \end{bmatrix}.$$

The motions with initial state 0 are of special interest, and we see that they are described by the equation

$$R(s)\xi = 0, \quad \xi \in \mathcal{H}^q.$$

Let $m = q - p$. We can find a proper rational $q \times m$ matrix G such that the sequence

$$0 \to O^m \xrightarrow{G} O^q \xrightarrow{R} k(s)^p$$

is exact. Tensoring this by $\mathcal{H}$ and using the fact that $\mathcal{H}$ is without torsion (and hence an inductive limit of finitely generated free modules), we obtain an exact sequence

$$0 \to \mathcal{H}^m \xrightarrow{G} \mathcal{H}^q \xrightarrow{R} \mathcal{M}^p.$$

It follows that the space of motions having 0 as initial state is equal to $G\mathcal{H}^m$. We note that G is necessarily left biproper since there is an embedding of O^q/GO^m into $k(s)^p$. (Obviously, the matrix G is uniquely determined up to biproper transformation from the right, and it is natural to call the equivalence class of G or, what is equivalent, the module GO^m the transfer function.) It easy to see from the "operational" equation above that there are finitely many linearly independent initial states. Further, because R has full row rank, the equation

$$R(s)w = [s \ldots s^n]Tp(R)x_0, \quad w \in k(s)^q,$$

is solvable for each $x_0 \in k^n$, and consequently to each initial state there corresponds at least one exponential solution.

Thus $\mathcal{B}$ satisfies the property that

$$G\mathcal{H}^m \subseteq \mathcal{B} \subseteq G\mathcal{H}^m + L(O)^q \quad \text{and} \quad \dim(\mathcal{B}/G\mathcal{H}^m) < +\infty$$

for some left biproper rational matrix G. The other property of $\mathcal{B}$ is obvious, and is as follows:

$$\frac{d}{dt}(\mathcal{B}) \subseteq \mathcal{B}.$$

Applying the main result of [11], we shall show that the two properties above characterize linear dynamical systems among all k-linear subspaces of $\mathcal{H}^q$.

It should be pointed out that the Willems problem has already been investigated before. In Soethoudt [19] a functional analytic solution is presented, which uses the notion of "locally specified" and which is in the spirit of Willems [21]. Our approach is *purely* algebraic. This is achieved by using an abstract, simplified version of the Heaviside–Mikusinski formalizm. We remark that this formalizm makes it possible to treat the continuous-time and the discrete-time cases simultaneously. Even more, it allows us to avoid completely the concept of time, and in this sense our approach is very similar to that of Oberst [14].

The paper addresses also the standard topics of controllability and autonomy, and state-space representations. Actually, we shall deal with the general singular case; the "classical" case will be deduced from it. We emphasize that the general singular theory is easier. Moreover, it is analogous to the Willems theory as developed in [20] and [21]. The reader will find an analogy between impulsive motions and trajectories defined on the left half-axis, "regular" motions and trajectories defined on the right half-axis, motions with zero initial state and trajectories defined on the whole axis.

Throughout, k is an arbitrary field, s an indeterminate, q a signal number. We put $t = s^{-1}$, and denote by O the ring of proper rational functions in $k(s)$. If X is a finite-dimensional k-linear space, then we write $X[s]$ and $O(X)$ to denote $k[s] \otimes X$ and $O \otimes X$, respectively. Given k-linear spaces V and W such that $V \subseteq W$, we denote by $[W : V]$ the dimension of W/V.

2. Some preliminaries

For convenience of the reader, we collect here some definitions and facts as given in [11].

A transfer function with input number m is a $k(s)$-linear subspace in $k(s)^q$ of dimension m. Equivalently, it can be defined also as an equivalence class of full column rank $q \times m$ rational matrices. (Two full column rank rational matrices are equivalent if one is obtained from the other by a nonsingular right transformation.) This is a natural definition when inputs and outputs are not classified a priori (see [6]). Note that possibly after permutation of signal variables a transfer function can be represented by a matrix of the form $\begin{bmatrix} I \\ G \end{bmatrix}$, where G is a uniquely determined rational matrix of size $(q-m)\times m$.

By a "classical" transfer function with input number m we mean any submodule $T \subseteq O^q$ of rank m such that O^q/T has no torsion. Equivalently, it can also be defined as an equivalence class of full column rank $q \times m$ left biproper rational matrices. (Two left biproper rational matrices are equivalent if one is obtained from the other by a biproper right transformation.) It should be emphasized that there is a canonical one-to-one correspondence between transfer functions and "classical" transfer functions which is given by $T \mapsto T \cap O^q$.

By the forward and backward shift operators we shall mean respectively k-linear endomorphisms $k[s]^q \to k[s]^q$ and $O^q \to O^q$ defined as

$$a_0s^n + \cdots + a_n \mapsto a_0s^{n-1} + \cdots + a_{n-1} \text{ and } (b_0 + b_1t + \cdots) \mapsto (b_1 + b_2t + \cdots).$$

A frequency response is a k-linear subspace $\Phi \subseteq k(s)^q$ satisfying the following properties: 1) there exists a transfer function T such that $T \subseteq \Phi$ and $[\Phi : T] < +\infty$; 2) Φ is invariant with respect to taking the polynomial and the strictly proper parts; 3) $\Phi \cap k[s]^q$ is invariant with respect to the forward shift, and $s\Phi \cap O^q$ with respect to the backward shift. The "T" is uniquely determined, and we call it the transfer function of Φ.

If Φ is a frequency response and T its transfer function, we say that Φ is controllable if $\Phi \subseteq T + k[s]^q$ and $\Phi \subseteq T + tO^q$. We say that Φ is regular if $\Phi \subseteq T + tO^q$.

We call a "classical" frequency response any k-linear subspace $\Phi \subseteq tO^q$ satisfying the following properties: 1) there exists a "classical" transfer function T such that $T \subseteq s\Phi$ and $[s\Phi : T] < +\infty$; 2) $s\Phi$ is invariant with respect to the backward shift. The "T" is uniquely determined, and we call it the transfer function of Φ.

Lemma 1. *There is a canonical one-to-one correspondence between regular frequency responses and "classical" ones; this is given by* $\Phi \mapsto \Phi \cap tO^q$.

A linear bundle is a congruence class of nonsingular rational matrices. (The congruence relation is the following: D_1 and D_2 are congruent if $D_2 = D_1B$ for some biproper B.) If Δ is a linear bundle and D its representative, then the rank $rk(\Delta)$ is defined as the size of D and the Chern number $\mathrm{ch}(\Delta)$ as minus the order at infinity of the determinant of D. The cohomology space $H^0(\Delta)$ is defined to be $k[s]^p \cap DO^p$, where p is the rank. The latter is a k-linear space of finite dimension.

An AR-model with output number p is a pair (Δ, R), where Δ is a linear bundle of rank p and R a full row rank polynomial $p \times q$ matrix such that $D^{-1}R$ is proper for any $D \in \Delta$. The number $q-p$ is called the input number. The space $X = H^0(t\Delta)$ is called the state space. The Chern number of Δ and the dimension of the state space are equal, and this common value is called the McMillan degree. The space $R^{-1}\{0\} = \{w \in k(s)^q | Rw = 0\}$ is called the transfer function and the space $R^{-1}(X) = \{w \in k(s)^q | Rw \in X\}$ the frequency response.

Two AR-models (Δ_1, R_1) and (Δ_2, R_2) are said to be equivalent if there exists a unimodular matrix U such that $R_2 = UR_1$ and $D_2^{-1}UD_1$ is biproper for $D_1 \in \Delta_1$ and $D_2 \in \Delta_2$.

Lemma 2. *The mapping that takes an AR-model to its frequency response establishes a one-to-one correspondence between equivalence classes of AR-models and frequency responses.*

If (Δ, R) is an AR-model and D a representative of Δ, then we say that (Δ, R) is controllable if R is right unimodular and $D^{-1}R$ right biproper. We say that (Δ, R) is regular if $D^{-1}R$ is right biproper.

Lemma 3. *An AR-model is controllable* (*resp. regular*) *if and only if its frequency response is controllable* (*resp. regular*)*.*

Lemma 4. *Let* (Δ, R) *be an AR-model with McMillan degree* d*. Suppose that* $n \geq \max\{d-1, 0\}$*. Then the model is controllable if and only if the linear map*

$$R : k[s]^q \cap s^n O^q \to H^0(s^n \Delta)$$

is surjective.

Remark. The "only if" part of the lemma can be slightly strengthened. Namely, if the model is controllable, then the linear map above is surjective for $n \geq d - 1$. (See the proof of Proposition 2 in [11].)

Lemma 5. *Let* (Δ, R) *be an AR-model. Then* $\deg(R) \leq \mathrm{ch}(\Delta)$*, and the model is regular if and only if the equality holds.*

Let (Δ, R) be an arbitrary AR-model with output number p. We certainly can find a nonsingular rational matrix D_1 such that $D_1 O^p = RO^q$. The linear bundle Δ_1 associated with D_1 does not depend on the choice of this latter. Obviously the AR-model (Δ_1, R) is regular; it is called the regular part of (Δ, R).

A "classical" AR-model with p outputs is a full row rank polynomial $p \times q$ matrix R. The space $X = k[s]^p \cap tRO^q$ is called the state space. As one knows, the degree (i.e., the maximum of degrees of full size minors) of R and the dimension of the state space are equal, and this common value is called the McMillan degree. The module $\{w \in O^q | Rw = 0\}$ is called the transfer function and the space $\{w \in tO^q | Rw \in X\}$ the frequency response. Note that the latter is equal to $\{w \in tO^q | Rw \text{ is polynomial}\}$, which is termed in Kuijper [3] as the rational behavior.

Lemma 6. *The mapping* $(\Delta, R) \mapsto R$ *establishes a one-to-one correspondence between regular AR-models and "classical" AR-models.*

(For Lemma 5 and Lemma 6 we refer the reader to [10].)

3. Operational Calculus

Assume that an operational calculus $(\mathcal{H}, L)$ is given. We mean that $\mathcal{H}$ is a non-torsion module over O and L an injective homomorphism of O into $\mathcal{H}$. Let us require the following

$$\mathcal{H} = t\mathcal{H} \oplus L(k) \tag{1}$$

to hold, that is, $\mathcal{H} = t\mathcal{H} + L(k)$ and $t\mathcal{H} \cap L(k) = \{0\}$. We call elements of $\mathcal{H}$ regular functions. (One may think of them as infinitely differentiable functions defined on $\mathbb{R}_+$.) Multiplication by $t = s^{-1}$ will be regarded as integration, and therefore elements of $\mathcal{H}$ multiplied by t should be thought as regular functions having initial value 0. The homomorphism L is interpreted as the (inverse) Laplace transform and elements of $L(k)$ as constant functions. The intuitive meaning of (1) is evident; we call this the Newton–Leibniz axiom. (We shall see in a moment that (1) indeed leads to the Newton–Leibniz formula.)

An obvious example of operational calculus can be obtained by taking $\mathcal{H} = O$ and $L = id$. Here are examples that are important for applications.

Example 1. Let $\mathcal{H} = k^{\mathbb{Z}_+}$. For $g \in O$, let

$$L(g) = (b_0, b_1, b_2, \dots),$$

where b_i are the coefficients in the expansion of g at infinity. Define

$$O \times \mathcal{H} \to \mathcal{H}, \quad (g, \eta) \mapsto L(g) * \eta.$$

(Here "*" stands for the convolution in $\mathcal{H}$.) This multiplication makes $\mathcal{H}$ into a module over O and L a homomorphism over O.

Example 2. Let $k = \mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{H} = C^\infty(\mathbb{R}_+, k)$. For $g \in O$, let

$$L(g)(x) = b_0 + b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} + \cdots), \quad x \geq 0,$$

where b_i are as above. Define

$$O \times \mathcal{H} \to \mathcal{H}, \quad (g, \eta) \mapsto (L(g) * \eta)'.$$

(Here "*" stands for the convolution in $\mathcal{H}$.) Clearly $\mathcal{H}$ becomes into a module over O and L a homomorphism over O.

Example 3. Let $k = \mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{H} = D'(\mathbb{R}_+, k)$, the space of distributions on $\mathbb{R}$ vanishing in $(-\infty, 0)$. Exactly as in the previous example, $\mathcal{H}$ gives rise to an operational calculus.

Remark. There are interesting examples of operational calculus (namely, those that come from continuous functions or locally integrable functions) where the Newton–Leibniz axiom does not hold.

Let $\mathcal{M}$ denote the fraction space of $\mathcal{H}$. (Note that $\mathcal{M}$ may be introduced as the localization with respect to the multiplicative set $\{t^n | n \geq 0\}$.) This is a linear space over $k(s)$. Its elements are called Mikusinski functions and will be regarded as generalized functions. Identifying $\mathcal{H}$ with its image in $\mathcal{M}$ under the canonical map $\xi \mapsto \xi/1$, we obviously have

$$\mathcal{H} \subseteq s\mathcal{H} \subseteq s^2\mathcal{H} \subseteq \cdots \quad \text{and} \quad \mathcal{M} = \cup s^n\mathcal{H}.$$

The homomorphism L can be evidently continued to a $k(s)$-linear map $k(s) \to \mathcal{M}$. This again will be denoted by L. We call elements of $L(k(s))$ Laplace functions, elements of $L(O)$ exponential functions, and elements of $\mathcal{J} = L(k[s])$ impulsive functions. We have an evident decomposition

$$\mathcal{J} = s\mathcal{J} \oplus L(k). \tag{2}$$

It is easily seen from (1) and (2) that there are canonical operators $\mathcal{H} \to \mathcal{H}$ and $\mathcal{J} \to \mathcal{J}$. We call them differentiation operators and denote respectively by d/dt and d/ds. There are also canonical linear maps $\mathcal{H} \to k$ and $\mathcal{J} \to k$ which we interpret respectively as the evaluation maps at times 0 and $+\infty$. Thus we have the Newton–Leibniz formulas

$$\eta = t\frac{d\eta}{dt} + L(\eta(0)) \quad \text{and} \quad \theta = s\frac{d\theta}{ds} + L(\theta(+\infty)),$$

where $\eta \in \mathcal{H}$ and $\theta \in \mathcal{J}$. These can be rewritten as

$$s\eta = \frac{d\eta}{dt} + s\eta(0) \quad \text{and} \quad t\theta = \frac{d\theta}{ds} + t\theta(+\infty).$$

These formulas can be easily generalized; namely, for each $n \geq 1$, we have the Taylor formulas

$$s^n\eta = \frac{d^n\eta}{dt^n} + s^n\eta(0) + \cdots + s\eta^{(n-1)}(0)$$

and

$$t^n\theta = \frac{d^n\theta}{ds^n} + t^n\theta(+\infty) + \cdots + t\theta^{(n-1)}(+\infty).$$

Lemma 7. $\mathcal{M} = \mathcal{H} \oplus s\mathcal{J}$.

Proof. That $\mathcal{M} = \mathcal{H} + s\mathcal{J}$ follows immediately from the first Taylor formula because every generalized function can be represented as $s^n\eta$ where $n \geq 0$ and $\eta \in \mathcal{H}$. Assume that $\mathcal{H} \cap s\mathcal{J} \neq \{0\}$. We then have

$$\eta = s^nL(a_1) + \cdots + sL(a_n),$$

where $\eta \in \mathcal{H}$ and $a_1, \ldots, a_n \in k$, with $a_1 \neq 0$. It follows that

$$t^n\eta - (t^{n-1}L(a_n) + \cdots + tL(a_2)) = a_1.$$

This contradicts (1). $\square$

The lemma implies in particular that there are canonical projection maps $\mathcal{M} \to \mathcal{H}$ and $\mathcal{M} \to s\mathcal{J}$. We shall denote them respectively by Π_+ and Π_-.

Suppose now we are given a polynomial matrix $A = A_0 + A_1 s + \cdots + A_n s^n$ and a proper rational matrix $B = B_0 + B_1 t + B_2 t^2 + \cdots$. We define differential operators $A(d/dt) : \mathcal{H}^q \to \mathcal{H}^p$ and $B(d/ds) : \mathcal{J}^q \to \mathcal{J}^p$ respectively as

$$A(d/dt) = A_0 + A_1 d/dt + \cdots + A_n d^n/dt^n$$

and

$$B(d/ds) = B_0 + B_1 d/ds + B_2 d^2/ds^2 + \cdots .$$

(Notice that $B(d/ds)\theta$ is well-defined for each $\theta \in \mathcal{J}^q$.) We define also the Toeplitz matrix $Tp(A)$ and the Hankel matrix $Hk(B)$ respectively as

$$Tp(A) = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ A_2 & A_3 & & 0 \\ \vdots & & & \vdots \\ A_n & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad Hk(B) = \begin{bmatrix} B_1 & B_2 & B_3 & \dots \\ B_2 & B_3 & B_4 & \dots \\ B_3 & B_4 & B_5 & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Lemma 8. *Let A and B be as above. Then*

$$Ker A(d/dt) = \{\eta \in \mathcal{H}^q | A\eta \in s\mathcal{J}^p\} \quad \text{and} \quad Ker B(d/ds) = \{\theta \in \mathcal{J}^q | B\theta \in t\mathcal{H}^p\}.$$

Proof. Let $\eta \in \mathcal{H}^q$ and $\theta \in \mathcal{J}^q$. Using the Taylor formulas, it can be easily shown that

$$A\eta = A(d/dt)\eta + [s \dots s^n] Tp(A) \bar{\eta}(0)$$

and

$$B\theta = B(d/ds)\theta + [t\ t^2\ \dots] Hk(B) \bar{\theta}(+\infty),$$

where

$$\bar{\eta}(0) = \begin{bmatrix} \eta(0) \\ \eta^{(1)}(0) \\ \vdots \\ \eta^{(n-1)}(0) \end{bmatrix} \quad \text{and} \quad \bar{\theta}(+\infty) = \begin{bmatrix} \theta(+\infty) \\ \theta^{(1)}(+\infty) \\ \theta^{(2)}(+\infty) \\ \vdots \end{bmatrix}.$$

The lemma follows. $\square$

We shall need the following

Lemma 9. *Let m and p be nonnegative integers, and assume we have exact sequences*

$$0 \to k[s]^m \xrightarrow{A_1} k[s]^{m+p} \xrightarrow{A_2} k[s]^p \to 0 \quad \text{and} \quad 0 \to O^m \xrightarrow{B_1} O^{m+p} \xrightarrow{B_2} O^p \to 0.$$

Then the sequences

$$0 \to \mathcal{H}^m \xrightarrow{A_1(d/dt)} \mathcal{H}^{m+p} \xrightarrow{A_2(d/dt)} \mathcal{H}^p \to 0$$

and

$$0 \to \mathcal{J}^m \xrightarrow{B_1(d/ds)} \mathcal{J}^{m+p} \xrightarrow{B_2(d/ds)} \mathcal{J}^p \to 0.$$

are exact.

Proof. Our sequences split, and consequently there exist polynomial matrices A_3, A_4 and proper rational matrices B_3, B_4 such that

$$[A_1\ A_3]\begin{bmatrix} A_4 \\ A_2 \end{bmatrix} = I \quad \text{and} \quad [B_1\ B_3]\begin{bmatrix} B_4 \\ B_2 \end{bmatrix} = I.$$

Therefore

$$[A_1(d/dt)\ A_3(d/dt)]\begin{bmatrix} A_4(d/dt) \\ A_2(d/dt) \end{bmatrix} = I$$

and

$$[B_1(d/ds)\ B_3(d/ds)]\begin{bmatrix} B_4(d/ds) \\ B_2(d/ds) \end{bmatrix} = I.$$

The lemma follows immediately. (Note that the sequences even are splittable.) □

Concluding the section, we note that one can easily introduce vector-valued functions. If X is a finite-dimensional k-linear space, we define $\mathcal{H}(X)$ to be $\mathcal{H} \otimes X$. Likewise, we set $\mathcal{M}(X) = \mathcal{M} \otimes X$ and $\mathcal{J}(X) = \mathcal{J} \otimes X$. (The tensor products are taken over k, of course.)

Remark. Operational calculus was developed by Heaviside, and then by Mikusinski [13]. The exposition above follows closely [6] and [8]. We recall that the starting point for Mikusinski was the ring structure of the function space. Our approach is based on the observation that when dealing with linear constant coefficient differential equations the module structure (over the ring of proper rational functions) is quite sufficient. The Newton–Leibniz axiom is new, and we hope that the reader will find it appealing. Note that the Mikusinski space of Example 1 is just the space of Laurent series considered in [16] and that of Example 2 is just the space of smooth-impulsive functions considered in [1] and [2].

4. Linear Systems

Given a transfer function T, let $T\mathcal{M}$ denote the image of the canonical injective homomorphism of $T \otimes \mathcal{M}$ into $k(s)^q \otimes \mathcal{M} = \mathcal{M}^q$. (The tensor products are taken over $k(s)$.) This is generated by elements of the form $f\xi$, with $f \in T$ and $\xi \in \mathcal{M}$. If G is a rational matrix representing T, then $T\mathcal{M} = G\mathcal{M}^m$ where m denotes the input number of T. To see this, we need only to tensor by $\mathcal{M}$ the commutative diagram

$$\begin{array}{ccc} k(s)^m & \xrightarrow{G} & k(s)^q \\ \downarrow & & || \\ T & \subseteq & k(s)^q \end{array}.$$

We claim that $L^{-1}(T\mathcal{M}) = T$. Indeed, choose a subspace $\mathcal{M}_1 \subset \mathcal{M}$ so that $\mathcal{M} = L(k(s) \oplus \mathcal{M}_1$. If G and m are as above, then $G\mathcal{M}_1^m \subseteq \mathcal{M}_1^q$, and consequently $G\xi \in L(k(s)^q$ if and only if $\xi \in L(k(s)^m$. This implies our claim. It follows in particular that the correspondence $T \mapsto T\mathcal{M}$ is injective. It should perhaps be noted that a $k(s)$-linear subspace $\mathcal{T} \subseteq \mathcal{M}^q$ is representable as $T\mathcal{M}$ if and only if it satisfies the following condition: if $f \in L^{-1}\mathcal{T}$, then $f\xi \in \mathcal{T}$ for each $\xi \in \mathcal{M}$.

Lemma 10. *let $\mathcal{B}$ be a k-linear subspace of $\mathcal{M}^q$. There may exist only one transfer function T such that*

$$T\mathcal{M} \subseteq \mathcal{B} \quad \textit{and} \quad [\mathcal{B} : T\mathcal{M}] < +\infty.$$

Proof. Assume that transfer functions T_1 and T_2 satisfy the condition of our lemma. Then this condition is clearly satisfied by $T = T_1 + T_2$ as well. It follows that

$$[T\mathcal{M} : T_1\mathcal{M}] < +\infty \quad \text{and} \quad [T\mathcal{M} : T_2\mathcal{M}] < +\infty.$$

Any $k(s)$-linear space (if it is not trivial of course) has infinite dimension over k. Hence, we must have $T_1\mathcal{M} = T\mathcal{M} = T_2\mathcal{M}$. As remarked above, it follows from this that $T_1 = T_2$. □

We are ready now to define a linear (dynamical) system. This is a k-linear subspace $\mathcal{B} \subseteq \mathcal{M}^q$ satisfying the following axioms:

(LS1) There exists a transfer function T such that

$$T\mathcal{M} \subseteq \mathcal{B} \subseteq Lk(s)^q + T\mathcal{M} \quad \text{and} \quad [\mathcal{B} : T\mathcal{M}] < +\infty;$$

(LS2) $\mathcal{B}$ is invariant with respect to Π_+ and Π_-;

(LS3) $\mathcal{B} \cap \mathcal{H}^q$ is invariant with respect to d/dt and $t\mathcal{B} \cap \mathcal{J}^q$ with respect to d/ds.

Comment. $\mathcal{M}^q$ is a very "huge" space, a universum (in Willems' terminology). Among all its subspaces those of the form $T\mathcal{M}$ surely are the most "conceivable". A linear system is a k-linear subspace that contains such a subspace and modulo it is spanned by a finite number of Laplace functions, and also satisfies certain natural properties of invariance.

Remark. The notion of a linear system can be introduced in the generality of arbitrary operational calculus. A linear system can be defined as a linear subspace $\mathcal{B}$ satisfying (LS1) and the folowing two properties: (LS2') $\mathcal{B} \cap Lk(s)^q$ is invariant with respect to Π_+ and Π_-; (LS3') $\mathcal{B} \cap LO^q$ is invariant with respect to d/dt and $t\mathcal{B} \cap Lk[s]^q$ with respect to d/ds. (Certainly, the operators Π_+ and Π_- are defined for Laplace functions, and the operators d/dt and d/ds are defined respectively for exponential and impulsive functions.)

Let $\mathcal{B}$ be a linear system. The transfer function T satisfying the property (LS1) is uniquely determined by the previous lemma; it is called the transfer function of $\mathcal{B}$. The space $t\mathcal{B}/T\mathcal{M}$ is called the state space. The number $[\mathcal{B} : T\mathcal{M}]$, which of course is the same as the dimension of the state space, is called the McMillan degree. If $\xi \in \mathcal{B}$, then $t\xi \operatorname{mod} T\mathcal{M}$ is called the initial state of ξ. So, elements from $T\mathcal{M}$ are considered as motions with zero initial state. (See Proposition 2 for the justification of all these definitions.) The condition $\mathcal{B} \subseteq T\mathcal{M} + Lk(s)^q$ means that there always exists in $\mathcal{B}$ a Laplace motion that has a given initial state. Finally, we remark that the property (LS2) amounts to saying that $\mathcal{B} = (\mathcal{B} \cap \mathcal{H}^q) \oplus (\mathcal{B} \cap s\mathcal{J}^q)$.

Assume an AR-model (Δ, R) is given, and let D be a representative of Δ. We then have differential equations

$$R(d/dt)\eta = 0, \quad \eta \in \mathcal{H}^q \quad \text{and} \quad (D^{-1}R)(d/ds)\theta = 0,\ \theta \in \mathcal{J}^q.$$

Let $\mathcal{B}_+$ denote the solution space of the first equation, and let $\mathcal{B}_-$ denote the solution space of the second one multiplied by s. Set $\mathcal{B} = \mathcal{B}_+ + \mathcal{B}_-$, and let X denote the state space of (Δ, R), i.e., the space $k[s]^p \cap tDO^p$.

Proposition 1. $\mathcal{B} = \{\xi \in \mathcal{M}^q |\ R\xi \in sL(X)\}$.

Proof. We remark that

$$sL(X) = sL(k[s]^p \cap tDO^p) = s\mathcal{J}^p \cap D\mathcal{H}^p,$$

where p is the output number of our AR-model. It follows that if $\xi = \xi_+ + \xi_-$, with $\xi_+ \in \mathcal{H}^q$ and $\xi_- \in s\mathcal{J}^q$, then

$$R\xi \in (s\mathcal{J}^p \cap D\mathcal{H}^p) \Longleftrightarrow R\xi_+ \in s\mathcal{J}^p \ \text{ and } \ R\xi_- \in D\mathcal{H}^p.$$

(This is because $D^{-1}R$ is proper and R is polynomial.) By Lemma 8,

$$\mathcal{B}_+ = \{\eta \in \mathcal{H}^q |\ R\eta \in s\mathcal{J}^p\} \quad \text{and} \quad \mathcal{B}_- = \{\theta \in s\mathcal{J}^q |\ D^{-1}R\theta \in \mathcal{H}^p\}.$$

The proposition follows. □

Remark. The proposition indicates in particular how to define the behavior of an AR-model when operational calculus does not satisfy the Newton-Leibniz axiom. It should be noted that most of the notions and results, which will be studied, can be generalized to this case.

We call $\mathcal{B}$ the behavior of (Δ, R) and its elements the motions of (Δ, R). By the previous proposition, if ξ is a motion of our model, then $R\xi = sL(x)$ for some $x \in X$. The state x is called the initial state of ξ.

Lemma 11. *Let T be the transfer function of (Δ, R). Then $T\mathcal{M}$ is the space of motions that have initial state 0.*

Proof. We have an exact sequence

$$0 \to T \to k(s)^q \to k(s)^p \to 0.$$

Tensoring this by $\mathcal{M}$, we get an exact sequence

$$0 \to T \otimes \mathcal{M} \to \mathcal{M}^q \to \mathcal{M}^p \to 0.$$

In view of the previous proposition, from this we obtain that

$$0 \to T\mathcal{M} \to \mathcal{B} \to sL(X) \to 0. \tag{3}$$

is an exact sequence. □

Proposition 2. *$\mathcal{B}$ is a linear system; the transfer function of $\mathcal{B}$ coincides with that of (Δ, R) and the state space of $\mathcal{B}$ is canonically isomorphic to that of (Δ, R).*

Proof. Obviously, $\mathcal{B}$ satisfies the properties (LS2) and (LS3). Let T be the transfer function of our model. The exact sequence (3) implies that $T\mathcal{M} \subseteq \mathcal{B}$ and $[\mathcal{B} : T\mathcal{M}] < +\infty$. Further, let ξ be a motion with initial state x. Clearly there exists $w \in k(s)^q$ such that $Rw = sx$. We then have $RL(w) = sL(x)$, and hence $\xi - L(w) \in T\mathcal{M}$. Thus $\mathcal{B}$ satisfies (LS1) and is indeed a linear system, and T is indeed its transfer function. Finally, it is clear that $t\mathcal{B}/T\mathcal{M}$ is canonically isomorphic to X. □

Let $\mathcal{B}$ be a linear system. It is easily seen that $\Phi = tL^{-1}(\mathcal{B})$ is a frequency response, and we call it the frequency response of $\mathcal{B}$. The transfer function of $\mathcal{B}$ clearly coincides with that of Φ.

Theorem 1. *The frequency response completely determines a linear system.*

Proof. Let $\mathcal{B}$ be a linear system. Let T denote its transfer function and Φ its frequency response. It is easily verified that $\mathcal{B} = L(\Phi) + T\mathcal{M}$; whence follows the validity of the theorem. □

We remark that if (Δ, R) is an AR-model and $\mathcal{B}$ its behavior, then the frequency response of $\mathcal{B}$ is equal to that of (Δ, R). (This is clear from the proof of Proposition 2.)

Theorem 2. *Two AR-models have the same behavior if and only if they are equivalent.*

Proof. Obviously, two equivalent AR-models have the same behavior. Conversely, if two AR-models have the same behavior, then they have the same frequency response and, in view of Lemma 2, must be equivalent. □

Theorem 3. *Every linear system allows an AR-representation.*

Proof. Let $\mathcal{B}$ be a linear system, and let Φ be its frequency response. By Lemma 2, there exists an AR-model having the frequency response Φ. By Theorem 1, the behavior of this AR-model must equal to $\mathcal{B}$. □

Remark. Theorem 2 was obtained first in Geerts and Schumacher [1,2] for the continuous-time case and in Ravi, Rosenthal and Schumacher [16] for the discrete-time case. In the discrete-time case there are alternative descriptions of singular linear systems (see [9], [12]). The discussion of the section is based in part on [8] which is a detailed version of [6].

5. State Models

A state model is a quintuple (X, Z, E, F, G), where X, Z are finite-dimensional linear spaces and $E, F : X \to Z$, $G : k^q \to Z$ linear maps satisfying the following properties:

(SM1) $sE - F$ has full column rank;

(SM2) $[sE - F\ G]$ has full row rank.

The space X is called the state space, the space Z the internal variable space, the number $\dim Z - \dim X$ the output number. The model is called observable if $sE - F$ is left unimodular and E has full column rank. (The model is called controllable if $[sE - F\ G]$ is right unimodular and $[E\ G]$ has full row rank.)

Example 4. Let $x' = Ax + Bu$, $v = Cx + Du$ be an ordinary classical linear system with m inputs and p outputs, and with state space X. Then

$$\left(X, X \oplus k^p, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} B & 0 \\ D & -I \end{bmatrix}\right)$$

is a state model.

Two state models $(X_1, Z_1, E_1, F_1, G_1)$ and $(X_2, Z_2, E_2, F_2, G_2)$ are said to be similar if there exist bijective linear maps $S : X_1 \to X_2$ and $T : Z_1 \to Z_2$ such that

$$TE_1 = E_2 S, \quad TF_1 = F_2 S \quad \text{and} \quad G_2 = SG_1.$$

For each $n \geq 1$, define two matrices of size $(n+1) \times n$

$$I'(n) = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad I''(n) = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

Lemma 12. *The sequences*

$$0 \to k[s]^n \overset{sI'(n)-I''(n)}{\to} k[s]^{n+1} \overset{[1 \ldots s^n]}{\to} k[s] \to 0$$

and

$$0 \to O^n \overset{I'(n)-tI''(n)}{\to} O^{n+1} \overset{[t^n \ldots 1]}{\to} O \to 0$$

are exact.

Proof. Straightforward and easy. $\square$

The lemma above can be generalized. Indeed, let Δ be a linear bundle with nonnegative Wiener-Hopf indices and let p be its size. Set $X = H^0(t\Delta)$ and $Z = H^0(\Delta)$, and define $E : X \to Z$ and $F : X \to Z$ as the multiplications by 1 and s, respectively. Further, let $Z[s] \to k[s]^p$ and $O(Z) \to O^p$ be the canonical homomorphisms that are determined respectively by the maps $Z \to k[s]^p$ and $Z \to O^p$. (The latter is the composition of $Z \to DO^p$ and the pre-multiplication by D^{-1}.) There holds

Proposition 3. *The sequences*

$$0 \to X[s] \overset{sE-F}{\to} Z[s] \to k[s]^p \to 0 \quad \textit{and} \quad 0 \to O(X) \overset{E-tF}{\to} O(Z) \to O^p \to 0 \quad (4)$$

are exact.

Proof. The previous lemma corresponds to the case where Δ has rank 1 and is represented by s^n. The general case can be deduced easily from this special one by using the Wiener-Hopf theorem. $\square$

Let now an AR-model (Δ, R) be given. Define X, Z, E and F as above, and define $G : k^q \to Z$ as the linear map $a \to Ra$ $(a \in k^q)$. We have commutative diagrams

$$\begin{array}{ccc} k[s]^q & & \\ \downarrow & \searrow & \\ Z[s] & \to & k[s]^p \end{array} \quad \text{and} \quad \begin{array}{ccc} O^q & & \\ \downarrow & \searrow & \\ O(Z) & \to & O^p \end{array} . \tag{5}$$

(The south-east arrows here are R and $D^{-1}R$ respectively.) Using these diagrams and the previous proposition, we see that (X, Z, E, F, G) is an observable state model. We call this the state representation of (Δ, R).

Theorem 4. *The map taking an AR-model to its state representation induces a one-to-one correspondence between equivalence classes of AR-models and similarity classes of observable state models.*

Proof. Let (X, Z, E, F, G) be an observable state model, with output number p. The observability property implies that the modules $Z[s]/(sE - F)X[s]$ and $O(Z)/(E - tF)O(X)$ have no torsion, and therefore are isomorphic to $k[s]^p$ and O^p, respectively. Consequently, there exist exact sequences as in (4). The homomorphisms $Z[s] \to k[s]^p$ and $O(Z) \to O^p$ determine $k(s)$-linear maps from $Z(s)$ to $k(s)^p$. They have the same kernel, and therefore one of them, say, the first is obtained from the other by multiplication by some nonsingular rational matrix D. Let R denote the composition of $G : k[s]^q \to Z[s]$ with $Z[s] \to k[s]^p$ and Δ the linear bundle associated with D. The rational matrix $D^{-1}R$ is proper, and the property (SM2) implies that R has full row rank. So (Δ, R) is an AR-model. One can see easily that the equivalence class of this model is well-defined: It does not depend on the choice of the homomorphisms $Z[s] \to k[s]^p$ and $O(Z) \to O^p$.

The reader can easily complete the proof. $\square$

Given a state model (X, Z, E, F, G), we define its behavior to be $\mathcal{B} = \mathcal{B}_+ + \mathcal{B}_-$, where $\mathcal{B}_+$ and $\mathcal{B}_-$ are defined by

$$\mathcal{B}_+ = \{\eta \in \mathcal{H}^q |\ E\frac{d\xi}{dt} - F\xi = G\eta \quad \text{for some} \quad \xi \in \mathcal{H}(X)\}$$

and

$$t\mathcal{B}_- = \{\theta \in \mathcal{J}^q |\ E\xi - F\frac{d\xi}{ds} = G\theta \quad \text{for some} \quad \xi \in \mathcal{J}(X)\}.$$

The following proposition implies in particular that this is a linear system.

Proposition 4. *The behavior of an AR-model is equal to the behavior of its state representation.*

Proof. Let (Δ, R) be an AR-model, and let (X, Z, E, F, G) be its state representation. Choose any $D \in \Delta$, and consider the diagrams

$$\begin{array}{ccccccccc} & & & & \mathcal{H}^q & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 \to & \mathcal{H}(X) & \to & & \mathcal{H}(Z) & \to & \mathcal{H}^p & \to 0 & \end{array}$$

and

$$\begin{array}{ccccccc} & & & \mathcal{J}^q & & & \\ & & & \downarrow & \searrow & & \\ 0 \to & \mathcal{J}(X) & \to & \mathcal{J}(Z) & \to & \mathcal{J}^p & \to 0 \end{array}.$$

These are commutative because so are (5). Next, by Lemma 9, they have exact rows.

The proof now is easily completed. □

In fact, there are two types of state models (see Kuijper [3]). What we have considered above are "right" ones, and in the remainder of this section we want to say a few words about "left" state models. As we shall see, there is a simple connection between the two types of state models.

A "left" state model is a quintuple (X, Y, K, L, M), where X, Y are finite-dimensional linear spaces and $K, L : Y \to X$, $M : Y \to k^q$ linear maps satisfying the following properties:

(1) $sK - L$ has full row rank;

(2) $\begin{bmatrix} sK - L \\ M \end{bmatrix}$ is left unimodular and $\begin{bmatrix} L \\ M \end{bmatrix}$ has full column rank.

The space X is called the state space, the space Y the internal variable space, the number $\dim Y - \dim X$ the input number.

Example 5. Let again $x' = Ax + Bu$, $v = Cx + Du$ be an ordinary classical linear system with m inputs and p outputs, and with state space X. Then,

$$\left(X, X \oplus k^m, [I\ 0], [A\ B], \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}\right)$$

is a "left" state model.

Suppose we are given "left" and "right" state models (X, Y, K, L, M) and (X, Z, E, F, G). We say that these form an exact pair, if the sequence

$$0 \to Y \xrightarrow{\begin{bmatrix} -L \\ K \\ M \end{bmatrix}} X \oplus X \oplus k^q \xrightarrow{\begin{bmatrix} F & E & G \end{bmatrix}} Z \to 0$$

is exact.

Example 6. The state models in the previous two examples form an exact pair.

Let (X, Z, E, F, G) be a "right" state model. Clearly, there exist a finite-dimensional linear space Y and linear maps $K, L : Y \to X$, $M : Y \to k^q$ such that the sequence above is exact. It can be shown (see the discussions in [6, Section 2.4] and [8, Section 6]) that (X, Y, K, L, M) is a "left" state model, called a left description. This certainly is uniquely determined. More precisely, if (X, Y_1, K_1, L_1, M_1) is another left description, then there exists a unique isomorphism $A : Y_1 \to Y$ such that $K_1 = KA$, $L_1 = LA$ and $M_1 = MA$.

Attaching to each "right" state model its left description, we get a functor. There is a functor to the opposite direction as well. These two functors are clearly inverse to each other. And thus, "left" and "right" state models are canonically equivalent objects.

Remark. State representations of AR-models have been studied extensively in recent years (see, for example, [1], [2], [3], [4], [5], [7], [12], [15], [16], [17], [20], [21]). The exposition above follows closely [5,7]. (It should be noted that "right" state models are not explicitly mentioned in [5,7]; in [5] they are used to establish the duality between "left" state models.)

6. Controllability and Autonomy

Let $\mathcal{B}$ be a linear system, and let T be its transfer function. We say that $\mathcal{B}$ is controllable if

$$\mathcal{B} \subseteq T\mathcal{M} + sLk[s]^q \quad \text{and} \quad \mathcal{B} \subseteq T\mathcal{M} + LO^q,$$

that is, if there always exists in $\mathcal{B}$ a purely impulsive motion with a given initial state as well as an exponential motion.

Theorem 5. *A linear system is controllable if and only if so is its AR-representation.*

Proof. Obviously a linear system is controllable if and only if its frequency response is controllable. The statement follows therefore from Lemma 3. □

Theorem 6. *Let $\mathcal{B}$ be a linear system with transfer function T and McMillan degree d. Suppose that $n \geq d$. Then $\mathcal{B}$ is controllable if and only if,*

$$\forall \xi_+ \in \mathcal{B} \cap \mathcal{H}^q \text{ and } \forall \xi_- \in \mathcal{B} \cap s\mathcal{J}^q,\ \exists \xi_0 \in T\mathcal{M} \text{ such that}$$
$$\Pi_+(s^n\xi_0) = \xi_+ \text{ and } \Pi_-(\xi_0) = \xi_-.$$

Proof. "If". Let ξ be a motion from $\mathcal{B}$, and let ξ_+ be its regular part. Choose $\xi_0 \in T\mathcal{M}$ such that $s^n\xi_0 \equiv \xi_+ \mathrm{mod} sJ^q$ (and, say, $\xi_0 \equiv 0 \mathrm{mod} \mathcal{H}^q$). Then clearly $\xi \equiv s^n\xi_0 \mathrm{mod} sJ^q$.

Let again ξ be a motion from $\mathcal{B}$, and let w be a Laplace motion having the same initial state as ξ. Choose $w_0 \in T\mathcal{M}$ such that $s^n w_0 \equiv 0 \mathrm{mod} sJ^q$ and $w_0 \equiv w_- \mathrm{mod}\mathcal{H}^q$, where w_- is the purely impulsive part of w. Surely w_0 is a Laplace motion and $w \equiv w_0 \mathrm{mod}\mathcal{H}^q$. Because $L(k(s)) \cap \mathcal{H} = L(O)$, we have that $w - w_0$ is an exponential motion. We conclude that $\xi \equiv \xi_0 \mathrm{mod} L(O^q)$, where $\xi_0 = (\xi - w) + w_0 \in T\mathcal{M}$.

"Only if". Let (Δ, R) be an AR-representation of $\mathcal{B}$. By the remark that follows Lemma 4, the linear map $k[s]^q \cap s^{n-1}O^q \to k[s]^p \cap s^{n-1}DO^p$ is surjective. We obtain that the linear map

$$sJ^q \cap s^n\mathcal{H}^q \to sJ^p \cap s^n D\mathcal{H}^p$$

is surjective. Take now $\xi_+ \in \mathcal{B} \cap \mathcal{H}^q$ and $\xi_- \in \mathcal{B} \cap s\mathcal{J}^q$, that is, $R\xi_+ \in sJ^p$ and $R\xi_- \in D\mathcal{H}^p$. It is easily verified that

$$R(\xi_+) \in sJ^p \cap s^n D\mathcal{H}^p \quad \text{and} \quad R(s^n\xi_-) \in sJ^p \cap s^n D\mathcal{H}^p.$$

Hence by the surjectivity of the above linear map, there exists $w \in sJ^q \cap s^n\mathcal{H}^q$ such that

$$Rw = R(\xi_+ + s^n\xi_-).$$

Then $\xi_+ + s^n\xi_- \equiv w \bmod T\mathcal{M}$, and thus there exists $\xi_0 \in T\mathcal{M}$ such that

$$s^n\xi_0 = \xi_+ + s^n\xi_- - w.$$

Clearly, $s^n\xi_- - w$ is purely impulsive and $t^n\xi_+ - t^n w$ is regular. Hence

$$s^n\xi_0 \equiv \xi_+ \bmod sJ^q \quad \text{and} \quad \xi_0 \equiv \xi_- \bmod \mathcal{H}^q.$$

The theorem is proved. □

A linear system $\mathcal{B}$ is called autonomous if it satisfies the following obviously equivalent conditions: 1) $\mathcal{B}$ has input number 0; 2) $\mathcal{B}$ has transfer function 0; 3) $\mathcal{B}$ is finite-dimensional.

Theorem 7. *Let $\mathcal{B}$ be a linear system, and let T be its transfer function. Then $\mathcal{B}$ is autonomous if and only if*

$$\xi_0 \in T\mathcal{M} \text{ and } \Pi_-(\xi_0) = 0 \Longrightarrow \xi_0 = 0.$$

Proof. The "only if" part is trivial. To prove the "if" part, assume that $T \neq 0$ and take any nonzero element $w \in T$. For sufficiently large n, $t^n w \in O^q$. Consequently, $\xi_0 = L(t^n w)$ is an exponential motion. It lies in $T\mathcal{M}$ and is regular, but is not zero. □

Remark. The reader notices an analogy of the controllability and autonomy properties formulated in Theorems 6 and 7 to the controllability and the autonomy properties as introduced in Willems [21]. The reason why we do not discuss observability is that our linear systems are a priori observable (see Theorem 4). This point of view is in agreement with the classical one (see [5]), but not with the one as developed in Willems [21].

7. "Classical" Case

Let $\mathcal{B}$ be a linear system and T its transfer function. We say that $\mathcal{B}$ is regular if

$$\mathcal{B} \subseteq T\mathcal{M} + LO^q,$$

that is, if there always exists in $\mathcal{B}$ an exponential motion with a given initial state. Certainly, the property of regularity is considerably weaker than that of controllability: Regularity is "controllability at infinity".

Theorem 8. *A linear system is regular if and only if so is its AR-representation.*

Proof. Obviously, a linear system is regular if and only if its frequency response is regular. The statement follows therefore from Lemma 3. □

Theorem 9. *Let $\mathcal{B}$ be a linear system with transfer function T. Then $\mathcal{B}$ is regular if and only if*

$$\forall \xi_- \in \mathcal{B} \cap s\mathcal{J}^q,\ \exists \xi_0 \in T\mathcal{M} \text{ such that } \Pi_-(\xi_0) = \xi_-.$$

Proof. The statement amounts to saying that $\mathcal{B} \subseteq T\mathcal{M} + \mathcal{H}^q$; in other words, that there always exists a regular motion with a given initial state. The "only if" part therefore is trivial.

"If". Let (Δ, R) be an AR-representation of $\mathcal{B}$, and let (Δ_1, R) be its regular part. The "regular" behavior of (Δ_1, R) is determined by the equation $R(d/dt)\xi = 0,\ \xi \in \mathcal{H}^q$ and therefore is equal to $\mathcal{B} \cap \mathcal{H}^q$. Letting X and X_1 denote respectively the state spaces of (Δ, R) and (Δ_1, R), we have surjective linear maps

$$\mathcal{B} \cap \mathcal{H}^q \to sL(X) \quad \text{and} \quad \mathcal{B} \cap \mathcal{H}^q \to sL(X_1).$$

(The first linear map is surjective by hypothesis; the second one is surjective because (Δ_1, R) is regular.) These two linear maps have the same kernel, namely, $T\mathcal{M} \cap \mathcal{H}^q$. We therefore obtain that $X \simeq X_1$. It follows that

$$\deg(R) = \mathrm{ch}(\Delta_1) = \dim(X_1) = \dim(X) = \mathrm{ch}(\Delta).$$

Using Lemma 5, we see that (Δ, R) is regular. Hence $\mathcal{B}$ is regular. □

A "classical" linear (dynamical) system is a k-linear subspace $\mathcal{B} \subseteq \mathcal{H}^q$ satisfying the following axioms:

(CLS1) There exists a "classical" transfer function T such that

$$T\mathcal{H} \subseteq \mathcal{B} \subseteq T\mathcal{H} + LO^q \quad \text{and} \quad [\mathcal{B} : T\mathcal{H}] < +\infty;$$

(CLS2) $\mathcal{B}$ is invariant with respect to d/dt.

(Above $T\mathcal{H}$ denotes the submodule in $\mathcal{H}^q$ generated by elements of the form $f\eta$, where $f \in T$ and $\eta \in \mathcal{H}$.) The "T" is uniquely determined; it is called the transfer function. (The uniqueness of T can be shown exactly as in Lemma 10.) The space $\{w \in tO^q | sL(w) \in \mathcal{B}\}$ is a "classical" frequency response, and is called the frequency response. The transfer function of the latter coincides with T.

Lemma 13. *Let T be a transfer function, and let $T_1 = T \cap O^q$. Then*

(a) *$T\mathcal{M} \cap \mathcal{H}^q = T_1\mathcal{H}$;*

(b) *$(T\mathcal{M} + Lk(s)^q) \cap \mathcal{H}^q = T_1\mathcal{H} + LO^q$.*

Proof. (a) Let m denote the input number of T, and choose a proper rational matrix G such that $GO^m = T_1$. Suppose that $G\xi \in \mathcal{H}^q$, where $\xi \in \mathcal{M}^m$. The matrix G is left biproper, that is, $FG = I$ for some proper rational matrix F. It follows that $\xi = F(G\xi) \in \mathcal{H}^m$.

(b) Let m and G be as above, and suppose that $G\xi + w \in \mathcal{H}^q$ where $\xi \in \mathcal{M}^m$ and $w \in Lk(s)^q$. We have

$$G\xi + w \equiv G\xi_- + w_- \mathrm{mod}(T_1\mathcal{H} + LO^q),$$

where ξ_- and w_- are the purely impulsive parts of ξ and w, respectively. We see that $G\xi_- + w_-$ is a regular function. On the other hand, this is a Laplace function, and hence an exponential function. The assertion follows. □

Lemma 14. *If $\mathcal{B}$ is a linear system, then $\mathcal{B} \cap \mathcal{H}^q$ is a "classical" linear system.*

Proof. Let T denote the transfer function of $\mathcal{B}$, and set $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{H}^q$ and $T_1 = T \cap O^q$. Using the previous lemma, it can be easily seen that

$$T_1\mathcal{H} \subseteq \mathcal{B}_1 \subseteq T_1\mathcal{H} + LO^q.$$

Further, consider the canonical linear map $\mathcal{B}_1 \to \mathcal{B}/T\mathcal{M}$. Its kernel is clearly equal to $\mathcal{B}_1 \cap T\mathcal{M}$. Using again the previous lemma, we obtain that

$$T_1\mathcal{H} \subseteq \mathcal{B}_1 \cap T\mathcal{M} \subseteq \mathcal{H}^q \cap T\mathcal{M} = T_1\mathcal{H}.$$

Thus the kernel coincides with $T_1\mathcal{H}$, and it follows that $[\mathcal{B}_1 : T_1\mathcal{H}] < +\infty$. □

Theorem 10. *The mapping*

$$\mathcal{B} \mapsto \mathcal{B} \cap \mathcal{H}^q$$

establishes a one-to-one correspondence between regular linear systems and "classical" linear systems.

Proof. In view of Theorem 9, if $\mathcal{B}$ is a regular linear system and T its transfer function, then $\mathcal{B} = (\mathcal{B} \cap \mathcal{H}^q) + T\mathcal{M}$, that is, $\mathcal{B}$ can be reconstructed from the knowledge of $\mathcal{B} \cap \mathcal{H}^q$. This implies the injectivity.

Let now $\mathcal{B}_1$ be an arbitrary "classical" linear system, and let T_1 be its transfer function. Let m denote the input number of our system and choose a proper rational matrix G generating T_1. Set $T = k(s)T_1$ and $\mathcal{B} = \mathcal{B}_1 + T\mathcal{M}$. We are going to show that $\mathcal{B}$ is a regular linear system.

Obviously, $T\mathcal{M} \subseteq \mathcal{B} \subseteq T\mathcal{M} + Lk(s)^q$. It is also obvious that the canonical linear map $\mathcal{B}_1 \to \mathcal{B}/T\mathcal{M}$ is surjective. Its kernel is equal to $T_1\mathcal{H}$, and hence $[\mathcal{B} : T\mathcal{M}] < +\infty$.

We remark that if $\xi \in \mathcal{B}_1$ and $n \geq 1$, then, by the Taylor formula, $\Pi_+(s^n\xi) = d^n\xi/dt^n$ and hence belongs to $\mathcal{B}_1$. It is clear that modulo $T_1\mathcal{H}$ every element of $T\mathcal{M}$ is a linear combination of elements of the form $s^nGL(a))$, with $n \geq 1$ and $a \in k^m$. Therefore, $\Pi_+(T\mathcal{M}) \subseteq \mathcal{B}_1$. We conclude that $\Pi_+(\mathcal{B}) \subseteq \mathcal{B}_1$. Automatically, $\mathcal{B}$ is invariant with respect to Π_- as well.

Certainly, $\mathcal{B} \cap \mathcal{H}^q = \mathcal{B}_1$, and so is invariant with respect to d/dt. Further, it is easily seen that $\mathcal{B} \cap s\mathcal{J}^q = \Pi_-(T\mathcal{M})$. As remarked above, $T\mathcal{M}/T_1\mathcal{H}$ is spanned by elements of the form $s^nGL(a))$, with $n \geq 1$ and $a \in k^m$. Using the Taylor formula and letting $G_0, G_1, G_2, \dots$ denote the coefficients of G in the expansion at infinity, we have

$$\Pi_-(s^n GL(a)) = (s^n G_0 + \cdots + sG_n)L(a).$$

It follows that $t\mathcal{B} \cap \mathcal{J}^q$ is spanned by elements of the form $L(s^{n-1}G_0 a + \cdots + G_n a)$ where $n \geq 1$ and $a \in k^m$. Obviously, this subspace is invariant with respect to d/ds.

Thus $\mathcal{B}$ is indeed a linear system and, obviously, is regular. □

From the previous theorem (and Lemma 1 and Lemma 6) we get the following theorems.

Theorem 1′. *The frequency response completely determines a "classical" linear system.*

Theorem 2′. *Two "classical" AR-models have the same behavior if and only if they are equivalent.*

Theorem 3′. *Every "classical" linear system allows an AR-representation.*

Remark. Theorem 1′ is due to Kuijper (see Theorem 3.12 in [3]), and Theorem 2′ to Schumacher (see Corollary 2.5 in [18]).

Appendix: Obtaining State Representations

Our purpose in this appendix is to derive explicit formulas for state representations of AR-models that are given properly. We shall do this very easily.

Let

$$(\{s^{n_1}, \dots, s^{n_p}\}, R) \tag{6}$$

be an AR-model, where $n_1, \dots, n_p$ are nonnegative integers and $\{s^{n_1}, \dots, s^{n_p}\}$ denotes the linear bundle represented by $diag(s^{n_1}, \dots, s^{n_p})$. (Note that, using the Wiener-Hopf theorem, an arbitrary AR-model can be brought to this form.) Write

$$R = \begin{bmatrix} r_{10}s^{n_1} + \cdots + r_{1,n_1} \\ \vdots \\ r_{p0}s^{n_p} + \cdots + r_{p,n_p} \end{bmatrix},$$

where $r_{ij} \in k^q$. For each $1 \leq i \leq p$, let

$$E_i = I'(n_i), \quad F_i = I''(n_i) \quad \text{and} \quad G_i = \begin{bmatrix} r_{i0} \\ \vdots \\ r_{i,n_i} \end{bmatrix}.$$

Proposition 5.

$$\left(\oplus k^{n_i}, \oplus k^{n_i+1}, \begin{bmatrix} E_1 & & \\ & \ddots & \\ & & E_p \end{bmatrix}, \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_p \end{bmatrix}, \begin{bmatrix} G_1 \\ \vdots \\ G_p \end{bmatrix} \right)$$

is a state representation of (6).

Proof. Follows immediately from the definition (see Section 5); it suffices to make identifications according to Serre's formulas (see [11]). □

We need the following

Lemma 15. *Let T be a polynomial $r \times l$ matrix, and let $n_1, \dots, n_r$ be non-negative integers. One can uniquely write*

$$T = \begin{bmatrix} s^{n_1} & & \\ & \ddots & \\ & & s^{n_r} \end{bmatrix} T_h + \begin{bmatrix} [s^{n_1-1} \ \dots \ 1] & & \\ & \ddots & \\ & & [s^{n_r-1} \ \dots \ 1] \end{bmatrix} \bar{T},$$

where T_h and $\bar{T}$ are constant matrices of sizes $r \times l$ and $\sum n_i \times l$, respectively.

Proof. Write

$$T = \begin{bmatrix} f_{10}s^{n_1} + \dots + f_{1,n_1} \\ \vdots \\ f_{r0}s^{n_r} + \dots + f_{r,n_r} \end{bmatrix},$$

where $f_{ij} \in k^l$. For each $i = 1, \dots, r$, let T_i denote the matrix with rows equal to $f_{i1}, \dots f_{i,n_i}$. Letting

$$T_h = \begin{bmatrix} f_{10} \\ \vdots \\ f_{r0} \end{bmatrix} \quad \text{and} \quad \bar{T} = \begin{bmatrix} T_1 \\ \vdots \\ T_p \end{bmatrix},$$

we get the required expression for T. The uniqueness is also easy to prove, and is left to the reader. □

Let

$$R(d/dt)w = 0 \tag{7}$$

be a "classical" AR-model. Assume without loss of generality that the polynomial matrix R is row reduced. Let p be the output number, and let $n_1, \dots, n_p$ be the row degrees of R. Set

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_p \end{bmatrix},$$

where

$$J_i = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \quad \text{and} \quad C_i = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

are matrices of sizes $n_i \times n_i$ and $1 \times n_i$, respectively. Let R_h and $\bar{R}$ be defined as in the previous lemma.

Proposition 6.
$$\begin{cases} x' &= Jx + \bar{R}w \\ 0 &= Cx + R_h w \end{cases}$$
is a state representation of (7).

Proof. Follows immediately from Proposition 5. It suffices to write the internal variable space $\oplus k^{n_i+1}$ as $\oplus k^{n_i} \oplus k^p$. $\square$

Let
$$P(d/dt)v = Q(d/dt)u \tag{8}$$
be a "classical" AR-model, where P is nonsingular and $P^{-1}Q$ is proper. Assume, without loss of generality, that P is "monic", i.e., the high order coefficient matrix of P is equal to I. Let $n_1, \dots, n_p$ be the row degrees of P. Since $P^{-1}Q$ is proper, the row degrees of Q are not greater than $n_1, \dots, n_p$. We may apply Lemma 15 to the matrices P, Q and $[Q \ \ -P]$. Let J and C be defined as in Proposition 6, and set
$$A = J - \bar{P}C, \quad B = \bar{Q} - \bar{P}Q_h, \quad D = Q_h.$$

Proposition 7.
$$\begin{cases} x' &= Ax + Bu \\ v &= Cx + Du \end{cases}$$
is a state representation of (8).

Proof. Let $X = \oplus k^{n_i}$, and observe that
$$\begin{bmatrix} I & -\bar{P} \\ 0 & I \end{bmatrix} : X \oplus k^p \to X \oplus k^p$$
is an isomorphism. By Proposition 2,
$$(X, X \oplus k^p, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} J \\ C \end{bmatrix}, \begin{bmatrix} \bar{Q} & -\bar{P} \\ Q_h & -I \end{bmatrix})$$
is a state representation of (8). Because
$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} I & -\bar{P} \\ 0 & I \end{bmatrix} \begin{bmatrix} J \\ C \end{bmatrix} \text{ and } \begin{bmatrix} B & 0 \\ D & -I \end{bmatrix} = \begin{bmatrix} I & -\bar{P} \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{Q} & -\bar{P} \\ Q_h & -I \end{bmatrix},$$
the proposition follows. $\square$

We illustrate the previous proposition by a very classical fact.

Example 7. A high order equation
$$\frac{d^n v}{dt^n} + a_1 \frac{d^{n-1} v}{dt^{n-1}} + \cdots + a_n = b_0 \frac{d^n u}{dt^n} + b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + u_n$$
may be described by the first order equation
$$\begin{cases} x' &= Ax + Bu \\ v &= Cx + Du \end{cases},$$

where

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ \vdots & & & \dots & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 - a_1 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \quad D = b_0.$$

Remark. Our interest in the topic was inspired by the paper [17] of Rosenthal and Shumacher, where state representations, with little or no computations, are provided. It should be noted that here we deal with "right" state representations, while in [17] "left" ones are considered. The reader notices that the "ABCD" form above is the same as that in Theorem 4.3 of [17]. (However, we do not make the assumption that $n_1, \dots, n_p \geq 1$.)

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References

1. A. H. W. Geerts and J. M. Schumacher, Impulsive-smooth behavior in multimode systems. Part I: State-space and polynomial representations. *Automatica* **32**(1996), 747–758.
2. A. H. W. Geerts and J. M. Schumacher, Impulsive-smooth behavior in multimode systems. Part II: Minimality and equivalence. *Automatica* **32** (1996), 819–832.
3. M. Kuijper, First order representations of linear systems. *Birkhäuser, Boston,* 1994.
4. M. Kuijper and J. M. Schumacher, Realization of autoregressive equations in pencil and descriptor form. *SIAM J. Control Optim.* **28**(1990), 1162–1189.
5. V. Lomadze, Finite-dimensional time-invariant linear dynamical systems: Algebraic theory. *Acta Appl. Math.* **19** (1990), 149–201.
6. V. Lomadze, On operational calculus. *Bull. Georgian Acad. Sci.* **140**(1990), 497-500.
7. V. Lomadze, On state space representations of AR-models. *Georgian Math. Journal* **7**(2000), 117–131.
8. V. Lomadze, Linear constant coefficient differential (or difference) equations. *J. Pure Appl. Algebra* **147**(2000), 143–157.
9. V. Lomadze, Singular linear behaviors and their AR-representations. *Math. Control Signals Systems* **14**(2001), 194–211.
10. V. Lomadze, On interconnections and control. *Math. Control Signals Systems* (submitted).
11. V. Lomadze, Convolutional codes and frequency responses. *Georgian Math. J.* **8**(2001), No. 3, 553–570.
12. V. Lomadze, M. S. Ravi, J. Rosenthal, and J. M. Schumacher, A behavioral approach to singular systems. *Acta Appl. Math.* **54**(1998), 331–344.

13. J. Mikusinski, Operational calculus. *Pergamon Press, London,* 1959.

14. U. Oberst, Multidimensional constant linear systems. *Acta Appl. Math.* **20**(1990), 1–175.

15. M. S. Ravi and J. Rosenthal, A general realization theory for higher order linear differential equations. *Systems Control Lett.* **25**(1995), 351–360.

16. M. S. Ravi, J. Rosenthal, and J. M. Schumacher, Homogeneous behaviors. *Math. Control Signals Systems* **10**(1997), 61–75.

17. J. Rosenthal and J. M. Schumacher, Realization by inspection. *IEEE Trans. Automat. Control* **42**(1997), 1257–1263.

18. J. M. Schumacher, Transformations of linear systems under external equivalence. *Linear Algebra Appl.* **102**(1988), 1–33.

19. H. Soethoudt, Introduction to a behavioral approach for continuous-time systems. *Ph.D. thesis, Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven,* 1993.

20. J. C. Willems, From time series to linear system -Part I. Finite dimensional linear time invariant systems. *Automatica* **22**(1986), 561–580.

21. J. C. Willems, Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Automat. Control* **36**(1991), 259–294.

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