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ON THE UNIQUENESS PROPERTY FOR PRODUCTS OF SYMMETRIC INVARIANT PROBABILITY MEASURES

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Abstract. Two symmetric invariant probability measures μ_1 and μ_2 are constructed such that each of them possesses the strong uniqueness property but their product $\mu_1 \times \mu_2$ turns out to be a symmetric invariant probability measure without the uniqueness property.

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Let E be a nonempty set, G be a family of transformations of E and let μ be a σ-finite measure defined on some σ-algebra of subsets of E. We recall that μ is invariant with respect to G (under G) if, for each set $X \in \mathrm{dom}(\mu)$ and for each transformation $g \in G$, the relations

$$g(X) \in \mathrm{dom}(\mu), \quad \mu(g(X)) = \mu(X)$$

hold true. In all typical situations the role of G is played by an appropriate group of transformations of E. In particular, suppose that a group $(\Gamma, \cdot)$ is given and let μ be a probability measure defined on some σ-algebra of subsets of Γ. Obviously, we may identify Γ with the group of all its left (right) translations. If our μ is invariant with respect to the latter group (according to the definition above), then we say, in short, that μ is a left (right) Γ-invariant measure. Clearly, for a commutative group Γ, the notion of a left Γ-invariant measure and the notion of a right Γ-invariant measure coincide, and in this way we get the concept of a Γ-invariant measure on Γ.

Recall also that a measure ν on a group Γ is symmetric if, for any set $X \in \mathrm{dom}(\nu)$, we have the equality

$$\nu(X^{-1}) = \nu(X).$$

For instance, the Haar measure on a compact topological group is always symmetric.

Let E be an arbitrary set, G be a family of transformations of E and let M be a class of measures on E invariant under G. We shall say that a measure $\mu \in M$ has the uniqueness property (in M) if, for every measure $\nu \in M$, the relation $\mathrm{dom}(\nu) = \mathrm{dom}(\mu)$ implies the equality $\nu = \mu$.

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For example, it is well known that the probability Haar measure on a compact topological group Γ possesses the uniqueness property, i.e., any Borel left (right) Γ-invariant probability measure on Γ necessarily coincides with the Haar measure (see, e.g., [1] or [2]).

Remark 1. Let G' be a family of transformations of E containing G and let M be a class of measures on E invariant with respect to G'. It can easily be seen that if a measure $\mu \in M$ regarded as a measure invariant under G has the uniqueness property (in M), then it also has the uniqueness property (in M) as a measure invariant under G'.

Let us introduce the concept of the strong uniqueness property for invariant measures.

Let E be again an arbitrary set, G be a family of transformations of E and let M be a class of measures on E invariant with respect to G. We shall say that a measure $\mu \in M$ has the strong uniqueness property (in M) if, for each set $X \in \mathrm{dom}(\mu)$ and for any measure $\nu \in M$, the relation $X \in \mathrm{dom}(\nu)$ implies the equality $\nu(X) = \mu(X)$.

Obviously, if a given measure μ on E has the strong uniqueness property (in M), then it also possesses the uniqueness property (in M). The converse assertion is not true, in general (in this connection, cf. Theorem 2 below).

In our further considerations, we will write (E, G, μ) if E is any set, G is a group of transformations of E and μ is a probability measure on E invariant with respect to G. The triple (E, G, μ) will be called a space equipped with an invariant probability measure.

It is evident that the invariance and symmetricity of probability measures are preserved under the product operation. The behaviour of the uniqueness property under the same operation will be discussed later on (see Theorem 1). Our main goal in this paper is to demonstrate that, on the one-dimensional torus $\mathbf{T}_1$ regarded as a commutative compact topological group, there exist two symmetric invariant probability measures with the strong uniqueness property, whose product does not possess the uniqueness property. In order to give such an example, we need some preliminary notions and constructions. These constructions will be based on the methods of extending invariant measures (the corresponding techniques was developed in [3], [4], [5], [6]).

As usual, we denote by ω the first infinite ordinal (cardinal) and by $\mathbf{c}$ the cardinality of the continuum. Let α stand for the first ordinal number whose cardinality is equal to $\mathbf{c}$. The symbol λ will denote the standard Lebesgue measure on the real line $\mathbf{R}$, i.e., the completion of the Haar measure on the locally compact topological group $\mathbf{R}$.

For further purposes, it is convenient to consider the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rationals. Let $H = \{e_\xi : \xi < \alpha\}$ be a Hamel basis in $\mathbf{R}$. Then, for any $x \in \mathbf{R}$, we have a unique representation

$$x = \sum_{\xi<\alpha} q_\xi e_\xi$$

where all q_ξ $(\xi < \alpha)$ are rational numbers and

$$\mathrm{card}(\{\xi < \alpha : q_\xi \neq 0\}) < \omega.$$

For each $x \in \mathbf{R} \setminus \{0\}$, denote by $\xi = \xi(x)$ the largest ordinal from the interval $[0, \alpha[$, satisfying the relation $q_\xi \neq 0$, and define

$$A = \{x \in \mathbf{R} : q_{\xi(x)} > 0\}, \quad B = \{x \in \mathbf{R} : q_{\xi(x)} < 0\}.$$

It is clear that

$$A \cap B = \varnothing, \quad A \cup B \cup \{0\} = \mathbf{R}, \quad -A = B.$$

The last relation means that the sets A and B are symmetric to each other (with respect to the origin). Actually, both the sets A and B are convex (more precisely, $\mathbf{Q}$-convex) cones in $\mathbf{R}$ considered as a vector space over $\mathbf{Q}$. Let us point out an interesting property of these sets (cf. [4]). It is not hard to verify that, for any $y \in \mathbf{R}$, the inequalities

$$\mathrm{card}(A \triangle (A + y)) < \mathbf{c}, \quad \mathrm{card}(B \triangle (B + y)) < \mathbf{c}$$

are valid. To see this, take a representation

$$y = \sum_{\xi < \alpha} q'_\xi e_\xi$$

of y (with respect to H). Then, for those $x \in \mathbf{R}$ which satisfy the relation $\xi(x) > \xi(y)$, we have

$$x \in A \Leftrightarrow x + y \in A,$$
$$x \in B \Leftrightarrow x + y \in B.$$

It remains to observe that

$$\mathrm{card}(\{x \in \mathbf{R} : \xi(x) \leq \xi(y)\}) < \mathbf{c},$$

which yields at once the required result. In other words, we obtain that the sets A and B are almost invariant under the group of all translations of $\mathbf{R}$.

We now assert that the sets A and B are nonmeasurable in the Lebesgue sense. Indeed, suppose to the contrary that at least one of these sets is Lebesgue measurable, i.e., belongs to $\mathrm{dom}(\lambda)$. Then, in view of the equality $-A = B$, the second set will be Lebesgue measurable, too. Since

$$\{0\} \cup A \cup B = \mathbf{R},$$

we infer that

$$\lambda(A) = \lambda(B) > 0.$$

On the other hand, the metrical transitivity of the Lebesgue measure, with respect to the group of all translations of $\mathbf{R}$, implies

$$\lambda(\mathbf{R} \setminus A) = 0 \quad \vee \quad \lambda(\mathbf{R} \setminus B) = 0$$

which leads to a contradiction. Thus, we conclude that each of the sets A and B is nonmeasurable in the Lebesgue sense. Moreover, an easy argument based on the same property of metrical transitivity of λ shows that both these sets are λ-thick in $\mathbf{R}$, i.e., we have

$$\lambda_*(A) = \lambda_*(B) = 0,$$

where λ_* denotes the inner measure associated with λ. However, the last relation enables us to consider the sets A and B as measurable ones with respect to some measure on $\mathbf{R}$ which extends λ and is invariant under the group of all motions of $\mathbf{R}$. Indeed, let us denote:

S = the σ-algebra of subsets of $\mathbf{R}$, generated by $\mathrm{dom}(\lambda) \cup \{A, B\}$;

J = the σ-ideal of all those subsets of $\mathbf{R}$ whose cardinalities are strictly less than $\mathbf{c}$;

S' = the σ-algebra of subsets of $\mathbf{R}$, generated by $S \cup J$.

Note that any set U from the σ-algebra S can be represented in the form

$$U = (A \cap X) \cup (B \cap Y) \cup (\{0\} \cap Z),$$

where

$$\{X, Y, Z\} \subset \mathrm{dom}(\lambda),$$

and such a representation is unique in the sense that an analogous equality

$$U = (A \cap X_1) \cup (B \cap Y_1) \cup (\{0\} \cap Z_1)$$

implies the relations

$$\lambda(X \triangle X_1) = 0, \quad \lambda(Y \triangle Y_1) = 0,$$

where $\triangle$ denotes the operation of the symmetric difference of two sets.

Now, define a functional

$$\lambda' : S \to \mathbf{R} \cup \{+\infty\}$$

by the formula

$$\lambda'(U) = (1/2)(\lambda(X) + \lambda(Y)).$$

The remark just made implies that this definition is correct. Moreover, an easy calculation shows that λ' turns out to be a measure on S extending λ. This measure, obviously, can be uniquely extended to a measure on the σ-algebra S', by putting

$$\lambda'(V) = 0 \quad (V \in J).$$

We preserve the same notation for the extended in this manner measure. Finally, we may assert that the obtained measure λ' on S' is invariant under all isometric transformations of $\mathbf{R}$. To see this, let us observe that the equality $-A = B$ and the definition of λ' immediately yield the invariance of λ' with respect to the symmetry

$$s : x \to -x \quad (x \in \mathbf{R}).$$

Further, the almost invariance of A and B under the group of all translations of $\mathbf{R}$ implies at once the translation-invariance of λ'. It remains to utilize the elementary geometric fact saying that the group of all motions of $\mathbf{R}$ is generated by s and all translations of $\mathbf{R}$.

Lemma 1. *The measure λ' has the strong uniqueness property.*

The proof of Lemma 1 can be found in Chapter 7 of monograph [5]. Let us stress once more that, in this lemma, λ' is regarded as a measure extending λ and invariant under the symmetry s and all translations of $\mathbf{R}$.

The following statement shows us that the uniqueness property is preserved under products of invariant measures.

Theorem 1. *Let (E_1, G_1, μ_1) and (E_2, G_2, μ_2) be two spaces equipped with invariant probability measures and suppose that both these measures possess the uniqueness property. Consider the product space*

$$(E, G, \mu) = (E_1 \times E_2, G_1 \times G_2, \mu_1 \times \mu_2).$$

Then the measure μ also possesses the uniqueness property.

Proof. Let ν be an arbitrary probability measure defined on $\mathrm{dom}(\mu_1 \times \mu_2)$ and invariant under the product group $G_1 \times G_2$. First, let us observe that a functional

$$\mu' : \mathrm{dom}(\mu_1) \to [0, 1]$$

defined by the formula

$$\mu'(X) = \nu(X \times E_2) \quad (X \in \mathrm{dom}(\mu_1))$$

is a probability measure on $\mathrm{dom}(\mu_1)$ invariant under the group G_1. Hence, in view of the uniqueness property of μ_1, we claim that μ' coincides with μ_1. So we have

$$\mu_1(X) = \nu(X \times E_2) \quad (X \in \mathrm{dom}(\mu_1)).$$

Fix now a set $X \in \mathrm{dom}(\mu_1)$ with $\mu_1(X) > 0$ and consider a functional

$$\mu'' : \mathrm{dom}(\mu_2) \to [0, 1]$$

defined by the formula

$$\mu''(Y) = \frac{1}{\mu_1(X)} \nu((X \times E_2) \cap (E_1 \times Y)) \quad (Y \in \mathrm{dom}(\mu_2)).$$

Taking into account the said above, it is not hard to check that μ'' is a probability measure on $\mathrm{dom}(\mu_2)$ invariant under the group G_2. Applying the uniqueness property of μ_2, we deduce that μ'' coincides with μ_2. In particular, we get

$$\mu_1(X)\mu_2(Y) = \nu((X \times E_2) \cap (E_1 \times Y)) = \nu(X \times Y)$$

for any $Y \in \mathrm{dom}(\mu_2)$. Now, it is easy to verify that the above formula remains true for arbitrary sets $X \in \mathrm{dom}(\mu_1)$ and $Y \in \mathrm{dom}(\mu_2)$. This shows us that $\nu = \mu_1 \times \mu_2$, and the theorem is proved. $\square$

In particular, if we have two commutative groups G_1 and G_2 endowed with probability G_1-invariant and G_2-invariant measures μ_1 and μ_2, respectively, then, according to the theorem just established, we may assert that the product measure $\mu_1 \times \mu_2$ possesses the uniqueness property, whenever both given measures have this property.

In many cases (important from the point of view of applications) invariant measures on groups turn out to be also invariant with respect to the symmetry. For instance, we have already mentioned the invariance with respect to the symmetry of the Haar measure on a compact topological group (the same holds true for the Haar measure on a commutative locally compact group). Evidently, if (G_1, μ_1) and (G_2, μ_2) are two symmetric probability measures on groups G_1 and G_2, respectively, then the product measure $\mu_1 \times \mu_2$ is a symmetric measure on the product group $G_1 \times G_2$.

Now, we are going to consider the uniqueness property for the product measure of two symmetric invariant probability measures given on commutative groups. It will be shown that, even in this case, the uniqueness property is not preserved under products.

To present the corresponding example, we need some additional constructions. Let us take the number π and let us extend the one-element set $\{\pi\}$ to a Hamel basis in $\mathbf{R}$. We denote the obtained Hamel basis by

$$H = \{e_\xi : \xi < \alpha\},$$

where α is the least ordinal of cardinality continuum, and we suppose in the sequel, without loss of generality, that $e_0 = \pi$. Let A and B be the two $\mathbf{Q}$-convex cones described earlier and associated with the Hamel basis H. Further, let

$$\phi : \mathbf{R} \to \mathbf{T}_1$$

be the canonical surjective group homomorphism defined by

$$\phi(x) = (\cos(x), \sin(x)) \quad (x \in \mathbf{R}),$$

where $\mathbf{T}_1$ is the unit circumference (i.e., the standard one-dimensional torus) in the plane $\mathbf{R}^2$. Then it is not hard to check that:

1) $s(\phi(A)) = \phi(B)$ where s denotes the symmetry of the group $\mathbf{T}_1$;

2) $\mathrm{card}(\phi(A) \cap \phi(B)) \leq \omega$;

3) the sets $\phi(A)$ and $\phi(B)$ are almost invariant with respect to the group of all rotations of the circumference $\mathbf{T}_1$ about its centre;

4) $\phi(A) \cup \phi(B) = \mathbf{T}_1$.

For the sake of simplicity, let us put

$$A' = \phi(A), \quad B' = \phi(B).$$

Also, let us denote by λ_1 the classical Lebesgue probability measure on $\mathbf{T}_1$ (evidently, it is invariant under the group of all rotations of $\mathbf{T}_1$ about its centre). A direct verification shows that both sets A' and B' are λ_1-thick in $\mathbf{T}_1$. By starting with the above-mentioned properties of the sets A' and B' and utilizing an argument similar to the one given earlier, it can easily be constructed a measure ν on $\mathbf{T}_1$ which satisfies the following conditions:

(1) ν extends λ_1 and is a rotation-invariant measure on $\mathbf{T}_1$;

(2) ν is invariant under the symmetry of $\mathbf{T}_1$;

(3) $\{A', B'\} \subset \mathrm{dom}(\nu)$ and $\nu(A') = \nu(B') = 1/2$.

As said above, the construction of ν can be carried out in the way similar to the construction of an invariant extension λ' of the Lebesgue measure on $\mathbf{R}$, which forces the sets A and B to be measurable.

Also, by starting with condition (3), the next proposition can be obtained.

Lemma 2. *The measure ν has the strong uniqueness property.*

The proof is completely similar to the proof of Lemma 1 (cf. Chapter 7 in [5]). Let us emphasize once more that, in the lemma above, ν is regarded as a symmetric $\mathbf{T}_1$-invariant measure extending λ_1.

Now, we are able to establish the following statement.

Theorem 2. *The product measure $\nu \times \nu$, considered as a symmetric $(T_1 \times T_1)$-invariant probability measure, does not possess the uniqueness property. More precisely, there exists a symmetric $(T_1 \times T_1)$-invariant probability measure on* $\mathrm{dom}(\nu \times \nu)$ *which differs from $\nu \times \nu$.*

Proof. Obviously, we may write

$$\{A' \times A', A' \times B', B' \times A', B' \times B'\} \subset \mathrm{dom}(\nu \times \nu).$$

At the same time, denoting $\lambda_2 = \lambda_1 \times \lambda_1$, we have

$$\lambda_2((A' \times A') \cup (A' \times B') \cup (B' \times A') \cup (B' \times B')) = 1.$$

Now, it is clear that the general form of an element W from $\mathrm{dom}(\nu \times \nu)$ is the following one:

$$W = ((A' \times A') \cap X) \cup ((A' \times B') \cap Y) \cup ((B' \times A') \cap Z) \cup ((B' \times B') \cap K)$$

where X, Y, Z, K are some elements of $\mathrm{dom}(\lambda_2)$ (of course, in this representation we omit some sets of $(\nu \times \nu)$-measure zero). Now, fix a real $r \in \,]0, 1[$ and define a measure μ on $\mathrm{dom}(\nu \times \nu)$ by the formula

$$\mu(W) = (1/2)(r\lambda_2(X) + r\lambda_2(K) + (1-r)\lambda_2(Y) + (1-r)\lambda_2(Z)).$$

The correctness of this definition is implied by the fact that all the sets

$$A' \times A', \quad A' \times B', \quad B' \times A', \quad B' \times B'$$

are thick in $\mathbf{T}_1 \times \mathbf{T}_1$ with respect to λ_2. Obviously, we have

$$\mu \neq \nu \times \nu$$

whenever $r \neq 1/2$. It can also be easily verified that μ is a symmetric $(\mathbf{T}_1 \times \mathbf{T}_1)$-invariant probability measure on $\mathrm{dom}(\nu \times \nu)$. This completes the proof of our theorem. □

Remark 2. The measure $\nu \times \nu$ has the uniqueness property if it is regarded as a G'-invariant probability measure, where G' stands for the group generated by the pairs (g, s) and (s, h), where g and h are arbitrary elements of $\mathbf{T}_1$ and s is the symmetry of $\mathbf{T}_1$. To get this result, it suffices to apply Theorem 1. Indeed, in our case, the group G' coincides with the product group

$$[\{s\} \cup \mathbf{T}_1] \times [\{s\} \cup \mathbf{T}_1]$$

where $[\{s\} \cup \mathbf{T}_1]$ denotes the group generated by s and $\mathbf{T}_1$.

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