Georgian Mathematical Journal
Volume 9 (2002), Number 1, 71–74

ON KAN FIBRATIONS FOR MALTSEV ALGEBRAS

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Abstract. We prove that any surjective homomorphism of Maltsev algebras is a Kan fibration.

2000 Mathematics Subject Classification: 55U10, 18C10.
Key words and phrases: Kan fibration, Maltsev algebras.

It is well known that simplicial groups are Kan complexes and, more generally, any surjective homomorphism of simplicial groups is a Kan fibration. Among other things, from the results in [1] the following statements follow:

Theorem 1. *Any simplicial model of a Maltsev theory is a Kan complex.*

Theorem 2. *If $\mathbb{T}$ is an algebraic theory such that any simplicial $\mathbb{T}$-model is a Kan complex, then $\mathbb{T}$ is a Maltsev theory.*

This note has two aims: firstly, to give direct proofs of these facts without using the category theory machinery and, secondly, to get sharper results. In particular we prove that any surjective homomorphism of Maltsev algebras is a Kan fibration.

A *Maltsev operation* in an algebraic theory $\mathbb{T}$ is a ternary operation $[\_,\_,\_]$ in $\mathbb{T}$ satisfying the identities

$$[a,a,b]=b \quad \text{and} \quad [a,b,b]=a.$$

An algebraic theory is called Maltsev if it possesses a Maltsev operation. Clearly, the theory of groups is a Maltsev theory by taking $[a,b,c]=ab^{-1}c$. More generally, the theory of loops is a Maltsev theory; this latter fact has already been used in the homotopy theory (see [3]). An example of another sort of a Maltsev theory is the theory of Heyting algebras.

We start by proving a stronger version of Theorem 2. Let

$$S^1=\Delta^1/\partial\Delta^1$$

be the smallest simplicial model of the circle. Moreover, let $S^1_{\mathbb{T}}$ be the simplicial $\mathbb{T}$-model which is obtained by applying degreewise the free $\mathbb{T}$-model functor to S^1.

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Proposition 1. *Let $\mathbb{T}$ be an algebraic theory such that $S^1_{\mathbb{T}}$ satisfies the $(1,2)$-th Kan condition in dimension 2. That is, for any 1-simplices x_1, x_2 with $d_1x_1 = d_1x_2$ there is a 2-simplex x with $d_1x = x_1$ and $d_2x = x_2$. Then $\mathbb{T}$ is a Maltsev theory.*

Proof. Denote the unique nondegenerate 1-simplex of S^1 by σ and the unique vertex $d_0\sigma = d_1\sigma$ by $*$. So $S^1_{\mathbb{T}}$ in dimension zero is the free $\mathbb{T}$-model generated by $*$. Similarly, $S^1_{\mathbb{T}}$ in dimension one is the free $\mathbb{T}$-model generated by s_0* and σ, and in dimension two it is the free $\mathbb{T}$-model generated by s_1s_0*, $s_0\sigma$, and $s_1\sigma$. Since $d_1s_0* = d_1\sigma = *$, the $(1,2)$-th Kan condition implies the existence of a 2-simplex x of $S^1_{\mathbb{T}}$ with $d_1x = s_0*$ and $d_2x = \sigma$. This means there is an element $x(s_1s_0*, s_0\sigma, s_1\sigma)$ in the free $\mathbb{T}$-model with three generators s_1s_0*, $s_0\sigma$, $s_1\sigma$ such that the equalities $x(d_1s_1s_0*, d_1s_0\sigma, d_1s_1\sigma) = s_0*$ and $x(d_2s_1s_0*, d_2s_0\sigma, d_2s_1\sigma) = \sigma$ hold in the free $\mathbb{T}$-algebra with two generators s_0*, σ. Applying standard simplicial identities we see that this means $x(s_0*, \sigma, \sigma) = s_0*$ and $x(s_0*, s_0*, \sigma) = \sigma$, i. e., that x is a Maltsev operation. □

The following Theorem shows that if $\mathbb{T}$ is a Maltsev theory, then all surjective homomorphisms of simplicial $\mathbb{T}$-models are Kan fibrations, which obviously implies Theorem 1. Our proof uses exactly the same inductive argument as the one given in [2] for simplicial groups (see page 130 in [2]) except that we put a new input for w_0.

Theorem 3. *Any surjective homomorphism $f : X \to Y$ of simplicial models of a Maltsev theory is a Kan fibration.*

Proof. For $n > 0$ and $0 \leqslant k \leqslant n$, given $y \in Y_n$ with $d_iy = f(x_i)$ for $i \neq k$, $0 \leqslant i \leqslant n$, where x_i are elements of X_{n-1} with matching faces, we have to find $x \in X_n$ with $f(x) = y$ and $d_ix = x_i$ for $i \neq k$. Take $x' \in f^{-1}(y)$ and then put

$$\begin{aligned} w_0 &= [s_0x_0, s_0d_0x', x'], \\ w_j &= [w_{j-1}, s_jd_jw_{j-1}, s_jx_j] \end{aligned}$$

for $0 < j < k$; in case $k < n$ put

$$\begin{aligned} w_n &= [w_{k-1}, s_{n-1}d_nw_{k-1}, s_{n-1}x_n], \\ w_j &= [w_{j+1}, s_{j-1}d_jw_{j+1}, s_{j-1}x_j] \end{aligned}$$

for $n > j > k$.

We then have

$$\begin{aligned} f(w_0) &= [s_0f(x_0), s_0d_0f(x'), f(x')] = [s_0d_0y, s_0d_0y, y] = y, \\ f(w_j) &= [f(w_{j-1}), s_jd_jf(w_{j-1}), s_jf(x_j)] \\ &= [y, s_jd_jy, s_jd_jy] = y \end{aligned}$$

for $0 < j < k$ and, if $k < n$, then

$$\begin{aligned} f(w_n) &= [f(w_{k-1}), s_{n-1}d_nf(w_{k-1}), s_{n-1}f(x_n)] = [y, s_{n-1}d_ny, s_{n-1}d_ny] = y, \\ f(w_j) &= [f(w_{j+1}), s_{j-1}d_jf(w_{j+1}), s_{j-1}f(x_j)] = [y, s_{j-1}d_jy, s_{j-1}d_jy] = y \end{aligned}$$

for $n > j > k$. Furthermore,

$$
\begin{aligned}
d_0 w_0 &= [d_0 s_0 x_0, d_0 s_0 d_0 x', d_0 x'] = [x_0, d_0 x', d_0 x'] = x_0, \\
d_i w_j &= [d_i w_{j-1}, d_i s_j d_j w_{j-1}, d_i s_j x_j] \\
&= \begin{cases} [x_i, s_{j-1} d_{j-1} d_i w_{j-1}, s_{j-1} d_i x_j] & \\ \quad = [x_i, s_{j-1} d_{j-1} x_i, s_{j-1} d_{j-1} x_i] = x_i, & i < j, \\ [d_j w_{j-1}, d_j w_{j-1}, x_j] = x_j, & i = j \end{cases}
\end{aligned}
$$

for $0 < j < k$ and, if $k < n$, then

$$
\begin{aligned}
d_i w_n &= [d_i w_{k-1}, d_i s_{n-1} d_n w_{k-1}, d_i s_{n-1} x_n] \\
&= \begin{cases} [x_i, s_{n-2} d_{n-1} d_i w_{k-1}, s_{n-2} d_i x_n] & \\ \quad = [x_i, s_{n-2} d_{n-1} x_i, s_{n-2} d_{n-1} x_i] = x_i, & 0 \leqslant i < k, \\ [d_n w_{k-1}, d_n w_{k-1}, x_n] = x_n, & i = n, \end{cases} \\
d_i w_j &= [d_i w_{j+1}, d_i s_{j-1} d_j w_{j+1}, d_i s_{j-1} x_j] \\
&= \begin{cases} [x_i, s_{j-2} d_{j-1} d_i w_{j+1}, s_{j-2} d_i x_j] & \\ \quad = [x_i, s_{j-2} d_{j-1} x_i, s_{j-2} d_{j-1} x_i] = x_i, & 0 \leqslant i < k, \\ [x_i, s_{j-1} d_j d_i w_{j+1}, s_{j-1} d_{i-1} x_j] & \\ \quad = [x_i, s_{j-1} d_j x_i, s_{j-1} d_j x_i] = x_i, & j < i \leqslant n, \\ [d_j w_{j+1}, d_j w_{j+1}, x_j] = x_j, & i = j \end{cases}
\end{aligned}
$$

for $n > j > k$.

Thus if one takes $x = w_{k-1}$ for $k = n$ and $x = w_{k+1}$ for $k < n$, one obtains $f(x) = y$ and $d_i x = x_i$ for $i \neq k$, $0 \leqslant i \leqslant n$, as desired. □

Acknowledgements

The research was supported by the TMR research network ERB FMRX CT-97-0107 and the INTAS grant #99-00817. The authors are grateful to the Max Planck Institute for Mathematics (Bonn), where they were given an opportunity to work jointly on the paper.

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(Received 15.08.2001)

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