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# ESTIMATES OF A STABILIZATION RATE AS $t \to \infty$ OF SOLUTIONS OF A NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

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**Abstract**. The asymptotic behavior as $t \to \infty$ of solutions of a nonlinear integro-differential equation is studied. The equation arises as a model describing the penetration of the electromagnetic field in to a substance.

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**1. Introduction. The main result.** This paper is devoted to the study of the stabilization of solutions of the first boundary value problem in a cylindrical domain $Q = (0,1) \times \{t > 0\}$ for the system of nonlinear integro-differential parabolic equations

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial U}{\partial x}\Big], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial V}{\partial x}\Big], \quad (x,t) \in Q, \tag{1}$$

$$U(0,t) = V(0,t) = 0, \;\; U(1,t) = \psi_1, \;\; V(1,t) = \psi_2, \quad t \geq 0, \tag{2}$$

$$U(x,0) = U_0(x), \;\; V(x,0) = V_0(x), \;\; x \in (0,1), \tag{3}$$

where

$$S(x,t) = 1 + \int_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big] d\tau, \tag{4}$$

or

$$S(t) = 1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big] dx\, d\tau. \tag{5}$$

Here $\psi_1 = Const$, $\psi_2 = Const$, $a(S)$, $U_0(x)$ and $V_0(x)$ are given functions.

The characteristic feature of equations (1), (4) and (1), (5) is the appearance of nonlinear members depending on the integral of searched functions in the coefficients of higher derivatives.

System (1), (4) arises as a model describing the penetration of the electromagnetic field into a substance [1].

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A lot of scientific works are devoted to the investigation of the problem given in [1] and to some of its generalizations. These questions are considered in [2]–[9] and in a number of other works as well.

The study of equations of type (1), (4) began started in [1]. In this work, in particular, theorems of the existence of a generalized solution of the first boundary value problem for $a(S) = S$ and the uniqueness for more general cases are proved. The case $a(S) = S^p$, $0 < p \le 1$, is studied in [2], where a theorem of the existence and uniqueness of a solution of problem (1)–(4) is proved. Investigations for multidimensional space cases are carried out in [3] for the first time.

In [4], [5] an operational scheme with the so-called conditionally closed operators is proposed. This scheme is applied for the solution of problems of (1)–(4) type [4], [5].

Note that investigations of equations of (1), (4) type are also carried out in [6], [7].

In the work [5] some generalization of equations of type (1), (4) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modelled by averaged integro-differential equations of type (1), (5).

The purpose of this note is to continue the study of the asymptotic behavior of solutions of the equations (1), (4) which began in [8], [9]. In the present paper estimates of stabilization rate as $t \to \infty$ of solutions of problems (1)–(4) and (1)–(3), (5) are obtained for the case $a(S) = S^p$, $\;0 < p \le 1$. We will use the scheme of [10] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied. We should note that boundary conditions (2) are used here taking into account the physical problem considered in [11].

We assume that $(U,V) = (U(x,t), V(x,t))$ is a solution of (1)–(4) on $[0,1] \times [0,\infty)$ such that $U, V, \dfrac{\partial U}{\partial x}, \dfrac{\partial V}{\partial x}, \dfrac{\partial U}{\partial t}, \dfrac{\partial V}{\partial t}, \dfrac{\partial^2 U}{\partial x^2}, \dfrac{\partial^2 V}{\partial x^2}$ are all in $C^0([0,\infty); L_2(0,1))$, while $\dfrac{\partial^2 U}{\partial t\,\partial x}, \dfrac{\partial^2 V}{\partial t\,\partial x}$ are in $C^0((0,\infty); L_2(0,1))$ and $\dfrac{\partial^2 U}{\partial t^2}, \dfrac{\partial^2 V}{\partial t^2}$ are in $L_{2,loc}((0,\infty); L_2(0,1))$ (see [1], [2], [4], [5], [10]).

The main purpose of this work is to prove the following statement.

**Theorem.** *Assume*

$$a(S) = S^p, \;\; 0 < p \le 1,$$
$$U_0(0) = V_0(0) = 0, \;\; U_0(1) = \psi_1, \;\; V_0(1) = \psi_2,$$
$$\psi_1^2 + \psi_2^2 \ne 0, \;\; U_0, \; V_0 \in W_2^2(0,1).$$

*Then for the solution of problem* (1)–(4) *the following estimates are true as* $t \to \infty$*:*

$$\frac{\partial U(x,t)}{\partial x} = \psi_1 + O(t^{-1-p}), \quad \frac{\partial V(x,t)}{\partial x} = \psi_2 + O(t^{-1-p}), \tag{6}$$

$$\frac{\partial U(x,t)}{\partial t} = O(t^{-1}), \quad \frac{\partial V(x,t)}{\partial t} = O(t^{-1}), \tag{7}$$

*uniformly in* $x$ *on* $[0,1]$.

The proof of the theorem is based on a priori estimates which are obtained with the help of a number of identities derived below.

**2. Proof of the theorem.** Now let us proved to obtaining a priori estimates.

**Lemma 1.** *For solving problem* $(1)-(4)$ *the following estimations are true:*

$$\int_0^t\int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 dx\,d\tau \le C, \quad \int_0^t\int_0^1 \left(\frac{\partial V}{\partial \tau}\right)^2 dx\,d\tau \le C, \quad t \ge 0. \tag{8}$$

*Proof.* Let us differentiate the first equation of system (1) with respect to $t$:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left[S^p \frac{\partial^2 U}{\partial t\,\partial x} + pS^{p-1}\left(\left[\frac{\partial U}{\partial x}\right]^3 + \frac{\partial U}{\partial x}\left[\frac{\partial V}{\partial x}\right]^2\right)\right]. \tag{9}$$

Multiplying equation (9) by $\partial U/\partial t$ and integrating with respect to $x$ on the interval $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 S^p\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx + p\int_0^1 S^{p-1}\left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial^2 U}{\partial t\,\partial x}dx +$$
$$+p\int_0^1 S^{p-1}\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial^2 U}{\partial t\,\partial x}dx = 0. \tag{10}$$

Integration from 0 to $t$ gives the identity

$$\frac{1}{2}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^t\int_0^1 S^p\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau + \frac{p}{4}\int_0^1 S^{p-1}\left(\frac{\partial U}{\partial x}\right)^4 dx -$$
$$-\frac{p(p-1)}{4}\int_0^t\int_0^1 S^{p-2}\left(\frac{\partial U}{\partial x}\right)^4\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]dx\,d\tau +$$
$$+\frac{p}{2}\int_0^t\int_0^1 S^{p-1}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial x}\right)^2 dx\,d\tau = \frac{1}{2}\int_0^1 \left(\frac{\partial U(x,0)}{\partial t}\right)^2 dx + \frac{p}{4}\int_0^1 \left(\frac{\partial U_0}{\partial x}\right)^4 dx.$$

It follows that

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + 2\int_0^t\int_0^1 \left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau + p\int_0^t\int_0^1 S^{p-1}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial x}\right)^2 dx\,d\tau \le C.$$

Here and below $c$, $C$ and $C_i$ denote the positive constants dependent only on $\psi_i = Const$, $i = 1, 2$, $U_0(x)$, $V_0(x)$ and consequently independent of $t$.

Similarly, using the second equation of system (1), we get

$$\int_0^1 \Big(\frac{\partial V}{\partial t}\Big)^2 dx + 2\int_0^t\int_0^1 \Big(\frac{\partial^2 V}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau + p\int_0^t\int_0^1 S^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^2 \frac{\partial}{\partial\tau}\Big(\frac{\partial V}{\partial x}\Big)^2 dx\,d\tau \le C$$

and, therefore

$$\int_0^1 \Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial V}{\partial t}\Big)^2\Big] dx + 2\int_0^t\int_0^1 \Big[\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 + \Big(\frac{\partial^2 V}{\partial\tau\,\partial x}\Big)^2\Big] dx\,d\tau$$
$$+p\int_0^t\int_0^1 S^{p-1}\frac{\partial}{\partial\tau}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^2\Big] dx\,d\tau \le C.$$

Note that

$$p\int_0^t\int_0^1 S^{p-1}\frac{\partial}{\partial\tau}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^2\Big] dx\,d\tau = p\int_0^1 S^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^2 dx$$
$$-p\int_0^1 \Big(\frac{\partial U_0}{\partial x}\Big)^2\Big(\frac{\partial V_0}{\partial x}\Big)^2 dx - p(p-1)\int_0^t\int_0^1 S^{p-2}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2$$
$$+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^2 dx\,d\tau.$$

We have

$$\int_0^1 \Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial V}{\partial t}\Big)^2\Big] dx + 2\int_0^t\int_0^1 \Big[\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 + \Big(\frac{\partial^2 V}{\partial\tau\,\partial x}\Big)^2\Big] dx\,d\tau \le C. \quad (11)$$

From this, taking into consideration the relations

$$\int_0^t\int_0^1 \Big(\frac{\partial U}{\partial\tau}\Big)^2 dx\,d\tau \le \int_0^t\int_0^1 \Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau,$$
$$\int_0^t\int_0^1 \Big(\frac{\partial V}{\partial\tau}\Big)^2 dx\,d\tau \le \int_0^t\int_0^1 \Big(\frac{\partial^2 V}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau,$$

we get a priori estimates (8). □

**Lemma 2.** *The following estimations are true:*

$$c\varphi^{\frac{1}{1+2p}}(t) \le S(x,t) \le C\varphi^{\frac{1}{1+2p}}(t), \quad 0\le x\le 1, \quad t\ge 0, \quad (12)$$

*where*

$$\varphi(t) = 1 + \int_0^t\int_0^1 S^{2p}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big] dx\,d\tau.$$

*Proof.* From (4) it follows that

$$\frac{\partial S}{\partial t} = \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2, \quad S(x,0) = 1.$$

Let us multiply this equation by $S^{2p}$:

$$\frac{1}{1+2p}\frac{\partial S^{1+2p}}{\partial t} = \left(\frac{\partial U}{\partial x}\right)^2 S^{2p} + \left(\frac{\partial V}{\partial x}\right)^2 S^{2p}.$$

Now let us introduce the notations:

$$\sigma_1 = S^p\frac{\partial U}{\partial x}, \quad \sigma_2 = S^p\frac{\partial V}{\partial x},$$

then (1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial \sigma_1}{\partial x}, \quad \frac{\partial V}{\partial t} = \frac{\partial \sigma_2}{\partial x}.$$

We have

$$\frac{1}{1+2p}\frac{\partial S^{1+2p}}{\partial t} = \sigma_1^2 + \sigma_2^2, \tag{13}$$

$$\sigma_1^2(x,t) = \int_0^1 \sigma_1^2(y,t)dy + 2\int_0^1\int_y^x \sigma_1(\xi,t)\frac{\partial U(\xi,t)}{\partial t}d\xi\, dy,$$

$$\sigma_2^2(x,t) = \int_0^1 \sigma_2^2(y,t)dy + 2\int_0^1\int_y^x \sigma_2(\xi,t)\frac{\partial V(\xi,t)}{\partial t}d\xi\, dy,$$

$$\varphi(t) = 1 + \int_0^t\int_0^1 (\sigma_1^2 + \sigma_2^2)dx\, d\tau.$$

From (8) and (13) we get

$$\frac{1}{1+2p}S^{1+2p} = \int_0^t (\sigma_1^2 + \sigma_2^2)d\tau + \frac{1}{1+2p}$$

$$= \int_0^t\int_0^1 \Big(\sigma_1^2(y,\tau) + \sigma_2^2(y,\tau)\Big)dy\, d\tau + 2\int_0^t\int_0^1\int_y^x \sigma_1(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial \tau}d\xi\, dy\, d\tau$$

$$+2\int_0^t\int_0^1\int_y^x \sigma_2(\xi,\tau)\frac{\partial V(\xi,\tau)}{\partial \tau}d\xi\, dy\, d\tau + \frac{1}{1+2p}$$

$$\le 2\int_0^t\int_0^1 \Big(\sigma_1^2(y,\tau) + \sigma_2^2(y,\tau)\Big)dy\, d\tau + C + \frac{1}{1+2p} \le (2 + C_1)\varphi(t),$$

i.e.,

$$S(x,t) \le C\varphi^{\frac{1}{1+2p}}(t). \tag{14}$$

Analogously,

$$\frac{1}{1+2p}S^{1+2p}=\int_0^t\int_0^1\Big(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau)\Big)(y,\tau)dy\,d\tau$$

$$+2\int_0^t\int_0^1\int_y^x\sigma_1(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial\tau}d\xi\,dy\,d\tau+2\int_0^t\int_0^1\int_y^x\sigma_2(\xi,\tau)\frac{\partial V(\xi,\tau)}{\partial\tau}d\xi\,dy\,d\tau$$

$$+\frac{1}{1+2p}\ge\frac{1}{2}\int_0^t\int_0^1\Big(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau)\Big)dy\,d\tau-C_2\ge\frac{1}{2}\varphi(t)-C_3. \tag{15}$$

From (4) it follows that $S(x,t)\ge 1$. So

$$C_3S^{1+2p}\ge C_3. \tag{16}$$

Taking into account (15) and (16) we easily get

$$\Big(\frac{1}{1+2p}+C_3\Big)S^{1+2p}\ge\frac{1}{2}\varphi(t),$$

or

$$S(x,t)\ge c\varphi^{\frac{1}{1+2p}}(t). \tag{17}$$

Finally, from (14) and (17) we obtain (12). $\square$

**Lemma 3.** *The following esimates are true*:

$$c\varphi^{\frac{2p}{1+2p}}(t)\le\int_0^1\Big(\sigma_1^2(x,t)+\sigma_2^2(x,t)\Big)dx\le C\varphi^{\frac{2p}{1+2p}}(t),\quad t\ge 0. \tag{18}$$

*Proof.* Taking into account (12), we get

$$\int_0^1(\sigma_1^2+\sigma_2^2)dx=\int_0^1 S^{2p}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]dx\ge c\varphi^{\frac{2p}{1+2p}}(t)\int_0^1\Big[\Big(\frac{\partial U}{\partial x}\Big)^2$$

$$+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]dx\ge c\varphi^{\frac{2p}{1+2p}}(t)\Big\{\Big[\int_0^1\frac{\partial U}{\partial x}dx\Big]^2+\Big[\int_0^1\frac{\partial V}{\partial x}dx\Big]^2\Big\}=(\psi_1^2+\psi_2^2)c\varphi^{\frac{2p}{1+2p}}(t),$$

or

$$\int_0^1\Big(\sigma_1^2(x,t)+\sigma_2^2(x,t)\Big)dx\ge c\varphi^{\frac{2p}{1+2p}}(t). \tag{19}$$

At the same time, from (11) we derive

$$\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2dx\le C,\quad\int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2dx\le C,\quad t\ge 0. \tag{20}$$

Let us multiply the first and second equations of system (1) by $U(x,t)$ and $V(x,t)$, respectively. Using the boundary conditions (2), we have

$$\int_0^1 U\frac{\partial U}{\partial t}dx+\int_0^1 S^p\Big(\frac{\partial U}{\partial x}\Big)^2 dx=\psi_1\sigma_1(1,t),$$
$$\int_0^1 V\frac{\partial V}{\partial t}dx+\int_0^1 S^p\Big(\frac{\partial V}{\partial x}\Big)^2 dx=\psi_2\sigma_2(1,t).$$

Using these equalities, (12), (19), (20) and the maximum principle

$$|U(x,t)|\le \max_{0\le y\le 1}|U_0(y)|,\ \ |V(x,t)|\le \max_{0\le y\le 1}|V_0(y)|,\ \ 0\le x\le 1,\ \ t\ge 0,$$

we have

$$\Big\{\int_0^1\Big(\sigma_1^2(x,t)+\sigma_2^2(x,t)\Big)dx\Big\}^2\le 2\Big\{\int_0^1\sigma_1^2(x,t)dx\Big\}^2+2\Big\{\int_0^1\sigma_2^2(x,t)dx\Big\}^2$$
$$\le 2C_1\varphi^{\frac{2p}{1+2p}}(t)\Bigg[\Big\{\int_0^1 S^p\Big(\frac{\partial U}{\partial x}\Big)^2dx\Big\}^2+\Big\{\int_0^1 S^p\Big(\frac{\partial V}{\partial x}\Big)^2dx\Big\}^2\Bigg]$$
$$\le 4C_1\varphi^{\frac{2p}{1+2p}}(t)\Bigg[(\psi_1\sigma_1(1,t))^2+\Big(\int_0^1 U\frac{\partial U}{\partial t}dx\Big)^2+(\psi_2\sigma_2(1,t))^2$$
$$+\Big(\int_0^1 V\frac{\partial V}{\partial t}dx\Big)^2\Bigg]\le 4C_1\varphi^{\frac{2p}{1+2p}}(t)\Bigg[(\psi_1^2+\psi_2^2)\Big(\sigma_1^2(1,t)+\sigma_2^2(1,t)\Big)$$
$$+C_2\Big\{\Big(\max_{0\le y\le 1}|U_0(y)|\Big)^2+\Big(\max_{0\le y\le 1}|V_0(y)|\Big)^2\Big\}\Bigg]$$
$$\le 8C_1\varphi^{\frac{2p}{1+2p}}(t)\Bigg[(\psi_1^2+\psi_2^2)\Big\{\int_0^1\sigma_1^2dx+\int_0^1\Big(\frac{\partial\sigma_1}{\partial x}\Big)^2dx+\int_0^1\sigma_2^2dx$$
$$+\int_0^1\Big(\frac{\partial\sigma_2}{\partial x}\Big)^2dx\Big\}+C_3\Bigg]\le 8C_1\varphi^{\frac{2p}{1+2p}}(t)\Bigg[(\psi_1^2+\psi_2^2)\int_0^1(\sigma_1^2+\sigma_2^2)dx+C_4\Bigg]$$
$$\le 8C_1\varphi^{\frac{2p}{1+2p}}(t)\Bigg[(\psi_1^2+\psi_2^2)\int_0^1(\sigma_1^2+\sigma_2^2)dx+\frac{C_4}{c\varphi^{\frac{2p}{1+2p}}(t)}\int_0^1(\sigma_1^2+\sigma_2^2)dx\Bigg]$$
$$\le C\varphi^{\frac{2p}{1+2p}}(t)\int_0^1\Big(\sigma_1^2(x,t)+\sigma_2^2(x,t)\Big)dx,$$

i.e.,

$$\int_0^1\Big(\sigma_1^2(x,t)+\sigma_2^2(x,t)\Big)dx\le C\varphi^{\frac{2p}{1+2p}}(t).$$

Finally, using this estimate and (19) we obtain (18). $\square$

**Lemma 4.** *The following inequalities take place:*

$$ct^{2p} \le \int_0^1 \Big(\sigma_1^2(x,t) + \sigma_2^2(x,t)\Big)dx \le Ct^{2p}, \;\; t \ge 1, \tag{21}$$

$$ct \le S(x,t) \le Ct, \;\; 0 \le x \le 1, \;\; t \ge 1. \tag{22}$$

*Proof.* From (18) taking into account the relation

$$\frac{d\varphi(t)}{dt} = \int_0^1 \Big(\sigma_1^2(x,t) + \sigma_1^2(x,t)\Big)dx$$

we get

$$c\varphi^{\frac{2p}{1+2p}}(t) \le \frac{d\varphi(t)}{dt} \le C\varphi^{\frac{2p}{1+2p}}(t).$$

From this we have $ct^{1+2p} \le \varphi(t) \le Ct^{1+2p}, \;\; t \ge 1$. Now taking into account (12) and (18) from the last inequality we obtain (21) and (22). $\square$

**Lemma 5.** *$\partial U/\partial t$ and $\partial V/\partial t$ statisfy the inequality*

$$\int_0^1 \bigg[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial V}{\partial t}\Big)^2\bigg]dx \le Ct^{-2}, \;\; t \ge 1. \tag{23}$$

*Proof.* By Schwarz's inequality, (10) yields

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + \int_0^1 S^p \Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx \le 2p^2 \int_0^1 S^{p-2}\Big(\frac{\partial U}{\partial x}\Big)^6 dx$$
$$+2p^2 \int_0^1 S^{p-2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^4 dx. \tag{24}$$

Now using (21), (22), the relations $\sigma_1 = S^p\dfrac{\partial U}{\partial x}$, $\sigma_2 = S^p\dfrac{\partial V}{\partial x}$ and

$$\int_0^1 \Big(\frac{\partial \sigma_1}{\partial x}\Big)^2 dx = -\int_0^1 \sigma_1 \frac{\partial^2\sigma_1}{\partial x^2}dx, \;\; \int_0^1 \Big(\frac{\partial \sigma_2}{\partial x}\Big)^2 dx = -\int_0^1 \sigma_2 \frac{\partial^2\sigma_2}{\partial x^2}dx,$$

from (24) we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2 dx + ct^p \int_0^1 \Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx \le C_1 \frac{t^{p-2}}{t^{6p}} \int_0^1 (\sigma_1^6 + \sigma_1^2\sigma_2^4)dx$$
$$\le C_1 t^{-5p-2}\int_0^1 \sigma_1^2(x,t)dx\Big(\big[\max_{0\le x\le 1}\sigma_1^2(x,t)\big]^2 + \big[\max_{0\le x\le 1}\sigma_2^2(x,t)\big]^2\Big)$$
$$\le C_2 t^{-3p-2}\bigg(\bigg\{\int_0^1 \sigma_1^2 dx + 2\bigg[\int_0^1 \sigma_1^2 dx\bigg]^{\frac12}\bigg[\int_0^1 \Big(\frac{\partial\sigma_1}{\partial x}\Big)^2 dx\bigg]^{\frac12}\bigg\}^2$$

$$+\left\{\int_0^1 \sigma_2^2 dx + 2\left[\int_0^1 \sigma_2^2 dx\right]^{\frac{1}{2}}\left[\int_0^1 \left(\frac{\partial \sigma_2}{\partial x}\right)^2 dx\right]^{\frac{1}{2}}\right\}^2\Bigg)$$

$$\le C_2 t^{-3p-2}\Bigg(\left\{\int_0^1 \sigma_1^2 dx + 2\left[\int_0^1 \sigma_1^2 dx\right]^{\frac{3}{4}}\left[\int_0^1 \left(\frac{\partial^2 \sigma_1}{\partial x^2}\right)^2 dx\right]^{\frac{1}{4}}\right\}^2$$

$$+\left\{\int_0^1 \sigma_2^2 dx + 2\left[\int_0^1 \sigma_2^2 dx\right]^{\frac{3}{4}}\left[\int_0^1 \left(\frac{\partial^2 \sigma_2}{\partial x^2}\right)^2 dx\right]^{\frac{1}{4}}\right\}^2\Bigg)$$

$$\le C_3 t^{p-2} + C_4 t^{-3p-2} t^{3p}\left(\left[\int_0^1 \left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx\right]^{\frac{1}{2}} + \left[\int_0^1 \left(\frac{\partial^2 V}{\partial t\,\partial x}\right)^2 dx\right]^{\frac{1}{2}}\right)$$

$$\le C_3 t^{p-2} + C_5 t^{-p-4} + \frac{ct^p}{4}\left(\int_0^1 \left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx + \int_0^1 \left(\frac{\partial^2 V}{\partial t\,\partial x}\right)^2 dx\right).$$

Hence we have

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \frac{c}{4} t^p \int_0^1 \left[3\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 - \left(\frac{\partial^2 V}{\partial t\,\partial x}\right)^2\right] dx \le C t^{p-2}, \quad t \ge 1.$$

Alalogously, we can show that

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx + \frac{c}{4} t^p \int_0^1 \left[3\left(\frac{\partial^2 V}{\partial t\,\partial x}\right)^2 - \left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2\right] dx \le C t^{p-2}, \quad t \ge 1.$$

As a consequence we get the following estimate:

$$\frac{d}{dt}\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx + \frac{c}{2} t^p \int_0^1 \left[\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 + \left(\frac{\partial^2 V}{\partial t\,\partial x}\right)^2\right] dx \le C t^{p-2}. \quad (25)$$

By the Poincarè inequality

$$\left\|\frac{\partial U}{\partial t}\right\| \le \left\|\frac{\partial^2 U}{\partial t\,\partial x}\right\|, \quad \left\|\frac{\partial V}{\partial t}\right\| \le \left\|\frac{\partial^2 V}{\partial t\,\partial x}\right\|,$$

(25) gives

$$\frac{d}{dt}\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx + \frac{ct^p}{2}\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx \le C t^{p-2}.$$

From this we obtain (23). □

Let us now estimate $\partial S/\partial x$ in $L_1(0,1)$.

**Lemma 6.** *For $\partial S/\partial x$ the following estimate is true*:

$$\int_0^1 \left|\frac{\partial S}{\partial x}\right| dx \le C t^{-p}, \quad t \ge 1. \quad (26)$$

*Proof.* We have

$$\frac{\partial}{\partial t}\Big[S^{2p}\frac{\partial S}{\partial x}\Big]=2\sigma_1\frac{\partial\sigma_1}{\partial x}+2\sigma_2\frac{\partial\sigma_2}{\partial x}=2\sigma_1\frac{\partial U}{\partial t}+2\sigma_2\frac{\partial V}{\partial t}. \tag{27}$$

From (21) and (23) it follows that

$$\Big|\int_0^1\sigma_1\frac{\partial U}{\partial t}dx\Big|\le C_1t^pt^{-1}=Ct^{p-1},\quad \Big|\int_0^1\sigma_2\frac{\partial V}{\partial t}dx\Big|\le C_1t^pt^{-1}=Ct^{p-1} \tag{28}$$

and, applying (22), (27) and (28), we get

$$S^{2p}\frac{\partial S}{\partial x}=\int_0^t\Big(2\sigma_1\frac{\partial U}{\partial\tau}+2\sigma_2\frac{\partial V}{\partial\tau}\Big)d\tau,$$

$$\int_0^1\Big|\frac{\partial S}{\partial x}\Big|dx\le\frac{1}{c}t^{-2p}\int_0^tC_1\tau^{p-1}d\tau=Ct^{-p}. \qquad \square$$

Now we are ready to prove the theorem. Let us estimate $\partial^2U/\partial x^2$ in $L_1(0,1)$. We have

$$\frac{\partial U}{\partial x}=\sigma_1S^{-p},\quad \frac{\partial\sigma_1}{\partial x}=\frac{\partial U}{\partial t},\quad \frac{\partial^2U}{\partial x^2}=\frac{\partial U}{\partial t}S^{-p}-p\sigma_1S^{-p-1}\frac{\partial S}{\partial x},$$

$$\sigma_1^2(x,t)\le\int_0^1\sigma_1^2(y,t)dy+2\int_0^1|\sigma_1(y,t)|\Big|\frac{\partial U(y,t)}{\partial t}\Big|dy\le C_1t^{2p}+C_2t^{-2}.$$

From the latter we get

$$\sigma_1(x,t)\le Ct^p,\quad t\ge1. \tag{29}$$

Applying now (22), (23), (26) and (29), we derive

$$\int_0^1\Big|\frac{\partial^2U(x,t)}{\partial x^2}\Big|dx\le\int_0^1\Big|\frac{\partial U}{\partial t}S^{-p}\Big|dx+p\int_0^1\Big|\sigma_1S^{-p-1}\frac{\partial S}{\partial x}\Big|dx$$
$$\le\Big[\int_0^1S^{-2p}dx\Big]^{\frac12}\Big[\int_0^1\Big|\frac{\partial U}{\partial t}\Big|^2dx\Big]^{\frac12}+p\int_0^1\Big|\sigma_1S^{-p-1}\frac{\partial S}{\partial x}\Big|dx\le Ct^{-1-p}.$$

Hence we have

$$\int_0^1\Big|\frac{\partial^2U(x,t)}{\partial x^2}\Big|dx\le Ct^{-1-p},\quad t\ge1.$$

From this estimate, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x}=\int_0^1\frac{\partial U(y,t)}{\partial y}dy+\int_0^1\int_y^x\frac{\partial^2U(\xi,t)}{\partial\xi^2}d\xi\,dy,$$

it follows that

$$\frac{\partial U(x,t)}{\partial x}-\psi_1=\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi\,dy\le\int_0^1\left|\frac{\partial^2 U(y,t)}{\partial y^2}\right|dy\le Ct^{-1-p}.$$

Thus the following asymptotic formula takes place:

$$\frac{\partial U(x,t)}{\partial x}=\psi_1+O(t^{-1-p}).$$

The same estimate is valid for $\partial V/\partial x$:

$$\frac{\partial V(x,t)}{\partial x}=\psi_2+O(t^{-1-p}).$$

Let us now establish the asymptotic behaviour of the derivatives $\partial U/\partial t$ and $\partial V/\partial t$. For this multiply (10) by $t^2$. By integrating on $(0,t)$ we have

$$\frac{t^2}{2}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2dx-\int_0^t\int_0^1\tau\left(\frac{\partial U}{\partial \tau}\right)^2dx\,d\tau+\int_0^t\int_0^1\tau^2S^p\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2dx\,d\tau$$
$$+p\int_0^t\int_0^1\tau^2S^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial \tau\,\partial x}dx\,d\tau+p\int_0^t\int_0^1\tau^2S^{p-1}\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2\frac{\partial^2 U}{\partial \tau\,\partial x}dx\,d\tau=0$$

and, using Schwarz's inequality, we conclude that

$$\frac{1}{2}\int_0^t\int_0^1\tau^2S^p\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2dx\,d\tau\le\int_0^t\int_0^1\tau\left(\frac{\partial U}{\partial \tau}\right)^2dx\,d\tau$$
$$+p^2\int_0^t\int_0^1\tau^2S^{p-2}\left(\frac{\partial U}{\partial x}\right)^6dx\,d\tau+p^2\int_0^t\int_0^1\tau^2S^{p-2}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial V}{\partial x}\right)^4dx\,d\tau.$$

From this using (6), (22) and (23) we get

$$\int_0^t\int_0^1\tau^2S^p\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2dx\,d\tau\le Ct^{p+1}. \tag{30}$$

Hence

$$\int_0^t\tau^{p+2}\int_0^1\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2dx\,d\tau\le Ct^{p+1}. \tag{31}$$

Analogously,

$$\int_0^t\int_0^1\tau^2S^p\left(\frac{\partial^2 V}{\partial \tau\,\partial x}\right)^2dx\,d\tau\le Ct^{p+1}, \tag{32}$$

$$\int_0^t\tau^{p+2}\int_0^1\left(\frac{\partial^2 V}{\partial \tau\,\partial x}\right)^2dx\,d\tau\le Ct^{p+1}. \tag{33}$$

Multiplying (9) by $t^3\partial^2 U/\partial t^2$, applying the formula of integrating by parts and a priori estimates (6), (22), (30), (31), (32) and (33), we get

$$\int_0^t\int_0^1 \tau^3\Big(\frac{\partial^2 U}{\partial \tau^2}\Big)^2 dx\,d\tau+\frac{1}{2}\int_0^t\int_0^1 \tau^3 S^p\frac{\partial}{\partial\tau}\Big[\frac{\partial^2 U}{\partial\tau\,\partial x}\Big]^2 dx\,d\tau$$

$$+p\int_0^t\int_0^1 \tau^3 S^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial}{\partial\tau}\Big[\frac{\partial^2 U}{\partial\tau\,\partial x}\Big]dx\,d\tau$$

$$+p\int_0^t\int_0^1 \tau^3 S^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial}{\partial\tau}\Big[\frac{\partial^2 U}{\partial\tau\,\partial x}\Big]dx\,d\tau=0,$$

$$\int_0^t\int_0^1 \tau^3\Big(\frac{\partial^2 U}{\partial \tau^2}\Big)^2 dx\,d\tau+\frac{1}{2}\int_0^1 t^3 S^p\Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx=\frac{3}{2}\int_0^t\int_0^1 \tau^2 S^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2 dx\,d\tau$$

$$+\frac{p}{2}\int_0^t\int_0^1 \tau^3 S^{p-1}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$-pt^3\int_0^1 S^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial t\,\partial x}dx+3p\int_0^t\int_0^1 \tau^2 S^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial\tau\,\partial x}dx\,d\tau$$

$$+p(p-1)\int_0^t\int_0^1 \tau^3 S^{p-2}\Big[\Big(\frac{\partial U}{\partial x}\Big)^5+\Big(\frac{\partial U}{\partial x}\Big)^3\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\frac{\partial^2 U}{\partial\tau\,\partial x}dx\,d\tau$$

$$+3p\int_0^t\int_0^1 \tau^3 S^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial^2 U}{\partial\,\tau\partial x}\Big)^2 dx\,d\tau-pt^3\int_0^1 S^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial t\,\partial x}dx$$

$$+3p\int_0^t\int_0^1 \tau^2 S^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial\tau\,\partial x}dx\,d\tau+p(p-1)\int_0^t\int_0^1 \tau^3 S^{p-2}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2$$

$$+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial x\,\partial\tau}dx\,d\tau+p\int_0^t\int_0^1 \tau^3 S^{p-1}\Big(\frac{\partial V}{\partial x}\Big)^2\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+2p\int_0^t\int_0^1 \tau^3 S^{p-1}\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial\tau\,\partial x}\frac{\partial^2 V}{\partial\tau\,\partial x}dx\,d\tau,$$

$$t^3\int_0^1 S^p\Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx\le 3\int_0^t\int_0^1 \tau^2 S^p\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau+C_1\int_0^t \tau^{p+2}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+\frac{t^3}{2}\int_0^1 S^p\Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx+C_2 t^3\int_0^1 S^{p-2}dx+C_3\int_0^t\int_0^1 \tau^2 S^p\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+C_4\int_0^t\int_0^1 \tau^2 S^{p-2}dx\,d\tau+\int_0^t\int_0^1 \tau^2 S^p\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+C_5\int_0^t\int_0^1 \tau^4 S^{p-4}dx\,d\tau+C_7t^{p-1}+C_6\int_0^t \tau^{p+2}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+C_8\int_0^t \tau^{p+2}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau+C_9\int_0^t\int_0^1 \tau^2 S^p\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+C_{10}\int_0^t\int_0^1 \tau^4 S^{p-4}dx\,d\tau+C_{11}\int_0^t \tau^{p+2}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau$$

$$+C_{12}\int_0^t \tau^{p+2}\int_0^1\Big(\frac{\partial^2 U}{\partial\tau\,\partial x}\Big)^2 dx\,d\tau\le C_{13}t^{p+1},$$

$$ct^{p+3}\int_0^1\Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx\le Ct^{p+1}.$$

From this we have

$$\int_0^1\Big(\frac{\partial^2 U}{\partial t\,\partial x}\Big)^2 dx\le Ct^{-2}. \tag{34}$$

Taking into account the relation

$$\frac{\partial U(x,t)}{\partial t}=\int_0^1\frac{\partial U(y,t)}{\partial t}dy+\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial t\,\partial\xi}d\xi\,dy\le Ct^{-1}$$

$$+\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial t\,\partial\xi}d\xi\,dy\le Ct^{-1}+\Big[\int_0^1\Big(\frac{\partial^2 U(y,t)}{\partial t\,\partial y}\Big)^2 dy\Big]^{\frac12},$$

from (34) we get

$$\frac{\partial U(x,t)}{\partial t}=O(t^{-1}).$$

Analogously, we can show that

$$\frac{\partial V(x,t)}{\partial t}=O(t^{-1}).$$

So the proof of the theorem is over.

Finally we note that estimates similar to (6) and (7) are true for the averaged integro-differential problem (1)–(3), (5).

## References


1. D. G. Gordeziani, T. A. Dzhangveladze, and T. K. Korshiya, Existence and uniqueness of the solution of a class of nonlinear parabolic problems. (Russian) *Differentsial'nye Uravneniya* **19**(1983), No. 7, 1197–1207.
2. T. A. Dzhangveladze, The first boundary value problem for a nonlinear equation of parabolic type. (Russian) *Dokl. Akad. Nauk SSSR* **269**(1983), No. 4, 839–842.
3. T. A. Dzhangveladze, A nonlinear integro-differential equation of parabolic type. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 1, 41–46.
4. G. I. Laptev, Quasilinear parabolic equations that have a Volterra operator in the coefficients. (Russian) *Mat. Sb.* (*N.S.*) **136(178)**(1988), No. 4, 530–545.
5. G. I. Laptev, Quasilinear evolution partial differential equations with operator coefficients. (Russian) *Doctoral Dissertation, Moscow,* 1990.
6. N. T. Long and A. P. N. Dinh, Nonlinear parabolic problem associated with the penetration of a magnetic field into a substance. *Math. Mech. Appl. Sci.* **16**(1993), 281–295.
7. N. T. Long and A. P. N. Dinh, Periodic solutions of a nonlinear parabolic equation associated with the penetration of a magnetic field into a substance. *Comput. Math. Appl.* **30**(1995), No. 1, 63–78.
8. T. Jangveladze and Z. Kiguradze, The asymptotic behaviour of the solution of one nonlinear integro-differential parabolic equation. *Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics* **10**(1995), No. 1, 36–38.
9. T. Jangveladze, On one class of nonlinear integro-differential equations. *Semin. I. Vekua Inst. Appl. Math., Rep.* **23**(1997), 51–87.
10. C. M. Dafermos and L. Hsiao, Adiabatic shearing of incompressible fluids with temperature-dependent viscosity. *Quart. Appl. Math.* **41**(1983) No. 1, 45–58.
11. T. A. Dzhangveladze, B. Ya. Lyubimov, and T. K. Korshiya, On the numerical solution of a class of nonisothermic problems of the diffusion of an electromagnetic field. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **18**(1986), 5–47.


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