

ESTIMATES OF A STABILIZATION RATE AS $t \rightarrow \infty$ OF
SOLUTIONS OF A NONLINEAR INTEGRO-DIFFERENTIAL
EQUATION

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Abstract. The asymptotic behavior as $t \rightarrow \infty$ of solutions of a nonlinear integro-differential equation is studied. The equation arises as a model describing the penetration of the electromagnetic field in to a substance.

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1. Introduction. The main result. This paper is devoted to the study of the stabilization of solutions of the first boundary value problem in a cylindrical domain $Q = (0, 1) \times \{t > 0\}$ for the system of nonlinear integro-differential parabolic equations

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left[a(S) \frac{\partial U}{\partial x} \right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x} \left[a(S) \frac{\partial V}{\partial x} \right], \quad (x, t) \in Q, \quad (1)$$

$$U(0, t) = V(0, t) = 0, \quad U(1, t) = \psi_1, \quad V(1, t) = \psi_2, \quad t \geq 0, \quad (2)$$

$$U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x), \quad x \in (0, 1), \quad (3)$$

where

$$S(x, t) = 1 + \int_0^t \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] d\tau, \quad (4)$$

or

$$S(t) = 1 + \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau. \quad (5)$$

Here $\psi_1 = Const$, $\psi_2 = Const$, $a(S)$, $U_0(x)$ and $V_0(x)$ are given functions.

The characteristic feature of equations (1), (4) and (1), (5) is the appearance of nonlinear members depending on the integral of searched functions in the coefficients of higher derivatives.

System (1), (4) arises as a model describing the penetration of the electromagnetic field into a substance [1].

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A lot of scientific works are devoted to the investigation of the problem given in [1] and to some of its generalizations. These questions are considered in [2]–[9] and in a number of other works as well.

The study of equations of type (1), (4) began started in [1]. In this work, in particular, theorems of the existence of a generalized solution of the first boundary value problem for $a(S) = S$ and the uniqueness for more general cases are proved. The case $a(S) = S^p$, $0 < p \leq 1$, is studied in [2], where a theorem of the existence and uniqueness of a solution of problem (1)–(4) is proved. Investigations for multidimensional space cases are carried out in [3] for the first time.

In [4], [5] an operational scheme with the so-called conditionally closed operators is proposed. This scheme is applied for the solution of problems of (1)–(4) type [4], [5].

Note that investigations of equations of (1), (4) type are also carried out in [6], [7].

In the work [5] some generalization of equations of type (1), (4) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modelled by averaged integro-differential equations of type (1), (5).

The purpose of this note is to continue the study of the asymptotic behavior of solutions of the equations (1), (4) which began in [8], [9]. In the present paper estimates of stabilization rate as $t \rightarrow \infty$ of solutions of problems (1)–(4) and (1)–(3), (5) are obtained for the case $a(S) = S^p$, $0 < p \leq 1$. We will use the scheme of [10] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied. We should note that boundary conditions (2) are used here taking into account the physical problem considered in [11].

We assume that $(U, V) = (U(x, t), V(x, t))$ is a solution of (1)–(4) on $[0, 1] \times [0, \infty)$ such that $U, V, \frac{\partial U}{\partial x}, \frac{\partial V}{\partial x}, \frac{\partial U}{\partial t}, \frac{\partial V}{\partial t}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 V}{\partial x^2}$ are all in $C^0([0, \infty); L_2(0, 1))$, while $\frac{\partial^2 U}{\partial t \partial x}, \frac{\partial^2 V}{\partial t \partial x}$ are in $C^0((0, \infty); L_2(0, 1))$ and $\frac{\partial^2 U}{\partial t^2}, \frac{\partial^2 V}{\partial t^2}$ are in $L_{2,loc}((0, \infty); L_2(0, 1))$ (see [1], [2], [4], [5], [10]).

The main purpose of this work is to prove the following statement.

Theorem. *Assume*

$$\begin{aligned} a(S) &= S^p, \quad 0 < p \leq 1, \\ U_0(0) = V_0(0) &= 0, \quad U_0(1) = \psi_1, \quad V_0(1) = \psi_2, \\ \psi_1^2 + \psi_2^2 &\neq 0, \quad U_0, V_0 \in W_2^2(0, 1). \end{aligned}$$

Then for the solution of problem (1)–(4) the following estimates are true as $t \rightarrow \infty$:

$$\frac{\partial U(x, t)}{\partial x} = \psi_1 + O(t^{-1-p}), \quad \frac{\partial V(x, t)}{\partial x} = \psi_2 + O(t^{-1-p}), \quad (6)$$

$$\frac{\partial U(x, t)}{\partial t} = O(t^{-1}), \quad \frac{\partial V(x, t)}{\partial t} = O(t^{-1}), \quad (7)$$

uniformly in x on $[0, 1]$.

The proof of the theorem is based on a priori estimates which are obtained with the help of a number of identities derived below.

2. Proof of the theorem. Now let us proved to obtaining a priori estimates.

Lemma 1. *For solving problem (1) – (4) the following estimations are true:*

$$\int_0^t \int_0^1 \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau \leq C, \quad \int_0^t \int_0^1 \left(\frac{\partial V}{\partial \tau} \right)^2 dx d\tau \leq C, \quad t \geq 0. \quad (8)$$

Proof. Let us differentiate the first equation of system (1) with respect to t :

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x} \left[S^p \frac{\partial^2 U}{\partial t \partial x} + p S^{p-1} \left(\left[\frac{\partial U}{\partial x} \right]^3 + \frac{\partial U}{\partial x} \left[\frac{\partial V}{\partial x} \right]^2 \right) \right]. \quad (9)$$

Multiplying equation (9) by $\partial U / \partial t$ and integrating with respect to x on the interval $(0, 1)$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx + p \int_0^1 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^3 \frac{\partial^2 U}{\partial t \partial x} dx + \\ + p \int_0^1 S^{p-1} \frac{\partial U}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial^2 U}{\partial t \partial x} dx = 0. \end{aligned} \quad (10)$$

Integration from 0 to t gives the identity

$$\begin{aligned} \frac{1}{2} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \int_0^t \int_0^1 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau + \frac{p}{4} \int_0^1 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^4 dx - \\ - \frac{p(p-1)}{4} \int_0^t \int_0^1 S^{p-2} \left(\frac{\partial U}{\partial x} \right)^4 \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau + \\ + \frac{p}{2} \int_0^t \int_0^1 S^{p-1} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial}{\partial \tau} \left(\frac{\partial U}{\partial x} \right)^2 dx d\tau = \frac{1}{2} \int_0^1 \left(\frac{\partial U(x, 0)}{\partial t} \right)^2 dx + \frac{p}{4} \int_0^1 \left(\frac{\partial U_0}{\partial x} \right)^4 dx. \end{aligned}$$

It follows that

$$\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + 2 \int_0^t \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau + p \int_0^t \int_0^1 S^{p-1} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial}{\partial \tau} \left(\frac{\partial U}{\partial x} \right)^2 dx d\tau \leq C.$$

Here and below c , C and C_i denote the positive constants dependent only on $\psi_i = \text{Const}$, $i = 1, 2$, $U_0(x)$, $V_0(x)$ and consequently independent of t .

Similarly, using the second equation of system (1), we get

$$\int_0^1 \left(\frac{\partial V}{\partial t} \right)^2 dx + 2 \int_0^t \int_0^1 \left(\frac{\partial^2 V}{\partial \tau \partial x} \right)^2 dx d\tau + p \int_0^t \int_0^1 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^2 \frac{\partial}{\partial \tau} \left(\frac{\partial V}{\partial x} \right)^2 dx d\tau \leq C$$

and, therefore

$$\begin{aligned} & \int_0^1 \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial V}{\partial t} \right)^2 \right] dx + 2 \int_0^t \int_0^1 \left[\left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 + \left(\frac{\partial^2 V}{\partial \tau \partial x} \right)^2 \right] dx d\tau \\ & + p \int_0^t \int_0^1 S^{p-1} \frac{\partial}{\partial \tau} \left[\left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau \leq C. \end{aligned}$$

Note that

$$\begin{aligned} & p \int_0^t \int_0^1 S^{p-1} \frac{\partial}{\partial \tau} \left[\left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau = p \int_0^1 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial V}{\partial x} \right)^2 dx \\ & - p \int_0^1 \left(\frac{\partial U_0}{\partial x} \right)^2 \left(\frac{\partial V_0}{\partial x} \right)^2 dx - p(p-1) \int_0^t \int_0^1 S^{p-2} \left[\left(\frac{\partial U}{\partial x} \right)^2 \right. \\ & \quad \left. + \left(\frac{\partial V}{\partial x} \right)^2 \right] \left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial V}{\partial x} \right)^2 dx d\tau. \end{aligned}$$

We have

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial V}{\partial t} \right)^2 \right] dx + 2 \int_0^t \int_0^1 \left[\left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 + \left(\frac{\partial^2 V}{\partial \tau \partial x} \right)^2 \right] dx d\tau \leq C. \quad (11)$$

From this, taking into consideration the relations

$$\begin{aligned} & \int_0^t \int_0^1 \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau \leq \int_0^t \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau, \\ & \int_0^t \int_0^1 \left(\frac{\partial V}{\partial \tau} \right)^2 dx d\tau \leq \int_0^t \int_0^1 \left(\frac{\partial^2 V}{\partial \tau \partial x} \right)^2 dx d\tau, \end{aligned}$$

we get a priori estimates (8). \square

Lemma 2. *The following estimations are true:*

$$c\varphi^{\frac{1}{1+2p}}(t) \leq S(x, t) \leq C\varphi^{\frac{1}{1+2p}}(t), \quad 0 \leq x \leq 1, \quad t \geq 0, \quad (12)$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 S^{2p} \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau.$$

Proof. From (4) it follows that

$$\frac{\partial S}{\partial t} = \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2, \quad S(x, 0) = 1.$$

Let us multiply this equation by S^{2p} :

$$\frac{1}{1+2p} \frac{\partial S^{1+2p}}{\partial t} = \left(\frac{\partial U}{\partial x} \right)^2 S^{2p} + \left(\frac{\partial V}{\partial x} \right)^2 S^{2p}.$$

Now let us introduce the notations:

$$\sigma_1 = S^p \frac{\partial U}{\partial x}, \quad \sigma_2 = S^p \frac{\partial V}{\partial x},$$

then (1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial \sigma_1}{\partial x}, \quad \frac{\partial V}{\partial t} = \frac{\partial \sigma_2}{\partial x}.$$

We have

$$\frac{1}{1+2p} \frac{\partial S^{1+2p}}{\partial t} = \sigma_1^2 + \sigma_2^2, \quad (13)$$

$$\sigma_1^2(x, t) = \int_0^1 \sigma_1^2(y, t) dy + 2 \int_0^1 \int_y^x \sigma_1(\xi, t) \frac{\partial U(\xi, t)}{\partial t} d\xi dy,$$

$$\sigma_2^2(x, t) = \int_0^1 \sigma_2^2(y, t) dy + 2 \int_0^1 \int_y^x \sigma_2(\xi, t) \frac{\partial V(\xi, t)}{\partial t} d\xi dy,$$

$$\varphi(t) = 1 + \int_0^t \int_0^1 (\sigma_1^2 + \sigma_2^2) dx d\tau.$$

From (8) and (13) we get

$$\begin{aligned} \frac{1}{1+2p} S^{1+2p} &= \int_0^t (\sigma_1^2 + \sigma_2^2) d\tau + \frac{1}{1+2p} \\ &= \int_0^t \int_0^1 (\sigma_1^2(y, \tau) + \sigma_2^2(y, \tau)) dy d\tau + 2 \int_0^t \int_0^1 \int_y^x \sigma_1(\xi, \tau) \frac{\partial U(\xi, \tau)}{\partial \tau} d\xi dy d\tau \\ &\quad + 2 \int_0^t \int_0^1 \int_y^x \sigma_2(\xi, \tau) \frac{\partial V(\xi, \tau)}{\partial \tau} d\xi dy d\tau + \frac{1}{1+2p} \\ &\leq 2 \int_0^t \int_0^1 (\sigma_1^2(y, \tau) + \sigma_2^2(y, \tau)) dy d\tau + C + \frac{1}{1+2p} \leq (2 + C_1) \varphi(t), \end{aligned}$$

i.e.,

$$S(x, t) \leq C \varphi^{\frac{1}{1+2p}}(t). \quad (14)$$

Analogously,

$$\begin{aligned} \frac{1}{1+2p} S^{1+2p} &= \int_0^t \int_0^1 (\sigma_1^2(y, \tau) + \sigma_2^2(y, \tau))(y, \tau) dy d\tau \\ + 2 \int_0^t \int_0^1 \int_y^x \sigma_1(\xi, \tau) \frac{\partial U(\xi, \tau)}{\partial \tau} d\xi dy d\tau &+ 2 \int_0^t \int_0^1 \int_y^x \sigma_2(\xi, \tau) \frac{\partial V(\xi, \tau)}{\partial \tau} d\xi dy d\tau \\ + \frac{1}{1+2p} &\geq \frac{1}{2} \int_0^t \int_0^1 (\sigma_1^2(y, \tau) + \sigma_2^2(y, \tau)) dy d\tau - C_2 \geq \frac{1}{2} \varphi(t) - C_3. \end{aligned} \quad (15)$$

From (4) it follows that $S(x, t) \geq 1$. So

$$C_3 S^{1+2p} \geq C_3. \quad (16)$$

Taking into account (15) and (16) we easily get

$$\left(\frac{1}{1+2p} + C_3 \right) S^{1+2p} \geq \frac{1}{2} \varphi(t),$$

or

$$S(x, t) \geq c\varphi^{\frac{1}{1+2p}}(t). \quad (17)$$

Finally, from (14) and (17) we obtain (12). \square

Lemma 3. *The following estimates are true:*

$$c\varphi^{\frac{2p}{1+2p}}(t) \leq \int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx \leq C\varphi^{\frac{2p}{1+2p}}(t), \quad t \geq 0. \quad (18)$$

Proof. Taking into account (12), we get

$$\begin{aligned} \int_0^1 (\sigma_1^2 + \sigma_2^2) dx &= \int_0^1 S^{2p} \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx \geq c\varphi^{\frac{2p}{1+2p}}(t) \int_0^1 \left[\left(\frac{\partial U}{\partial x} \right)^2 \right. \\ &+ \left. \left(\frac{\partial V}{\partial x} \right)^2 \right] dx \geq c\varphi^{\frac{2p}{1+2p}}(t) \left\{ \left[\int_0^1 \frac{\partial U}{\partial x} dx \right]^2 + \left[\int_0^1 \frac{\partial V}{\partial x} dx \right]^2 \right\} = (\psi_1^2 + \psi_2^2) c\varphi^{\frac{2p}{1+2p}}(t), \end{aligned}$$

or

$$\int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx \geq c\varphi^{\frac{2p}{1+2p}}(t). \quad (19)$$

At the same time, from (11) we derive

$$\int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx \leq C, \quad \int_0^1 \left(\frac{\partial V}{\partial t} \right)^2 dx \leq C, \quad t \geq 0. \quad (20)$$

Let us multiply the first and second equations of system (1) by $U(x, t)$ and $V(x, t)$, respectively. Using the boundary conditions (2), we have

$$\begin{aligned} \int_0^1 U \frac{\partial U}{\partial t} dx + \int_0^1 S^p \left(\frac{\partial U}{\partial x} \right)^2 dx &= \psi_1 \sigma_1(1, t), \\ \int_0^1 V \frac{\partial V}{\partial t} dx + \int_0^1 S^p \left(\frac{\partial V}{\partial x} \right)^2 dx &= \psi_2 \sigma_2(1, t). \end{aligned}$$

Using these equalities, (12), (19), (20) and the maximum principle

$$|U(x, t)| \leq \max_{0 \leq y \leq 1} |U_0(y)|, \quad |V(x, t)| \leq \max_{0 \leq y \leq 1} |V_0(y)|, \quad 0 \leq x \leq 1, \quad t \geq 0,$$

we have

$$\begin{aligned} \left\{ \int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx \right\}^2 &\leq 2 \left\{ \int_0^1 \sigma_1^2(x, t) dx \right\}^2 + 2 \left\{ \int_0^1 \sigma_2^2(x, t) dx \right\}^2 \\ &\leq 2C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[\left\{ \int_0^1 S^p \left(\frac{\partial U}{\partial x} \right)^2 dx \right\}^2 + \left\{ \int_0^1 S^p \left(\frac{\partial V}{\partial x} \right)^2 dx \right\}^2 \right] \\ &\leq 4C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[(\psi_1 \sigma_1(1, t))^2 + \left(\int_0^1 U \frac{\partial U}{\partial t} dx \right)^2 + (\psi_2 \sigma_2(1, t))^2 \right. \\ &\quad \left. + \left(\int_0^1 V \frac{\partial V}{\partial t} dx \right)^2 \right] \leq 4C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[(\psi_1^2 + \psi_2^2) (\sigma_1^2(1, t) + \sigma_2^2(1, t)) \right. \\ &\quad \left. + C_2 \left\{ \left(\max_{0 \leq y \leq 1} |U_0(y)| \right)^2 + \left(\max_{0 \leq y \leq 1} |V_0(y)| \right)^2 \right\} \right] \\ &\leq 8C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[(\psi_1^2 + \psi_2^2) \left\{ \int_0^1 \sigma_1^2 dx + \int_0^1 \left(\frac{\partial \sigma_1}{\partial x} \right)^2 dx + \int_0^1 \sigma_2^2 dx \right. \right. \\ &\quad \left. \left. + \int_0^1 \left(\frac{\partial \sigma_2}{\partial x} \right)^2 dx \right\} + C_3 \right] \leq 8C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[(\psi_1^2 + \psi_2^2) \int_0^1 (\sigma_1^2 + \sigma_2^2) dx + C_4 \right] \\ &\leq 8C_1 \varphi^{\frac{2p}{1+2p}}(t) \left[(\psi_1^2 + \psi_2^2) \int_0^1 (\sigma_1^2 + \sigma_2^2) dx + \frac{C_4}{c \varphi^{\frac{2p}{1+2p}}(t)} \int_0^1 (\sigma_1^2 + \sigma_2^2) dx \right] \\ &\leq C \varphi^{\frac{2p}{1+2p}}(t) \int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx, \end{aligned}$$

i.e.,

$$\int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx \leq C \varphi^{\frac{2p}{1+2p}}(t).$$

Finally, using this estimate and (19) we obtain (18). \square

Lemma 4. *The following inequalities take place:*

$$ct^{2p} \leq \int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx \leq Ct^{2p}, \quad t \geq 1, \quad (21)$$

$$ct \leq S(x, t) \leq Ct, \quad 0 \leq x \leq 1, \quad t \geq 1. \quad (22)$$

Proof. From (18) taking into account the relation

$$\frac{d\varphi(t)}{dt} = \int_0^1 (\sigma_1^2(x, t) + \sigma_2^2(x, t)) dx$$

we get

$$c\varphi^{\frac{2p}{1+2p}}(t) \leq \frac{d\varphi(t)}{dt} \leq C\varphi^{\frac{2p}{1+2p}}(t).$$

From this we have $ct^{1+2p} \leq \varphi(t) \leq Ct^{1+2p}$, $t \geq 1$. Now taking into account (12) and (18) from the last inequality we obtain (21) and (22). \square

Lemma 5. *$\partial U/\partial t$ and $\partial V/\partial t$ satisfy the inequality*

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial V}{\partial t} \right)^2 \right] dx \leq Ct^{-2}, \quad t \geq 1. \quad (23)$$

Proof. By Schwarz's inequality, (10) yields

$$\begin{aligned} \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx &\leq 2p^2 \int_0^1 S^{p-2} \left(\frac{\partial U}{\partial x} \right)^6 dx \\ &+ 2p^2 \int_0^1 S^{p-2} \left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial V}{\partial x} \right)^4 dx. \end{aligned} \quad (24)$$

Now using (21), (22), the relations $\sigma_1 = S^p \frac{\partial U}{\partial x}$, $\sigma_2 = S^p \frac{\partial V}{\partial x}$ and

$$\int_0^1 \left(\frac{\partial \sigma_1}{\partial x} \right)^2 dx = - \int_0^1 \sigma_1 \frac{\partial^2 \sigma_1}{\partial x^2} dx, \quad \int_0^1 \left(\frac{\partial \sigma_2}{\partial x} \right)^2 dx = - \int_0^1 \sigma_2 \frac{\partial^2 \sigma_2}{\partial x^2} dx,$$

from (24) we get

$$\begin{aligned} \frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + ct^p \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx &\leq C_1 \frac{t^{p-2}}{t^{6p}} \int_0^1 (\sigma_1^6 + \sigma_1^2 \sigma_2^4) dx \\ &\leq C_1 t^{-5p-2} \int_0^1 \sigma_1^2(x, t) dx \left(\left[\max_{0 \leq x \leq 1} \sigma_1^2(x, t) \right]^2 + \left[\max_{0 \leq x \leq 1} \sigma_2^2(x, t) \right]^2 \right) \\ &\leq C_2 t^{-3p-2} \left(\left\{ \int_0^1 \sigma_1^2 dx + 2 \left[\int_0^1 \sigma_1^2 dx \right]^{\frac{1}{2}} \left[\int_0^1 \left(\frac{\partial \sigma_1}{\partial x} \right)^2 dx \right]^{\frac{1}{2}} \right\}^2 \right) \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \int_0^1 \sigma_2^2 dx + 2 \left[\int_0^1 \sigma_2^2 dx \right]^{\frac{1}{2}} \left[\int_0^1 \left(\frac{\partial \sigma_2}{\partial x} \right)^2 dx \right]^{\frac{1}{2}} \right\}^2 \\
 & \leq C_2 t^{-3p-2} \left(\left\{ \int_0^1 \sigma_1^2 dx + 2 \left[\int_0^1 \sigma_1^2 dx \right]^{\frac{3}{4}} \left[\int_0^1 \left(\frac{\partial^2 \sigma_1}{\partial x^2} \right)^2 dx \right]^{\frac{1}{4}} \right\}^2 \right. \\
 & \quad \left. + \left\{ \int_0^1 \sigma_2^2 dx + 2 \left[\int_0^1 \sigma_2^2 dx \right]^{\frac{3}{4}} \left[\int_0^1 \left(\frac{\partial^2 \sigma_2}{\partial x^2} \right)^2 dx \right]^{\frac{1}{4}} \right\}^2 \right) \\
 & \leq C_3 t^{p-2} + C_4 t^{-3p-2} t^{3p} \left(\left[\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx \right]^{\frac{1}{2}} + \left[\int_0^1 \left(\frac{\partial^2 V}{\partial t \partial x} \right)^2 dx \right]^{\frac{1}{2}} \right) \\
 & \leq C_3 t^{p-2} + C_5 t^{-p-4} + \frac{ct^p}{4} \left(\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx + \int_0^1 \left(\frac{\partial^2 V}{\partial t \partial x} \right)^2 dx \right).
 \end{aligned}$$

Hence we have

$$\frac{d}{dt} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx + \frac{c}{4} t^p \int_0^1 \left[3 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 - \left(\frac{\partial^2 V}{\partial t \partial x} \right)^2 \right] dx \leq Ct^{p-2}, \quad t \geq 1.$$

Analogously, we can show that

$$\frac{d}{dt} \int_0^1 \left(\frac{\partial V}{\partial t} \right)^2 dx + \frac{c}{4} t^p \int_0^1 \left[3 \left(\frac{\partial^2 V}{\partial t \partial x} \right)^2 - \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 \right] dx \leq Ct^{p-2}, \quad t \geq 1.$$

As a consequence we get the following estimate:

$$\frac{d}{dt} \int_0^1 \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial V}{\partial t} \right)^2 \right] dx + \frac{c}{2} t^p \int_0^1 \left[\left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 + \left(\frac{\partial^2 V}{\partial t \partial x} \right)^2 \right] dx \leq Ct^{p-2}. \quad (25)$$

By the Poincarè inequality

$$\left\| \frac{\partial U}{\partial t} \right\| \leq \left\| \frac{\partial^2 U}{\partial t \partial x} \right\|, \quad \left\| \frac{\partial V}{\partial t} \right\| \leq \left\| \frac{\partial^2 V}{\partial t \partial x} \right\|,$$

(25) gives

$$\frac{d}{dt} \int_0^1 \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial V}{\partial t} \right)^2 \right] dx + \frac{ct^p}{2} \int_0^1 \left[\left(\frac{\partial U}{\partial t} \right)^2 + \left(\frac{\partial V}{\partial t} \right)^2 \right] dx \leq Ct^{p-2}.$$

From this we obtain (23). \square

Let us now estimate $\partial S / \partial x$ in $L_1(0, 1)$.

Lemma 6. *For $\partial S / \partial x$ the following estimate is true:*

$$\int_0^1 \left| \frac{\partial S}{\partial x} \right| dx \leq Ct^{-p}, \quad t \geq 1. \quad (26)$$

Proof. We have

$$\frac{\partial}{\partial t} \left[S^{2p} \frac{\partial S}{\partial x} \right] = 2\sigma_1 \frac{\partial \sigma_1}{\partial x} + 2\sigma_2 \frac{\partial \sigma_2}{\partial x} = 2\sigma_1 \frac{\partial U}{\partial t} + 2\sigma_2 \frac{\partial V}{\partial t}. \quad (27)$$

From (21) and (23) it follows that

$$\left| \int_0^1 \sigma_1 \frac{\partial U}{\partial t} dx \right| \leq C_1 t^p t^{-1} = Ct^{p-1}, \quad \left| \int_0^1 \sigma_2 \frac{\partial V}{\partial t} dx \right| \leq C_1 t^p t^{-1} = Ct^{p-1} \quad (28)$$

and, applying (22), (27) and (28), we get

$$\begin{aligned} S^{2p} \frac{\partial S}{\partial x} &= \int_0^t \left(2\sigma_1 \frac{\partial U}{\partial \tau} + 2\sigma_2 \frac{\partial V}{\partial \tau} \right) d\tau, \\ \int_0^1 \left| \frac{\partial S}{\partial x} \right| dx &\leq \frac{1}{c} t^{-2p} \int_0^t C_1 \tau^{p-1} d\tau = Ct^{-p}. \quad \square \end{aligned}$$

Now we are ready to prove the theorem. Let us estimate $\partial^2 U / \partial x^2$ in $L_1(0, 1)$. We have

$$\begin{aligned} \frac{\partial U}{\partial x} &= \sigma_1 S^{-p}, \quad \frac{\partial \sigma_1}{\partial x} = \frac{\partial U}{\partial t}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} S^{-p} - p\sigma_1 S^{-p-1} \frac{\partial S}{\partial x}, \\ \sigma_1^2(x, t) &\leq \int_0^1 \sigma_1^2(y, t) dy + 2 \int_0^1 |\sigma_1(y, t)| \left| \frac{\partial U(y, t)}{\partial t} \right| dy \leq C_1 t^{2p} + C_2 t^{-2}. \end{aligned}$$

From the latter we get

$$\sigma_1(x, t) \leq Ct^p, \quad t \geq 1. \quad (29)$$

Applying now (22), (23), (26) and (29), we derive

$$\begin{aligned} \int_0^1 \left| \frac{\partial^2 U(x, t)}{\partial x^2} \right| dx &\leq \int_0^1 \left| \frac{\partial U}{\partial t} S^{-p} \right| dx + p \int_0^1 \left| \sigma_1 S^{-p-1} \frac{\partial S}{\partial x} \right| dx \\ &\leq \left[\int_0^1 S^{-2p} dx \right]^{\frac{1}{2}} \left[\int_0^1 \left| \frac{\partial U}{\partial t} \right|^2 dx \right]^{\frac{1}{2}} + p \int_0^1 \left| \sigma_1 S^{-p-1} \frac{\partial S}{\partial x} \right| dx \leq Ct^{-1-p}. \end{aligned}$$

Hence we have

$$\int_0^1 \left| \frac{\partial^2 U(x, t)}{\partial x^2} \right| dx \leq Ct^{-1-p}, \quad t \geq 1.$$

From this estimate, taking into account the relation

$$\frac{\partial U(x, t)}{\partial x} = \int_0^1 \frac{\partial U(y, t)}{\partial y} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi, t)}{\partial \xi^2} d\xi dy,$$

it follows that

$$\frac{\partial U(x, t)}{\partial x} - \psi_1 = \int_0^1 \int_y^x \frac{\partial^2 U(\xi, t)}{\partial \xi^2} d\xi dy \leq \int_0^1 \left| \frac{\partial^2 U(y, t)}{\partial y^2} \right| dy \leq Ct^{-1-p}.$$

Thus the following asymptotic formula takes place:

$$\frac{\partial U(x, t)}{\partial x} = \psi_1 + O(t^{-1-p}).$$

The same estimate is valid for $\partial V/\partial x$:

$$\frac{\partial V(x, t)}{\partial x} = \psi_2 + O(t^{-1-p}).$$

Let us now establish the asymptotic behaviour of the derivatives $\partial U/\partial t$ and $\partial V/\partial t$. For this multiply (10) by t^2 . By integrating on $(0, t)$ we have

$$\begin{aligned} & \frac{t^2}{2} \int_0^1 \left(\frac{\partial U}{\partial t} \right)^2 dx - \int_0^t \int_0^1 \tau \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau + \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\ & + p \int_0^t \int_0^1 \tau^2 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^3 \frac{\partial^2 U}{\partial \tau \partial x} dx d\tau + p \int_0^t \int_0^1 \tau^2 S^{p-1} \frac{\partial U}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial^2 U}{\partial \tau \partial x} dx d\tau = 0 \end{aligned}$$

and, using Schwarz's inequality, we conclude that

$$\begin{aligned} & \frac{1}{2} \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \leq \int_0^t \int_0^1 \tau \left(\frac{\partial U}{\partial \tau} \right)^2 dx d\tau \\ & + p^2 \int_0^t \int_0^1 \tau^2 S^{p-2} \left(\frac{\partial U}{\partial x} \right)^6 dx d\tau + p^2 \int_0^t \int_0^1 \tau^2 S^{p-2} \left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial V}{\partial x} \right)^4 dx d\tau. \end{aligned}$$

From this using (6), (22) and (23) we get

$$\int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \leq Ct^{p+1}. \quad (30)$$

Hence

$$\int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \leq Ct^{p+1}. \quad (31)$$

Analogously,

$$\int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 V}{\partial \tau \partial x} \right)^2 dx d\tau \leq Ct^{p+1}, \quad (32)$$

$$\int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 V}{\partial \tau \partial x} \right)^2 dx d\tau \leq Ct^{p+1}. \quad (33)$$

Multiplying (9) by $t^3 \partial^2 U / \partial t^2$, applying the formula of integrating by parts and a priori estimates (6), (22), (30), (31), (32) and (33), we get

$$\begin{aligned}
& \int_0^t \int_0^1 \tau^3 \left(\frac{\partial^2 U}{\partial \tau^2} \right)^2 dx d\tau + \frac{1}{2} \int_0^t \int_0^1 \tau^3 S^p \frac{\partial}{\partial \tau} \left[\frac{\partial^2 U}{\partial \tau \partial x} \right]^2 dx d\tau \\
& \quad + p \int_0^t \int_0^1 \tau^3 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^3 \frac{\partial}{\partial \tau} \left[\frac{\partial^2 U}{\partial \tau \partial x} \right] dx d\tau \\
& \quad + p \int_0^t \int_0^1 \tau^3 S^{p-1} \frac{\partial U}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial}{\partial \tau} \left[\frac{\partial^2 U}{\partial \tau \partial x} \right] dx d\tau = 0, \\
& \int_0^t \int_0^1 \tau^3 \left(\frac{\partial^2 U}{\partial \tau^2} \right)^2 dx d\tau + \frac{1}{2} \int_0^1 t^3 S^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx = \frac{3}{2} \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
& \quad + \frac{p}{2} \int_0^t \int_0^1 \tau^3 S^{p-1} \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
& \quad - pt^3 \int_0^1 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^3 \frac{\partial^2 U}{\partial t \partial x} dx + 3p \int_0^t \int_0^1 \tau^2 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^3 \frac{\partial^2 U}{\partial \tau \partial x} dx d\tau \\
& \quad + p(p-1) \int_0^t \int_0^1 \tau^3 S^{p-2} \left[\left(\frac{\partial U}{\partial x} \right)^5 + \left(\frac{\partial U}{\partial x} \right)^3 \left(\frac{\partial V}{\partial x} \right)^2 \right] \frac{\partial^2 U}{\partial \tau \partial x} dx d\tau \\
& \quad + 3p \int_0^t \int_0^1 \tau^3 S^{p-1} \left(\frac{\partial U}{\partial x} \right)^2 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau - pt^3 \int_0^1 S^{p-1} \frac{\partial U}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial^2 U}{\partial t \partial x} dx \\
& \quad + 3p \int_0^t \int_0^1 \tau^2 S^{p-1} \frac{\partial U}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial^2 U}{\partial \tau \partial x} dx d\tau + p(p-1) \int_0^t \int_0^1 \tau^3 S^{p-2} \left[\left(\frac{\partial U}{\partial x} \right)^2 \right. \\
& \quad \left. + \left(\frac{\partial V}{\partial x} \right)^2 \right] \frac{\partial U}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 \frac{\partial^2 U}{\partial x \partial \tau} dx d\tau + p \int_0^t \int_0^1 \tau^3 S^{p-1} \left(\frac{\partial V}{\partial x} \right)^2 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
& \quad + 2p \int_0^t \int_0^1 \tau^3 S^{p-1} \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} \frac{\partial^2 U}{\partial \tau \partial x} \frac{\partial^2 V}{\partial \tau \partial x} dx d\tau, \\
& t^3 \int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx \leq 3 \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau + C_1 \int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
& \quad + \frac{t^3}{2} \int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx + C_2 t^3 \int_0^1 S^{p-2} dx + C_3 \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau
\end{aligned}$$

$$\begin{aligned}
 & +C_4 \int_0^t \int_0^1 \tau^2 S^{p-2} dx d\tau + \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
 & +C_5 \int_0^t \int_0^1 \tau^4 S^{p-4} dx d\tau + C_7 t^{p-1} + C_6 \int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
 & +C_8 \int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau + C_9 \int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
 & +C_{10} \int_0^t \int_0^1 \tau^4 S^{p-4} dx d\tau + C_{11} \int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \\
 & +C_{12} \int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx d\tau \leq C_{13} t^{p+1}, \\
 & ct^{p+3} \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx \leq Ct^{p+1}.
 \end{aligned}$$

From this we have

$$\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x} \right)^2 dx \leq Ct^{-2}. \quad (34)$$

Taking into account the relation

$$\begin{aligned}
 \frac{\partial U(x, t)}{\partial t} & = \int_0^1 \frac{\partial U(y, t)}{\partial t} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi, t)}{\partial t \partial \xi} d\xi dy \leq Ct^{-1} \\
 & + \int_0^1 \int_y^x \frac{\partial^2 U(\xi, t)}{\partial t \partial \xi} d\xi dy \leq Ct^{-1} + \left[\int_0^1 \left(\frac{\partial^2 U(y, t)}{\partial t \partial y} \right)^2 dy \right]^{\frac{1}{2}},
 \end{aligned}$$

from (34) we get

$$\frac{\partial U(x, t)}{\partial t} = O(t^{-1}).$$

Analogously, we can show that

$$\frac{\partial V(x, t)}{\partial t} = O(t^{-1}).$$

So the proof of the theorem is over.

Finally we note that estimates similar to (6) and (7) are true for the averaged integro-differential problem (1)–(3), (5).

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