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CESÀRO MEANS OF TRIGONOMETRIC FOURIER SERIES

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Abstract. L. Zhizhiashvili proved that if $f \in H_p^\omega$ for some p, $1 \le p \le \infty$, and $\alpha \in (0,1)$, then the L^p-deviation of f from its Cesàro mean is $O(n^\alpha \omega(1/n))$ where $\omega(\cdot)$ is a modulus of continuity. In this paper we show that this estimation is non-amplifiable for $p = 1$.

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1. Introduction

Let L^p, $1 \le p < \infty$, denote the collection of all measurable, 2π-periodic functions defined on the $[0, 2\pi)$ with the norm

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \right)^{1/p} < \infty,$$

we write L^∞ instead of C, the space of continuous and 2π-periodic functions given on $[0, 2\pi)$ with the norm $\|f\|_\infty = \|f\|_C = \sup_{x \in [0,2\pi)} |f(x)|$.

Let $f \in L^p$, $1 \le p \le \infty$. The expression

$$\omega(\delta, f)_p = \sup_{|h| \le \delta} \|f(\cdot + h) - f(\cdot)\|_p$$

is called the L^p-modulus of continuity. If $\omega(\delta)$ is a modulus of continuity, then H_p^ω denotes the class of functions $f \in L^p$ for which $\omega(\delta, f)_p = O(\omega(\delta))$ as $\delta \to 0+$. In particular, $H_p^\omega \equiv H_p^{\delta^\beta}$ for $\omega(\delta) = \delta^\beta$ $(\beta > 0)$ and $H_p^\omega \equiv H_p^\delta$ for $\omega(\delta) = \delta$.

The Cesàro (C, α)-means of trigonometric Fourier series are defined as follows:

$$\sigma_n^\alpha(f, x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha (a_k \cos kx + b_k \sin kx),$$

where a_0, a_k, b_k, $k = 1, 2, \ldots$, are the Fourier coefficients and

$$A_0^\alpha = 1, A_n^\alpha = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}, \quad \alpha \ne -1, -2, -3, \ldots.$$

It is well-known that ([1], Ch. 3)

$$c_1 n^\alpha < A_n^\alpha < c_2 n^\alpha \tag{1}$$

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with some positive constants c_1 and c_2 depending on α.

For the uniform negative order Cesàro summability of trigonometric Fourier series the following result of Zygmund [2] is well-known: if $f \in H_\infty^{\delta^\beta}$ and $\alpha \in (0,\beta)$, then the trigonometric Fourier series of the function f is uniformly $(C,-\alpha)$-summable to f.

L. Zhizhiashvili ([3], Part. 1, Ch. 4) established approximate properties of Cesàro $(C,-\alpha)$-means with $\alpha \in (-\infty,1)$ of trigonometric Fourier series. In particular, for $\alpha \in (0,1)$ he proved the following assertion.

Theorem A. *Let $\alpha \in (0,1)$ and $f \in H_p^\omega$ for some p, $1 \le p \le \infty$, then*

$$\left\| f - \sigma_n^{-\alpha}(f) \right\|_p = O\Big(\omega(1/n)\, n^\alpha\Big). \tag{2}$$

T. Akhobadze [4] has proved that this estimation is non-amplifiable for $p = \infty$. In the case $p = 1$ the question of the non-amplifiability of estimation (2) was open (see [3], p. 151). A complete answer to the question is given in this paper.

2. Formulation of the Main Results

Let $\{l_k : k \ge 1\}$ be a subsequence of natural numbers such that $\overline{\lim\limits_{l\to\infty}}\, \omega(1/l)\, l^\alpha = \lim\limits_{k\to\infty} \omega(1/l_k)\, l_k^\alpha$.

Theorem 1. a) *Let $\frac{\omega(\delta)}{\delta} \uparrow \infty$ as $\delta \to 0+$ and $\alpha \in (0,1)$. Then there exists a function $f_0 \in H_\infty^\omega$ such that*

$$\overline{\lim_{k\to\infty}} \frac{\left\| f_0 - \sigma_{l_k}^{-\alpha}(f_0) \right\|_1}{\omega(1/l_k)\, l_k^\alpha} > 0.$$

b) *Let $\alpha \in (0,1)$. Then there exists a function $g_0 \in H_1^\delta$ such that*

$$\overline{\lim_{n\to\infty}} \frac{\left\| g_0 - \sigma_n^{-\alpha}(g_0) \right\|_1}{n^{\alpha-1}} > 0.$$

Theorem 2. *Let $\alpha \in (0,1)$. Then there exists a function $f \in H_1^\omega$ such that*

$$\overline{\lim_{k\to\infty}} \frac{\left\| f - \sigma_{l_k}^{-\alpha}(f) \right\|_1}{\omega(1/l_k)\, l_k^\alpha} > 0.$$

Corollary 1. *Let $\alpha \in (0,1)$. For all trigonometric Fourier series of the class H_1^ω to be L^1-summable by the Cesàro $(C,-\alpha)$-method it is necessary and sufficient that*

$$\lim_{n\to\infty} \omega(1/n)\, n^\alpha = 0.$$

Corollary 2. *Let $\alpha \in (0,1)$. Then there exists a continuous function f for which $\omega(\delta, f) = O(\delta^\alpha)$ and*

$$\overline{\lim_{n\to\infty}} \left\| f - \sigma_n^{-\alpha}(f) \right\|_1 > 0.$$

3. Proofs

Proof of Theorem 1. a) We can define the sequence $\{n_k : k\geq 1\} \subset \{l_k : k\geq 1\}$ satisfying the properties

$$\omega\left(\frac{1}{n_{k+1}}\right) \leq \frac{1}{2}\omega\left(\frac{1}{n_k}\right), \tag{3}$$

$$\sum_{j=1}^{k} \omega\left(\frac{1}{n_j}\right) n_j \leq \omega\left(\frac{1}{n_{k+1}}\right) n_{k+1}. \tag{4}$$

Consider the function f_0 defined by

$$f_0(x) = \sum_{k=1}^{\infty} \omega\left(\frac{1}{n_k}\right) \sin n_k x.$$

From (3) it is evident that $f_0 \in C$. First we shall prove that $f_0 \in H_{\infty}^{\omega}$. Let $h \in [1/n_{s+1}, 1/n_s)$. Since $\frac{\omega(\delta_1)}{\delta_1} \leq 2\frac{\omega(\delta_2)}{\delta_2}$, $0 < \delta_2 < \delta_1$, from (3) and (4) it follows that

$$|f_0(x+h) - f_0(x)| = 2\left|\sum_{k=1}^{\infty} \omega\left(\frac{1}{n_k}\right) \sin\frac{n_k h}{2} \cos\frac{2x+h}{2}\right|$$
$$\leq \sum_{k=1}^{s} \omega\left(\frac{1}{n_k}\right) n_k h + 2\sum_{k=s+1}^{\infty} \omega\left(\frac{1}{n_k}\right)$$
$$\leq 2\omega\left(\frac{1}{n_s}\right) n_s h + 4\omega\left(\frac{1}{n_{s+1}}\right) \leq 4\omega(h) + 4\omega(h) = 8\omega(h).$$

Since $b_{n_k}(f_0) = \omega\left(\frac{1}{n_k}\right)$, we obtain

$$\left\|\sigma_{n_k}^{-\alpha}(f_0) - f_0\right\|_1 \geq \left|\int_0^{2\pi} \left[\sigma_{n_k}^{-\alpha}(f_0;x) - f_0(x)\right] \sin n_k x dx\right|$$
$$\geq \left|\int_0^{2\pi} \sigma_{n_k}^{-\alpha}(f_0;x) \sin n_k x dx\right| - \pi\left|b_{n_k}(f_0)\right|$$
$$= \left|\frac{1}{A_{n_k}^{-\alpha}} \sum_{j=0}^{n_k} A_{n_{k-j}}^{-\alpha} \int_0^{2\pi} \left(a_j(f_0)\cos jx + b_j(f_0)\sin jx\right) \sin n_k x dx\right|$$
$$-\pi\left|b_{n_k}(f_0)\right| = \frac{\pi}{A_{n_k}^{-\alpha}}\left|b_{n_k}(f_0)\right| - \pi\left|b_{n_k}(f_0)\right|$$
$$= \frac{\pi}{A_{n_k}^{-\alpha}}\omega\left(\frac{1}{n_k}\right) - \pi\omega\left(\frac{1}{n_k}\right). \tag{5}$$

b) Let

$$g_0(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k}.$$

It is well-known ([1], Ch. I) that $g_0(x)$ is the function of bounded variation. According to the Hardy and Littlewood Theorem [5] we conclude that $g_0(x) \in H_1^\delta$.

Since $b_n(g_0) = \frac{1}{n}$, we write

$$\left\|\sigma_n^{-\alpha}(g_0) - g_0\right\|_1 \geq \left|\int_0^{2\pi} \left[\sigma_n^{-\alpha}(g_0; x) - g_0(x)\right] \sin nx dx\right|$$

$$\geq \left|\int_0^{2\pi} \sigma_n^{-\alpha}(g_0; x) \sin nx dx\right| - \pi \left|b_n(g_0)\right|$$

$$= \frac{\pi}{A_n^{-\alpha}} \left|b_n(g_0)\right| - \pi \left|b_n(g_0)\right| = \frac{\pi}{A_n^{-\alpha}} \frac{1}{n} - \frac{\pi}{n}. \tag{6}$$

Owing to (1), (5), and (6) the proof of Theorem 1 complete. $\square$

The validity of Theorem 2 and Corollary 2 follows immediately from Theorem 1. As for Corollary 1, it follows from Theorem A and Theorem 1.

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