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A STRONG SOLUTION OF AN EVOLUTION PROBLEM WITH INTEGRAL CONDITIONS

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Abstract. The paper is devoted to proving the existence and uniqueness of a strong solution of a mixed problem with integral boundary conditions for a certain singular parabolic equation. A functional analysis method is used. The proof is based on an energy inequality and on the density of the range of the operator generated by the studied problem.

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1. Posing of the Problem

In the domain $Q = \Omega \times (0, T)$, with $\Omega = (0, a) \times (0, b)$, where $a < \infty$, $b < \infty$ and $T < \infty$. We shall determine a solution u, in Q , of the differential equation

$$\mathcal{L}u = u_t - \frac{1}{x}(xu_x)_x - \frac{1}{x^2}u_{yy} = f(x, y, t), \quad (x, y, t) \in Q, \tag{1}$$

satisfying the initial condition

$$\ell u = u(x, y, 0) = \varphi(x, y), \quad 0 < x < a, \quad 0 < y < b, \tag{2}$$

the classical conditions

$$u(a, y, t) = 0, \quad 0 < t < T, \quad 0 < y < b, \tag{3}$$

$$u_y(x, b, t) = 0, \quad 0 < t < T, \quad 0 < x < a, \tag{4}$$

and the integral conditions

$$\int_0^a xu(x, y, t)dx = 0, \quad \int_0^b u(x, y, t)dy = 0. \tag{5}$$

For consistency, we have

$$\varphi(a, y) = 0, \quad \varphi_y(x, b) = 0,$$

$$\int_0^a x\varphi(x, y)dx = 0, \quad \int_0^b \varphi(x, y)dy = 0.$$

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Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. Cannon [5] used the potential method, combining a Dirichlet and an integral condition for an equation of the parabolic type. L. A. Mouravey and V. Philinovoski [12] used the maximum principal, combining a Neumann and an integral condition for the heat equation. Ionkin [10] used the Fourier method for the same purpose.

Mixed problems for one-dimensional second order parabolic equations, for which a local and an integral condition are combined, can be found in the papers by Cannon, Estiva, and van der Hoeck [6], Cannon and Van der hoeck [7]–[8], Kamynin [11], Yurchuk [16], Bouziani [2], Peter Shi [15], Mesloub and Bouziani [13]. Problems with purely integral conditions are studied by Bouziani [3], and Benouar and Bouziani [4], Mesloub and Bouziani [14]. In this paper, we prove the existence and uniqueness of a strong solution for the problem (1)–(5). The result and the method used here are a further elaboration of those from the paper by Benouar and Yurchuk [1].

We introduce appropriate function spaces. Let $L^2(Q)$ be the Hilbert space of square integrable functions having the norm and scalar product denoted respectively by $\|\cdot\|_{L^2(Q)}$ and $(\cdot,\cdot)_{L^2(Q)}$. Let $V^{1,0}(Q)$ be a subspace of $L^2(Q)$ with the finite norm

$$\|u\|^2_{V^{1,0}(Q)} = \|u\|^2_{L^2(Q)} + \|u_x\|^2_{L^2(Q)}\,,$$

having the scalar product defined by

$$(u,v)_{V^{1,0}(Q)} = (u,v)_{L^2(Q)} + (u_x,v_x)_{L^2(Q)}.$$

In general, a function in the space $V^{k,m}(Q)$, with k,m nonnegative integers, possesses x-derivatives up to kth order in $L^2(Q)$, and t-derivatives up to mth order in $L^2(Q)$. To problem (1)–(5) we associate the operator $L = (\mathcal{L}, \ell)$ with the domain of definition

$$D(L) = \left\{u \in L^2(Q) |\ u_t,\ u_x,\ u_y,\ u_{xx},\ u_{yy},\ u_{xt} \in L^2(Q)\right\}$$

satisfying (3)–(5). The operator L is considered from E to F, where E is the Banach space consisting of functions $u \in L^2(Q)$ satisfying the boundary conditions (3)–(5) and having the finite norm

$$\begin{aligned}\|u\|^2_E = &\int\limits_Q \left(x^3(\Im_y u_t)^2 + x(\Im_{xy}(\xi u_t))^2 + x^3u_x^2 + xu_y^2\right) dxdydt \\ &+ \sup_{0\le\tau\le T}\int\limits_\Omega \left(\left(x+x^3\right)u^2(\cdot,\cdot,\tau) + x^3(\Im_y u_x(\cdot,\cdot,\tau))^2\right) dxdy,\end{aligned}$$

where $\Im_x u = \int\limits_0^x u(\xi,y,t)d\xi$, $\Im_y u = \int\limits_0^y u(x,\eta,t)d\eta$, $\Im_{xy} u = \Im_x(\Im_y u)$ (below we will use also the notation $\Im_{yy} u = \Im_y^2 u = \Im_y(\Im_y u))$, $\Im_{xyy} u = \Im_x(\Im_{yy} u))$ and F is the Hilbert space of vector-valued functions $\mathcal{F} = (f,\varphi)$ having the norm

$$\|\mathcal{F}\|^2_F = \|f\|^2_{L^2(Q)} + \|\varphi\|^2_{V^{1,0}(\Omega)}\,.$$

Remark 1.1. The weights appearing in this paper arise because of singular coefficients and for the annihilation of inconvenient terms during integration by parts.

2. A Priori Estimate and Its Consequences

Theorem 2.1. *For any function $u \in D(L)$, we have the following a priori estimate:*

$$\|u\|_E \leq c\,\|Lu\|_F\,, \tag{6}$$

where c is a positive constant independent of the solution u.

Proof. In we take the inner product in $L^2(Q^\tau)$ of equation (1) and the operator

$$Mu = -x^3\Im_y^2 u_t + 2x^2\Im_y^2 u_x + x^3\Im_{xyy}(\xi u_t) + x^3\Im_y u_y,$$

where $Q^\tau = \Omega \times (0,\tau)$, then, in light of the initial condition (2), the boundary conditions (3)–(5), and a standard integration by parts, we get

$$\begin{aligned}
&\int\limits_{Q^\tau} x^3(\Im_y u_t)^2 dxdydt + \frac{1}{2}\int\limits_{\Omega} x^3(\Im_y u_x(\cdot,\cdot,\tau))^2 dxdy \\
&\quad + \frac{1}{2}\int\limits_{\Omega} xu^2(\cdot,\cdot,\tau)dxdy + a^2\int\limits_0^b\int\limits_0^\tau(\Im_y u_x(a,y,t))^2 dtdy \\
&\quad + \int\limits_0^b\int\limits_0^\tau u^2(0,y,t)dtdy + \int\limits_{Q^\tau} x(\Im_{xy}(\xi u_t))^2 dxdydt \\
&\quad + \frac{1}{2}\int\limits_{\Omega} x^3u^2(\cdot,\cdot,\tau)dxdy + \int\limits_{Q^\tau} x^3u_x^2 dxdydt + \int\limits_{Q^\tau} xu_y^2 dxdydt \\
&= \frac{1}{2}\int\limits_{\Omega} x^3(\Im_y\varphi_x)^2 dxdy + \frac{1}{2}\int\limits_{\Omega} x^3\varphi^2 dxdy + \frac{1}{2}\int\limits_{\Omega} x\varphi^2 dxdy \\
&\quad + \int\limits_{Q^\tau} x^4\Im_y u_x\cdot\Im_y u_t dxdydt + 2\int\limits_{Q^\tau} x^2\Im_y u_x\cdot\Im_{xy}(\xi u_t)dxdydt \\
&\quad - \int\limits_{Q^\tau} xu_y\Im_{xy}(\xi u_t)dxdydt + 2\int\limits_{Q^\tau} x^2u_y\Im_y u_x dxdydt \\
&\quad - \int\limits_{Q^\tau} x^3\mathcal{L}u\Im_y^2 u_t dxdydt + 2\int\limits_{Q^\tau} x^2\mathcal{L}u\Im_y^2 u_x dxdydt \\
&\quad + \int\limits_{Q^\tau} x^3\mathcal{L}u\Im_y u_y dxdydt + \int\limits_{Q^\tau} x^3\mathcal{L}u\Im_{xyy}(\xi u_t)dxdydt.
\end{aligned} \tag{7}$$

By virtue of the elementary inequality

$$\int_0^a (\Im_x u)^2 dx \leq \frac{a^2}{2} \int_0^a u^2 dx \tag{8}$$

(see [3]) and the Cauchy's ε-inequality

$$\alpha\beta \leq \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2, \tag{9}$$

we can estimate the terms on the right-hand side of (7) as follows:

$$\begin{aligned} \int_{Q^\tau} x^4 \Im_y u_x \cdot \Im_y u_t dxdydt \leq \frac{\varepsilon_1 a^2}{2} \int_{Q^\tau} x^3 (\Im_y u_x)^2 dxdydt \\ + \frac{1}{2\varepsilon_1} \int_{Q^\tau} x^3 (\Im_y u_t)^2 dxdydt, \end{aligned} \tag{10}$$

$$\begin{aligned} 2\int_{Q^\tau} x^2 \Im_y u_x \cdot \Im_{xy}(\xi u_t) dxdydt \leq \varepsilon_2 \int_{Q^\tau} x^3 (\Im_y u_x)^2 dxdydt \\ + \frac{1}{\varepsilon_2} \int_{Q^\tau} x (\Im_{xy}(\xi u_t)^2 dxdydt, \end{aligned} \tag{11}$$

$$\begin{aligned} -\int_{Q^\tau} x u_y \Im_{xy}(\xi u_t) dxdydt \leq \frac{\varepsilon_3}{2} \int_{Q^\tau} x {u_y}^2 dxdydt \\ + \frac{1}{2\varepsilon_3} \int_{Q^\tau} x (\Im_{xy}(\xi u_t))^2 dxdydt, \end{aligned} \tag{12}$$

$$\begin{aligned} 2\int_{Q^\tau} x^2 u_y \Im_y u_x dxdydt \leq \varepsilon_4 \int_{Q^\tau} x {u_y}^2 dxdydt \\ + \frac{1}{\varepsilon_4} \int_{Q^\tau} x^3 (\Im_y u_x)^2 dxdydt, \end{aligned} \tag{13}$$

$$\begin{aligned} -\int_{Q^\tau} x^3 \mathcal{L}u \Im_y^2 u_t dxdydt \leq \frac{a^3}{2\varepsilon_5} \int_{Q^\tau} f^2 dxdydt \\ + \frac{\varepsilon_5 b^2}{4} \int_{Q^\tau} x^3 (\Im_y u_t)^2 dxdydt, \end{aligned} \tag{14}$$

$$\begin{aligned} 2\int_{Q^\tau} x^2 \mathcal{L}u \Im_y^2 u_x dxdydt \leq \frac{a}{\varepsilon_6} \int_{Q^\tau} f^2 dxdydt \\ + \frac{\varepsilon_6 b^2}{2} \int_{Q^\tau} x^3 (\Im_y u_x)^2 dxdydt, \end{aligned} \tag{15}$$

$$\int\limits_{Q^\tau} x^3 \mathcal{L}u \Im_y u_y dxdydt \le \frac{a^5}{2\varepsilon_7}\int\limits_{Q^\tau} f^2 dxdydt + \frac{\varepsilon_7 b^2}{4}\int\limits_{Q^\tau} x u_y^2 dxdydt, \tag{16}$$

$$\int\limits_{Q^\tau} x^3 \mathcal{L}u \Im_{xyy}(\xi u_t) dxdydt \le \frac{a^5}{2\varepsilon_8}\int\limits_{Q^\tau} f^2 dxdydt + \frac{\varepsilon_8 b^2}{4}\int\limits_{Q^\tau} x(\Im_{xy}(\xi u_t))^2 dxdydt, \tag{17}$$

$$\frac{1}{2}\int\limits_{\Omega} x^3(\Im_y \varphi_x)^2 dxdy \le \frac{a^3b^2}{4}\int\limits_{\Omega}\varphi_x^2 dxdy, \tag{18}$$

then taking $\varepsilon_1 = 2$, $\varepsilon_2 = 8$, $\varepsilon_3 = 1$, $\varepsilon_4 = \frac{1}{8}$, $\varepsilon_5 = \frac{1}{b^2}$, $\varepsilon_6 = \frac{1}{b^2}$, $\varepsilon_7 = \frac{1}{2b^2}$, $\varepsilon_8 = \frac{1}{2b^2}$.

Substituting (10)–(18) into (7), and taking into account that the fourth and fifth terms in (7) are positive, we get

$$\begin{aligned}
&\int\limits_{Q^\tau}\Big(x^3(\Im_y u_t)^2 + x(\Im_{xy}(\xi u_t))^2 + x^3u_x^2 + xu_y^2\Big)dxdydt \\
&\quad + \int\limits_{\Omega}\Big(x^3(\Im_y u_x(\cdot,\cdot,\tau))^2 + (x+x^3)u^2(\cdot,\cdot,\tau)\Big)dxdy \\
&\le k\left\{\int\limits_{Q^\tau} x^3(\Im_y u_x)^2 dxdydt + \int\limits_{Q^\tau} f^2 dxdydt + \int\limits_{\Omega}(\varphi^2+\varphi_x^2)dxdy\right\},
\end{aligned} \tag{19}$$

where

$$k = \max(2a^3+2a,\ 8a^5b^2+4ab^2+2a^3b^2,\ 4a^2+66).$$

We now conclude from (19) and Gronwall's lemma that

$$\begin{aligned}
&\int\limits_{Q^\tau}\Big(x^3(\Im_y u_t)^2 + x(\Im_{xy}(\xi u_t))^2 + x^3u_x^2 + xu_y^2\Big)dxdydt \\
&\quad + \int\limits_{\Omega}\Big(x^3(\Im_y u_x(\cdot,\cdot,\tau))^2 + (x+x^3)u^2(\cdot,\cdot,\tau)\Big)dxdy \\
&\quad\quad \le ke^{kT}\left\{\int\limits_{Q^\tau} f^2 dxdydt + \int\limits_{\Omega}(\varphi^2+\varphi_x^2)dxdy\right\}.
\end{aligned} \tag{20}$$

Since the right-hand side of (20) does not depend on τ, we take the least upper bound in its left-hand side with respect to τ from 0 to T, thus obtaining (6), where $c = \sqrt{ke^{kT}}$. $\square$

It can be proved in a standard way that the operator $L : E \to F$ is closable. Let $\overline{L}$ be the closure of this operator, with the domain of definition $D(\overline{L})$.

Definition 2.1. A solution of the operator equation

$$\overline{L}u = \mathcal{F}$$

is called a strong solution of problem (1)–(5).

The a priori estimate (6) can be extended to strong solutions, i.e., we have the estimate

$$\|u\|_E \leq c \left\|\overline{L}u\right\|_F, \quad \forall u \in D(\overline{L}). \tag{21}$$

Inequality (21) implies the following corollaries.

Corollary 2.1. *A strong solution of* (1)–(5) *is unique and depends continuously on* $\mathcal{F} = (f, \varphi)$.

Corollary 2.2. *The range* $R(\overline{L})$ *of* $\overline{L}$ *is closed in* F *and* $\overline{R(L)} = R(\overline{L})$.

The latter corollary shows that to prove that problem (1)–(5) has a strong solution for arbitrary $\mathcal{F} = (f, \varphi)$, it suffices to prove that the set $R(L)$ is dense in F.

3. Solvability of the Problem

Theorem 3.1. *If, for some function* $\omega \in L^2(Q)$ *and for all elements* $u \in D_0(L) = \{u | u \in D(L) : \ell u = 0\}$, *we have*

$$\int_Q \mathcal{L}u \cdot \omega dx dy dt = 0, \tag{22}$$

then ω *vanishes almost everywhere in* Q.

Proof. Using the fact that relation (22) holds for any function $u \in D_0(L)$, we can express it in a special form. First define the function h by the relation

$$\begin{aligned} h(x,y,t) + \int_t^T \left(x^6 \Im_{xyy}(\xi u_\tau) - 10x^3 \Im_y^2 u_\tau + 20x^2 \Im_y^2 u_x \right) d\tau \\ = \int_t^T \omega d\tau. \end{aligned} \tag{23}$$

Let u_t be a solution of the equation

$$-x^5 \Im_y^2 u_t = h, \tag{24}$$

and let the function u to be given by

$$u = \begin{cases} 0, & 0 \leq t \leq s, \\ \int_s^t u_\tau d\tau, & s \leq t \leq T. \end{cases} \tag{25}$$

From the above relations we have

$$\omega(x,y,t) = x^5\Im_y^2 u_{tt} + x^6\Im_{xyy}(\xi u_t) - 10x^3\Im_y^2 u_t + 20x^2\Im_y^2 u_x. \qquad \square \tag{26}$$

Lemma 3.2. *The function ω represented by* (26) *belongs to* $L^2(Q)$.

Proof. Using a Poincaré type inequality of form (8), we easily prove that the last three terms of (26) are in $L^2(Q)$. To show that the term $x^5\Im_y^2 u_{tt}$ is in $L^2(Q)$, we use t-averaging operators ρ_ε of the form

$$(\rho_\varepsilon g)(x,t) = \frac{1}{\varepsilon}\int_0^T w\left(\frac{\nu - t}{\varepsilon}\right) g(x,t)d\nu,$$

where $w \in C_0^\infty(0,T)$, $w \geq 0$, $\int\limits_{-\infty}^{+\infty} w(t)dt = 1$.

Applying the operators ρ_ε and $\partial/\partial t$ to equation (24), and then estimating, we obtain

$$\begin{aligned}\int_Q \left(x^5\frac{\partial}{\partial t}\rho_\varepsilon\Im_y^2 u_t\right)^2 dxdydt \leq 2\int_Q \left(\frac{\partial}{\partial t}\rho_\varepsilon h\right)^2 dxdydt \\ + 2\int_Q \left[\frac{\partial}{\partial t}\left(\rho_\varepsilon x^5\Im_y^2 u_t - x^5\rho_\varepsilon\Im_y^2 u_t\right)\right]^2 dxdydt.\end{aligned}$$

Using the properties of ρ_ε introduced in [9], it follows that

$$\int_Q \left(x^5\frac{\partial}{\partial t}\rho_\varepsilon\Im_y^2 u_t\right)^2 dxdydt \leq 2\int_Q \left(\frac{\partial}{\partial t}\rho_\varepsilon h\right)^2 dxdydt.$$

Since $\rho_\varepsilon g \underset{\varepsilon\to 0}{\longrightarrow} g$ in $L^2(Q)$, and $\int\limits_Q \left(x^5\frac{\partial}{\partial t}\rho_\varepsilon\Im_y^2 u_t\right)^2 dxdydt$ is bounded, we conclude that $\omega \in L^2(Q)$. $\square$

We now return to the proof of Theorem 3.1. Replacing ω in relation (22) by its representation (26), invoking the special form of u given by (24) and (25) and the boundary conditions (3)–(5), and then carrying out appropriate integrations by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega x^5(\Im_y u_t(\cdot,\cdot,s))^2 dxdy + \frac{3}{2}\int_\Omega x^2(\Im_x(\xi u(\cdot,\cdot,T)))^2 dxdy \\ &+ 5\int_\Omega x^3(\Im_y u_x(\cdot,\cdot,T))^2 dxdy + 5\int_\Omega xu^2(\cdot,\cdot,T)dxdy \\ &+ \int_{Q_s} x^5(\Im_y u_{tx})^2 dxdydt + 2\int_{Q_s} x^3(\Im_y u_t)^2 dxdydt \\ &+ \frac{5}{2}\int_{Q_s} x^4(\Im_{xy}(\xi u_t))^2 dxdydt + \int_{Q_s} x^3 u_t^2 dxdydt\end{aligned}$$

$$
\begin{aligned}
&+10a^2\int_s^T\int_0^b(\Im_y u_x(a,y,t))^2dydt+10\int_s^T\int_0^b u^2(0,y,t)dydt\\
&=-\int_{Q_s}x^7\Im_y u\Im_y u_{tx}dxdydt-25\int_{Q_s}x^4\Im_y u\Im_{xy}(\xi u_t)dxdydt\\
&-\int_{Q_s}x^4u_t\Im_x(\xi u)dxdydt-12\int_{Q_s}x^6\Im_y u\Im_y u_t dxdydt, \qquad (27)
\end{aligned}
$$

where $Q_s=\Omega\times[s,T]$.

We now estimate each term on the right-hand side of (27) by using inequalities (8) and (9) and, taking into account that the last two terms on the left-hand side are positive, we get

$$
\begin{aligned}
&\int_\Omega x^5(\Im_y u_t(\cdot,\cdot,s))^2dxdy+\int_\Omega x^2(\Im_x(\xi u(\cdot,\cdot,T)))^2dxdy\\
&+\int_\Omega x^3(\Im_y u_x(\cdot,\cdot,T))^2dxdy+\int_\Omega xu^2(\cdot,\cdot,T)dxdy\\
&+\int_{Q_s}x^5(\Im_y u_{tx})^2dxdydt+\int_{Q_s}x^3(\Im_y u_t)^2dxdydt\\
&+\int_{Q_s}x^4(\Im_{xy}(\xi u_t))^2dxdydt+\int_{Q_s}x^3u_t^2dxdydt\\
&\le c\left\{\int_{Q_s}x^3(\Im_y u_x)^2dxdydt+\int_{Q_s}x^5(\Im_y u_t)^2dxdydt\right.\\
&\left.+\int_{Q_s}x^2(\Im_x(\xi u))^2dxdydt+\int_{Q_s}xu^2dxdydt\right\}, \qquad (28)
\end{aligned}
$$

where

$$c=\max\left\{12+12a^2+a^4,12a^4+a^6,a^3,\frac{5^4a^3b^2}{8}\right\}.$$

To use the essential inequality (28), we note that the constant c is independent of s. However, the function u in (28) does depend on s. To avoid this difficulty, we introduce a new function by the formula

$$\eta(x,y,t)=\int_t^T u_\tau(x,y,\tau)d\tau.$$

Then

$$u(x,y,t)=\eta(x,y,s)-\eta(x,y,t)$$

and we have

$$\int\limits_{Q_s} \left(x^5(\Im_y u_{tx})^2 + x^3(\Im_y u_t)^2 + x^4(\Im_{xy}(\xi u_t))^2 + x^3 u_t^2\right) dxdydt$$

$$+ \int\limits_{\Omega} x^5(\Im_y u_t(\cdot,\cdot,s))^2 dxdy + (1-2c(T-s)) \left\{ \int\limits_{\Omega} x\eta^2(\cdot,\cdot,s)dxdy \right.$$

$$\left. + \int\limits_{\Omega} x^2(\Im_x(\xi\eta(\cdot,\cdot,s)))^2 dxdy + \int\limits_{\Omega} x^3(\Im_y \eta_x(\cdot,\cdot,s))^2 dxdy \right\}$$

$$\leq 2c \left\{ \int\limits_{Q_s} \left(x^5(\Im_y \eta_t)^2 + x^3(\Im_y \eta_x)^2 + x^2(\Im_x(\xi\eta))^2 + x\eta^2\right) dxdydt \right\}. \tag{29}$$

If we choose $s_0 > 0$ such that $1 - 2c(T - s_0) = 1/2$, then (29) implies

$$\int\limits_{Q_s} \left(x^5(\Im_y u_{tx})^2 + x^3(\Im_y u_t)^2 + x^4(\Im_{xy}(\xi u_t))^2 + x^3 u_t^2\right) dxdydt$$

$$+ \left(\int\limits_{\Omega} x^5(\Im_y u_t(\cdot,\cdot,s))^2 + \int\limits_{\Omega} x\eta^2(\cdot,\cdot,s) \right.$$

$$\left. + \int\limits_{\Omega} x^2(\Im_x(\xi\eta(\cdot,\cdot,s))^2 + \int\limits_{\Omega} x^3(\Im_y \eta_x(\cdot,\cdot,s))^2 \right) dxdy$$

$$\leq 4c \left\{ \int\limits_{Q_s} \left(x^5(\Im_y \eta_t(x,y,t))^2 + x^3(\Im_y \eta_x(x,y,t))^2 \right. \right.$$

$$\left. \left. + x^2(\Im_x(\xi\eta(x,y,t)))^2 + x\eta^2(x,y,t)\right) dxdydt \right\} \tag{30}$$

for all $s \in [T - s_0, T]$.

If we denote the sum of the four integral terms on the right-hand side of (30) by $\alpha(s)$, we obtain

$$\int\limits_{Q_s} \left(x^5(\Im_y u_{tx})^2 + x^3(\Im_y u_t)^2\right) dxdydt$$

$$+ \int\limits_{Q_s} \left(x^4(\Im_{xy}(\xi u_t))^2 + x^3 u_t^2\right) dxdydt - \frac{d\alpha(s)}{ds}$$

$$\leq 4c\alpha(s).$$

Consequently,

$$-\frac{d}{ds}\left(\alpha(s)e^{4cs}\right) \leq 0. \tag{31}$$

Taking into account that $\alpha(T)=0$, (31) gives

$$\alpha(s)e^{4cs}\le 0. \tag{32}$$

It follows from (32) that $\omega=0$ almost everywhere in $Q_{T-s_0}=\Omega\times[T-s_0,T]$. Proceeding in this way step by step along the cylinders of heigth s_0, we prove that $\omega=0$ almost everywhere in Q. This completes the proof of Theorem 3.1. □

Now to conclude, we have to prove

Theorem 3.3. *The range of the operator L coincides with F.*

Proof. Since F is a Hilbert space, $R(L)=F$ is equivalent to the orthogonality of the vector $W=(\omega,\omega_0)\in F$ to the set $R(L)$, i.e., if and only if the relation

$$\int_Q \mathcal{L}u.\omega dxdydt+\int_\Omega\Big(\ell u\cdot\omega_0+\frac{d\ell u}{dx}\Big)dxdy=0, \tag{33}$$

where u runs over E and $W=(\omega,\omega_0)\in F$, implies that $W=0$.

Putting $u\in D(L_0)$ in (33), we get

$$\int_Q \mathcal{L}u\cdot\omega dxdydt=0.$$

Hence Theorem 3.1 implies that $\omega=0$. Thus (33) becomes

$$\int_\Omega\Big(\ell u\cdot\omega_0+\frac{d\ell u}{dx}\Big)dxdy=0,\quad \forall u\in D(L). \tag{34}$$

Since the range $R(\ell)$ of the trace operator ℓ is everywhere dense in $V^{1,0}(\Omega)$, then it follows from (34) that $\omega_0=0$. Hence $W=0$. This completes the proof of Theorem 3.3. □

References

1. N. E. Benouar and N. I. Yurchuk, Mixed problem with an integral condition for parabolic equations with the Bessel operator. (Russian) *Differentsial'nye Uravneniya* **27**(1991), No. 12, 2094–2098.
2. A. Bouziani, Solution forte d'un problème mixte avec conditions non locales pour une classe d'équations paraboliques. *Maghreb Math. Rev.* **6**(1997), No. 1, 1–17.
3. A. Bouziani, Mixed problem with boundary integral conditions for a certain parabolic equation. *J. Appl. Math. Stochastic Anal.* **9**(1996), No. 3, 323–330.
4. A. Bouziani and N. E. Benouar, Probleme mixte avec conditions intégrales pour une classe d'équations paraboliques. *C. R. Acad. Sci. Paris Sér. 1 Math.* **321**(1995), 1177–1182.
5. J. R. Cannon, The solution of heat equation subject to the specification of energy. *Quart. Appl. Math.* **21**(1963), No. 2, 155–160.

6. J. R. Cannon, S. P. Esteve, and J. Van der Hoeck, A galerkin procedure for the diffusion equation subject to the specification of mass. *SIAM J. Numer. Anal.* **24**(1987), 499–515.
7. J. R. Cannon and J. Van der Hoeck, The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy. *Boll. Un. Mat. Ital. Suppl.* **1**(1981), 253–282.
8. J. R. Cannon and J. Van der Hoeck, an implicit finite difference scheme for the diffusion of mass in a portion of the domain. *Numerical solutions of partial differential equations* (*J. Noye, ed.*), 527–539, *North-Holland, Amsterdam,* 1982.
9. L. Garding, Cauchy's problem for hyperbolic equation. *Mimeogr. Lecture Notes, University of Chicago, Chicago,* 1958.
10. N. I. Ionkin, Solution of boundary value problem in heat conduction theory with non local boundary conditions. (Russian) *Differentsial'nye Uravneniya* **13**(1977), 294–304.
11. N. I. Kamynin, A boundary value problem in the theory of heat conduction with non classical boundary condition. *Zh. Vychisl. Mat. Mat. Fiz.* **43**(1964), No. 6, 1006–1024.
12. L. A. Mouravey and V. Philipovski, Sur un problème avec une condition aux limites nonlocale pour une equation parabolique. (Russian) *Mat. Sb.* **182**(1991), No. 10, 1479–1512.
13. S. Mesloub and A. Bouziani, Mixed problem with a weighted integral condition for a parabolic equation with Bessel opeartor. *J. Appl. Math. Stochastic. Anal.* (to appear).
14. S. Mesloub and A. Bouziani, Problème mixte avec conditions aux limites intégrales pour une classe d'équations paraboliques bidimensionnelles. *Acad. Roy. Belg. Bull. Cl. Sci.* (6) **9**(1998), No. 1–6, 61–72.
15. P. Shi, Weak solution to an evolution problem with a non local constraint. *SIAM J. Math. Anal.* **24**(1993), No. 1, 46–58.
16. N. I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations. (Russian) *Differentsial'nye Uravneniya* **22**(1986), No. 19, 2117–2126.

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