

A STRONG SOLUTION OF AN EVOLUTION PROBLEM WITH INTEGRAL CONDITIONS

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Abstract. The paper is devoted to proving the existence and uniqueness of a strong solution of a mixed problem with integral boundary conditions for a certain singular parabolic equation. A functional analysis method is used. The proof is based on an energy inequality and on the density of the range of the operator generated by the studied problem.

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1. POSING OF THE PROBLEM

In the domain $Q = \Omega \times (0, T)$, with $\Omega = (0, a) \times (0, b)$, where $a < \infty$, $b < \infty$ and $T < \infty$. We shall determine a solution u , in Q , of the differential equation

$$\mathcal{L}u = u_t - \frac{1}{x}(xu_x)_x - \frac{1}{x^2}u_{yy} = f(x, y, t), \quad (x, y, t) \in Q, \quad (1)$$

satisfying the initial condition

$$\ell u = u(x, y, 0) = \varphi(x, y), \quad 0 < x < a, \quad 0 < y < b, \quad (2)$$

the classical conditions

$$u(a, y, t) = 0, \quad 0 < t < T, \quad 0 < y < b, \quad (3)$$

$$u_y(x, b, t) = 0, \quad 0 < t < T, \quad 0 < x < a, \quad (4)$$

and the integral conditions

$$\int_0^a xu(x, y, t)dx = 0, \quad \int_0^b u(x, y, t)dy = 0. \quad (5)$$

For consistency, we have

$$\begin{aligned} \varphi(a, y) &= 0, & \varphi_y(x, b) &= 0, \\ \int_0^a x\varphi(x, y)dx &= 0, & \int_0^b \varphi(x, y)dy &= 0. \end{aligned}$$

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. Cannon [5] used the potential method, combining a Dirichlet and an integral condition for an equation of the parabolic type. L. A. Mouravey and V. Philinovsky [12] used the maximum principal, combining a Neumann and an integral condition for the heat equation. Ionkin [10] used the Fourier method for the same purpose.

Mixed problems for one-dimensional second order parabolic equations, for which a local and an integral condition are combined, can be found in the papers by Cannon, Estiva, and van der Hoeck [6], Cannon and Van der hoeck [7]–[8], Kamynin [11], Yurchuk [16], Bouziani [2], Peter Shi [15], Mesloub and Bouziani [13]. Problems with purely integral conditions are studied by Bouziani [3], and Benouar and Bouziani [4], Mesloub and Bouziani [14]. In this paper, we prove the existence and uniqueness of a strong solution for the problem (1)–(5). The result and the method used here are a further elaboration of those from the paper by Benouar and Yurchuk [1].

We introduce appropriate function spaces. Let $L^2(Q)$ be the Hilbert space of square integrable functions having the norm and scalar product denoted respectively by $\|\cdot\|_{L^2(Q)}$ and $(\cdot, \cdot)_{L^2(Q)}$. Let $V^{1,0}(Q)$ be a subspace of $L^2(Q)$ with the finite norm

$$\|u\|_{V^{1,0}(Q)}^2 = \|u\|_{L^2(Q)}^2 + \|u_x\|_{L^2(Q)}^2,$$

having the scalar product defined by

$$(u, v)_{V^{1,0}(Q)} = (u, v)_{L^2(Q)} + (u_x, v_x)_{L^2(Q)}.$$

In general, a function in the space $V^{k,m}(Q)$, with k, m nonnegative integers, possesses x -derivatives up to k th order in $L^2(Q)$, and t -derivatives up to m th order in $L^2(Q)$. To problem (1)–(5) we associate the operator $L = (\mathcal{L}, \ell)$ with the domain of definition

$$D(L) = \left\{ u \in L^2(Q) \mid u_t, u_x, u_y, u_{xx}, u_{yy}, u_{xt} \in L^2(Q) \right\}$$

satisfying (3)–(5). The operator L is considered from E to F , where E is the Banach space consisting of functions $u \in L^2(Q)$ satisfying the boundary conditions (3)–(5) and having the finite norm

$$\begin{aligned} \|u\|_E^2 &= \int_Q \left(x^3 (\mathfrak{S}_y u_t)^2 + x (\mathfrak{S}_{xy} (\xi u_t))^2 + x^3 u_x^2 + x u_y^2 \right) dx dy dt \\ &+ \sup_{0 \leq \tau \leq T} \int_{\Omega} \left((x + x^3) u^2(\cdot, \cdot, \tau) + x^3 (\mathfrak{S}_y u_x(\cdot, \cdot, \tau))^2 \right) dx dy, \end{aligned}$$

where $\mathfrak{S}_x u = \int_0^x u(\xi, y, t) d\xi$, $\mathfrak{S}_y u = \int_0^y u(x, \eta, t) d\eta$, $\mathfrak{S}_{xy} u = \mathfrak{S}_x(\mathfrak{S}_y u)$ (below we will use also the notation $\mathfrak{S}_{yy} u = \mathfrak{S}_y^2 u = \mathfrak{S}_y(\mathfrak{S}_y u)$, $\mathfrak{S}_{xyy} u = \mathfrak{S}_x(\mathfrak{S}_{yy} u)$) and F is the Hilbert space of vector-valued functions $\mathcal{F} = (f, \varphi)$ having the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 + \|\varphi\|_{V^{1,0}(\Omega)}^2.$$

Remark 1.1. The weights appearing in this paper arise because of singular coefficients and for the annihilation of inconvenient terms during integration by parts.

2. A PRIORI ESTIMATE AND ITS CONSEQUENCES

Theorem 2.1. *For any function $u \in D(L)$, we have the following a priori estimate:*

$$\|u\|_E \leq c \|Lu\|_F, \quad (6)$$

where c is a positive constant independent of the solution u .

Proof. In we take the inner product in $L^2(Q^\tau)$ of equation (1) and the operator

$$Mu = -x^3 \mathfrak{S}_y^2 u_t + 2x^2 \mathfrak{S}_y^2 u_x + x^3 \mathfrak{S}_{xyy}(\xi u_t) + x^3 \mathfrak{S}_y u_y,$$

where $Q^\tau = \Omega \times (0, \tau)$, then, in light of the initial condition (2), the boundary conditions (3)–(5), and a standard integration by parts, we get

$$\begin{aligned} & \int_{Q^\tau} x^3 (\mathfrak{S}_y u_t)^2 dx dy dt + \frac{1}{2} \int_{\Omega} x^3 (\mathfrak{S}_y u_x(\cdot, \cdot, \tau))^2 dx dy \\ & + \frac{1}{2} \int_{\Omega} x u^2(\cdot, \cdot, \tau) dx dy + a^2 \int_0^b \int_0^\tau (\mathfrak{S}_y u_x(a, y, t))^2 dt dy \\ & + \int_0^b \int_0^\tau u^2(0, y, t) dt dy + \int_{Q^\tau} x (\mathfrak{S}_{xy}(\xi u_t))^2 dx dy dt \\ & + \frac{1}{2} \int_{\Omega} x^3 u^2(\cdot, \cdot, \tau) dx dy + \int_{Q^\tau} x^3 u_x^2 dx dy dt + \int_{Q^\tau} x u_y^2 dx dy dt \\ & = \frac{1}{2} \int_{\Omega} x^3 (\mathfrak{S}_y \varphi_x)^2 dx dy + \frac{1}{2} \int_{\Omega} x^3 \varphi^2 dx dy + \frac{1}{2} \int_{\Omega} x \varphi^2 dx dy \\ & + \int_{Q^\tau} x^4 \mathfrak{S}_y u_x \cdot \mathfrak{S}_y u_t dx dy dt + 2 \int_{Q^\tau} x^2 \mathfrak{S}_y u_x \cdot \mathfrak{S}_{xy}(\xi u_t) dx dy dt \\ & - \int_{Q^\tau} x u_y \mathfrak{S}_{xy}(\xi u_t) dx dy dt + 2 \int_{Q^\tau} x^2 u_y \mathfrak{S}_y u_x dx dy dt \\ & - \int_{Q^\tau} x^3 \mathcal{L} u \mathfrak{S}_y^2 u_t dx dy dt + 2 \int_{Q^\tau} x^2 \mathcal{L} u \mathfrak{S}_y^2 u_x dx dy dt \\ & + \int_{Q^\tau} x^3 \mathcal{L} u \mathfrak{S}_y u_y dx dy dt + \int_{Q^\tau} x^3 \mathcal{L} u \mathfrak{S}_{xyy}(\xi u_t) dx dy dt. \end{aligned} \quad (7)$$

By virtue of the elementary inequality

$$\int_0^a (\mathfrak{S}_x u)^2 dx \leq \frac{a^2}{2} \int_0^a u^2 dx \quad (8)$$

(see [3]) and the Cauchy's ε -inequality

$$\alpha\beta \leq \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2, \quad (9)$$

we can estimate the terms on the right-hand side of (7) as follows:

$$\begin{aligned} \int_{Q^\tau} x^4 \mathfrak{S}_y u_x \cdot \mathfrak{S}_y u_t dx dy dt &\leq \frac{\varepsilon_1 a^2}{2} \int_{Q^\tau} x^3 (\mathfrak{S}_y u_x)^2 dx dy dt \\ &\quad + \frac{1}{2\varepsilon_1} \int_{Q^\tau} x^3 (\mathfrak{S}_y u_t)^2 dx dy dt, \end{aligned} \quad (10)$$

$$\begin{aligned} 2 \int_{Q^\tau} x^2 \mathfrak{S}_y u_x \cdot \mathfrak{S}_{xy}(\xi u_t) dx dy dt &\leq \varepsilon_2 \int_{Q^\tau} x^3 (\mathfrak{S}_y u_x)^2 dx dy dt \\ &\quad + \frac{1}{\varepsilon_2} \int_{Q^\tau} x (\mathfrak{S}_{xy}(\xi u_t))^2 dx dy dt, \end{aligned} \quad (11)$$

$$\begin{aligned} - \int_{Q^\tau} x u_y \mathfrak{S}_{xy}(\xi u_t) dx dy dt &\leq \frac{\varepsilon_3}{2} \int_{Q^\tau} x u_y^2 dx dy dt \\ &\quad + \frac{1}{2\varepsilon_3} \int_{Q^\tau} x (\mathfrak{S}_{xy}(\xi u_t))^2 dx dy dt, \end{aligned} \quad (12)$$

$$\begin{aligned} 2 \int_{Q^\tau} x^2 u_y \mathfrak{S}_y u_x dx dy dt &\leq \varepsilon_4 \int_{Q^\tau} x u_y^2 dx dy dt \\ &\quad + \frac{1}{\varepsilon_4} \int_{Q^\tau} x^3 (\mathfrak{S}_y u_x)^2 dx dy dt, \end{aligned} \quad (13)$$

$$\begin{aligned} - \int_{Q^\tau} x^3 \mathcal{L}u \mathfrak{S}_y^2 u_t dx dy dt &\leq \frac{a^3}{2\varepsilon_5} \int_{Q^\tau} f^2 dx dy dt \\ &\quad + \frac{\varepsilon_5 b^2}{4} \int_{Q^\tau} x^3 (\mathfrak{S}_y u_t)^2 dx dy dt, \end{aligned} \quad (14)$$

$$\begin{aligned} 2 \int_{Q^\tau} x^2 \mathcal{L}u \mathfrak{S}_y^2 u_x dx dy dt &\leq \frac{a}{\varepsilon_6} \int_{Q^\tau} f^2 dx dy dt \\ &\quad + \frac{\varepsilon_6 b^2}{2} \int_{Q^\tau} x^3 (\mathfrak{S}_y u_x)^2 dx dy dt, \end{aligned} \quad (15)$$

$$\begin{aligned} \int_{Q^\tau} x^3 \mathcal{L}u \mathfrak{S}_y u_y dx dy dt &\leq \frac{a^5}{2\varepsilon_7} \int_{Q^\tau} f^2 dx dy dt \\ &+ \frac{\varepsilon_7 b^2}{4} \int_{Q^\tau} x u_y^2 dx dy dt, \end{aligned} \quad (16)$$

$$\begin{aligned} \int_{Q^\tau} x^3 \mathcal{L}u \mathfrak{S}_{xy}(\xi u_t) dx dy dt &\leq \frac{a^5}{2\varepsilon_8} \int_{Q^\tau} f^2 dx dy dt \\ &+ \frac{\varepsilon_8 b^2}{4} \int_{Q^\tau} x (\mathfrak{S}_{xy}(\xi u_t))^2 dx dy dt, \end{aligned} \quad (17)$$

$$\frac{1}{2} \int_{\Omega} x^3 (\mathfrak{S}_y \varphi_x)^2 dx dy \leq \frac{a^3 b^2}{4} \int_{\Omega} \varphi_x^2 dx dy, \quad (18)$$

then taking $\varepsilon_1 = 2$, $\varepsilon_2 = 8$, $\varepsilon_3 = 1$, $\varepsilon_4 = \frac{1}{8}$, $\varepsilon_5 = \frac{1}{b^2}$, $\varepsilon_6 = \frac{1}{b^2}$, $\varepsilon_7 = \frac{1}{2b^2}$, $\varepsilon_8 = \frac{1}{2b^2}$.

Substituting (10)–(18) into (7), and taking into account that the fourth and fifth terms in (7) are positive, we get

$$\begin{aligned} &\int_{Q^\tau} \left(x^3 (\mathfrak{S}_y u_t)^2 + x (\mathfrak{S}_{xy}(\xi u_t))^2 + x^3 u_x^2 + x u_y^2 \right) dx dy dt \\ &+ \int_{\Omega} \left(x^3 (\mathfrak{S}_y u_x(\cdot, \cdot, \tau))^2 + (x + x^3) u^2(\cdot, \cdot, \tau) \right) dx dy \\ &\leq k \left\{ \int_{Q^\tau} x^3 (\mathfrak{S}_y u_x)^2 dx dy dt + \int_{Q^\tau} f^2 dx dy dt + \int_{\Omega} (\varphi^2 + \varphi_x^2) dx dy \right\}, \end{aligned} \quad (19)$$

where

$$k = \max(2a^3 + 2a, 8a^5 b^2 + 4ab^2 + 2a^3 b^2, 4a^2 + 66).$$

We now conclude from (19) and Gronwall's lemma that

$$\begin{aligned} &\int_{Q^\tau} \left(x^3 (\mathfrak{S}_y u_t)^2 + x (\mathfrak{S}_{xy}(\xi u_t))^2 + x^3 u_x^2 + x u_y^2 \right) dx dy dt \\ &+ \int_{\Omega} \left(x^3 (\mathfrak{S}_y u_x(\cdot, \cdot, \tau))^2 + (x + x^3) u^2(\cdot, \cdot, \tau) \right) dx dy \\ &\leq k e^{kT} \left\{ \int_{Q^\tau} f^2 dx dy dt + \int_{\Omega} (\varphi^2 + \varphi_x^2) dx dy \right\}. \end{aligned} \quad (20)$$

Since the right-hand side of (20) does not depend on τ , we take the least upper bound in its left-hand side with respect to τ from 0 to T , thus obtaining (6), where $c = \sqrt{k e^{kT}}$. \square

It can be proved in a standard way that the operator $L : E \rightarrow F$ is closable. Let \bar{L} be the closure of this operator, with the domain of definition $D(\bar{L})$.

Definition 2.1. A solution of the operator equation

$$\bar{L}u = \mathcal{F}$$

is called a strong solution of problem (1)–(5).

The a priori estimate (6) can be extended to strong solutions, i.e., we have the estimate

$$\|u\|_E \leq c \|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}). \quad (21)$$

Inequality (21) implies the following corollaries.

Corollary 2.1. *A strong solution of (1)–(5) is unique and depends continuously on $\mathcal{F} = (f, \varphi)$.*

Corollary 2.2. *The range $R(\bar{L})$ of \bar{L} is closed in F and $\overline{R(\bar{L})} = R(\bar{L})$.*

The latter corollary shows that to prove that problem (1)–(5) has a strong solution for arbitrary $\mathcal{F} = (f, \varphi)$, it suffices to prove that the set $R(L)$ is dense in F .

3. SOLVABILITY OF THE PROBLEM

Theorem 3.1. *If, for some function $\omega \in L^2(Q)$ and for all elements $u \in D_0(L) = \{u | u \in D(L) : \ell u = 0\}$, we have*

$$\int_Q \mathcal{L}u \cdot \omega dx dy dt = 0, \quad (22)$$

then ω vanishes almost everywhere in Q .

Proof. Using the fact that relation (22) holds for any function $u \in D_0(L)$, we can express it in a special form. First define the function h by the relation

$$\begin{aligned} h(x, y, t) + \int_t^T \left(x^6 \mathfrak{S}_{xyy}(\xi u_\tau) - 10x^3 \mathfrak{S}_y^2 u_\tau + 20x^2 \mathfrak{S}_y^2 u_x \right) d\tau \\ = \int_t^T \omega d\tau. \end{aligned} \quad (23)$$

Let u_t be a solution of the equation

$$-x^5 \mathfrak{S}_y^2 u_t = h, \quad (24)$$

and let the function u to be given by

$$u = \begin{cases} 0, & 0 \leq t \leq s, \\ \int_s^t u_\tau d\tau, & s \leq t \leq T. \end{cases} \quad (25)$$

From the above relations we have

$$\omega(x, y, t) = x^5 \mathfrak{S}_y^2 u_{tt} + x^6 \mathfrak{S}_{xyy}(\xi u_t) - 10x^3 \mathfrak{S}_y^2 u_t + 20x^2 \mathfrak{S}_y^2 u_x. \quad \square \quad (26)$$

Lemma 3.2. *The function ω represented by (26) belongs to $L^2(Q)$.*

Proof. Using a Poincaré type inequality of form (8), we easily prove that the last three terms of (26) are in $L^2(Q)$. To show that the term $x^5 \mathfrak{S}_y^2 u_{tt}$ is in $L^2(Q)$, we use t -averaging operators ρ_ε of the form

$$(\rho_\varepsilon g)(x, t) = \frac{1}{\varepsilon} \int_0^T w\left(\frac{\nu - t}{\varepsilon}\right) g(x, t) d\nu,$$

where $w \in C_0^\infty(0, T)$, $w \geq 0$, $\int_{-\infty}^{+\infty} w(t) dt = 1$.

Applying the operators ρ_ε and $\partial/\partial t$ to equation (24), and then estimating, we obtain

$$\begin{aligned} \int_Q \left(x^5 \frac{\partial}{\partial t} \rho_\varepsilon \mathfrak{S}_y^2 u_t \right)^2 dx dy dt &\leq 2 \int_Q \left(\frac{\partial}{\partial t} \rho_\varepsilon h \right)^2 dx dy dt \\ &+ 2 \int_Q \left[\frac{\partial}{\partial t} \left(\rho_\varepsilon x^5 \mathfrak{S}_y^2 u_t - x^5 \rho_\varepsilon \mathfrak{S}_y^2 u_t \right) \right]^2 dx dy dt. \end{aligned}$$

Using the properties of ρ_ε introduced in [9], it follows that

$$\int_Q \left(x^5 \frac{\partial}{\partial t} \rho_\varepsilon \mathfrak{S}_y^2 u_t \right)^2 dx dy dt \leq 2 \int_Q \left(\frac{\partial}{\partial t} \rho_\varepsilon h \right)^2 dx dy dt.$$

Since $\rho_\varepsilon g \rightarrow g$ in $L^2(Q)$, and $\int_Q \left(x^5 \frac{\partial}{\partial t} \rho_\varepsilon \mathfrak{S}_y^2 u_t \right)^2 dx dy dt$ is bounded, we conclude that $\omega \in L^2(Q)$. \square

We now return to the proof of Theorem 3.1. Replacing ω in relation (22) by its representation (26), invoking the special form of u given by (24) and (25) and the boundary conditions (3)–(5), and then carrying out appropriate integrations by parts, we obtain

$$\begin{aligned} &\frac{1}{2} \int_\Omega x^5 (\mathfrak{S}_y u_t(\cdot, \cdot, s))^2 dx dy + \frac{3}{2} \int_\Omega x^2 (\mathfrak{S}_x(\xi u(\cdot, \cdot, T)))^2 dx dy \\ &+ 5 \int_\Omega x^3 (\mathfrak{S}_y u_x(\cdot, \cdot, T))^2 dx dy + 5 \int_\Omega x u^2(\cdot, \cdot, T) dx dy \\ &+ \int_{Q_s} x^5 (\mathfrak{S}_y u_{tx})^2 dx dy dt + 2 \int_{Q_s} x^3 (\mathfrak{S}_y u_t)^2 dx dy dt \\ &+ \frac{5}{2} \int_{Q_s} x^4 (\mathfrak{S}_{xy}(\xi u_t))^2 dx dy dt + \int_{Q_s} x^3 u_t^2 dx dy dt \end{aligned}$$

$$\begin{aligned}
& + 10a^2 \int_s^T \int_0^b (\mathfrak{S}_y u_x(a, y, t))^2 dy dt + 10 \int_s^T \int_0^b u^2(0, y, t) dy dt \\
& = - \int_{Q_s} x^7 \mathfrak{S}_y u \mathfrak{S}_y u_{tx} dx dy dt - 25 \int_{Q_s} x^4 \mathfrak{S}_y u \mathfrak{S}_{xy}(\xi u_t) dx dy dt \\
& \quad - \int_{Q_s} x^4 u_t \mathfrak{S}_x(\xi u) dx dy dt - 12 \int_{Q_s} x^6 \mathfrak{S}_y u \mathfrak{S}_y u_t dx dy dt, \tag{27}
\end{aligned}$$

where $Q_s = \Omega \times [s, T]$.

We now estimate each term on the right-hand side of (27) by using inequalities (8) and (9) and, taking into account that the last two terms on the left-hand side are positive, we get

$$\begin{aligned}
& \int_{\Omega} x^5 (\mathfrak{S}_y u_t(\cdot, \cdot, s))^2 dx dy + \int_{\Omega} x^2 (\mathfrak{S}_x(\xi u(\cdot, \cdot, T)))^2 dx dy \\
& \quad + \int_{\Omega} x^3 (\mathfrak{S}_y u_x(\cdot, \cdot, T))^2 dx dy + \int_{\Omega} x u^2(\cdot, \cdot, T) dx dy \\
& \quad + \int_{Q_s} x^5 (\mathfrak{S}_y u_{tx})^2 dx dy dt + \int_{Q_s} x^3 (\mathfrak{S}_y u_t)^2 dx dy dt \\
& \quad + \int_{Q_s} x^4 (\mathfrak{S}_{xy}(\xi u_t))^2 dx dy dt + \int_{Q_s} x^3 u_t^2 dx dy dt \\
& \leq c \left\{ \int_{Q_s} x^3 (\mathfrak{S}_y u_x)^2 dx dy dt + \int_{Q_s} x^5 (\mathfrak{S}_y u_t)^2 dx dy dt \right. \\
& \quad \left. + \int_{Q_s} x^2 (\mathfrak{S}_x(\xi u))^2 dx dy dt + \int_{Q_s} x u^2 dx dy dt \right\}, \tag{28}
\end{aligned}$$

where

$$c = \max \left\{ 12 + 12a^2 + a^4, 12a^4 + a^6, a^3, \frac{5^4 a^3 b^2}{8} \right\}.$$

To use the essential inequality (28), we note that the constant c is independent of s . However, the function u in (28) does depend on s . To avoid this difficulty, we introduce a new function by the formula

$$\eta(x, y, t) = \int_t^T u_\tau(x, y, \tau) d\tau.$$

Then

$$u(x, y, t) = \eta(x, y, s) - \eta(x, y, t)$$

and we have

$$\begin{aligned}
& \int_{Q_s} \left(x^5 (\mathfrak{S}_y u_{tx})^2 + x^3 (\mathfrak{S}_y u_t)^2 + x^4 (\mathfrak{S}_{xy} (\xi u_t))^2 + x^3 u_t^2 \right) dx dy dt \\
& + \int_{\Omega} x^5 (\mathfrak{S}_y u_t(\cdot, \cdot, s))^2 dx dy + (1 - 2c(T - s)) \left\{ \int_{\Omega} x \eta^2(\cdot, \cdot, s) dx dy \right. \\
& \quad \left. + \int_{\Omega} x^2 (\mathfrak{S}_x (\xi \eta(\cdot, \cdot, s)))^2 dx dy + \int_{\Omega} x^3 (\mathfrak{S}_y \eta_x(\cdot, \cdot, s))^2 dx dy \right\} \\
& \leq 2c \left\{ \int_{Q_s} \left(x^5 (\mathfrak{S}_y \eta_t)^2 + x^3 (\mathfrak{S}_y \eta_x)^2 + x^2 (\mathfrak{S}_x (\xi \eta))^2 + x \eta^2 \right) dx dy dt \right\}. \quad (29)
\end{aligned}$$

If we choose $s_0 > 0$ such that $1 - 2c(T - s_0) = 1/2$, then (29) implies

$$\begin{aligned}
& \int_{Q_s} \left(x^5 (\mathfrak{S}_y u_{tx})^2 + x^3 (\mathfrak{S}_y u_t)^2 + x^4 (\mathfrak{S}_{xy} (\xi u_t))^2 + x^3 u_t^2 \right) dx dy dt \\
& \quad + \left(\int_{\Omega} x^5 (\mathfrak{S}_y u_t(\cdot, \cdot, s))^2 + \int_{\Omega} x \eta^2(\cdot, \cdot, s) \right. \\
& \quad \left. + \int_{\Omega} x^2 (\mathfrak{S}_x (\xi \eta(\cdot, \cdot, s)))^2 + \int_{\Omega} x^3 (\mathfrak{S}_y \eta_x(\cdot, \cdot, s))^2 \right) dx dy \\
& \leq 4c \left\{ \int_{Q_s} \left(x^5 (\mathfrak{S}_y \eta_t(x, y, t))^2 + x^3 (\mathfrak{S}_y \eta_x(x, y, t))^2 \right. \right. \\
& \quad \left. \left. + x^2 (\mathfrak{S}_x (\xi \eta(x, y, t)))^2 + x \eta^2(x, y, t) \right) dx dy dt \right\} \quad (30)
\end{aligned}$$

for all $s \in [T - s_0, T]$.

If we denote the sum of the four integral terms on the right-hand side of (30) by $\alpha(s)$, we obtain

$$\begin{aligned}
& \int_{Q_s} \left(x^5 (\mathfrak{S}_y u_{tx})^2 + x^3 (\mathfrak{S}_y u_t)^2 \right) dx dy dt \\
& \quad + \int_{Q_s} \left(x^4 (\mathfrak{S}_{xy} (\xi u_t))^2 + x^3 u_t^2 \right) dx dy dt - \frac{d\alpha(s)}{ds} \\
& \leq 4c\alpha(s).
\end{aligned}$$

Consequently,

$$-\frac{d}{ds} \left(\alpha(s) e^{4cs} \right) \leq 0. \quad (31)$$

Taking into account that $\alpha(T) = 0$, (31) gives

$$\alpha(s)e^{4cs} \leq 0. \quad (32)$$

It follows from (32) that $\omega = 0$ almost everywhere in $Q_{T-s_0} = \Omega \times [T - s_0, T]$. Proceeding in this way step by step along the cylinders of height s_0 , we prove that $\omega = 0$ almost everywhere in Q . This completes the proof of Theorem 3.1.

□

Now to conclude, we have to prove

Theorem 3.3. *The range of the operator L coincides with F .*

Proof. Since F is a Hilbert space, $R(L) = F$ is equivalent to the orthogonality of the vector $W = (\omega, \omega_0) \in F$ to the set $R(L)$, i.e., if and only if the relation

$$\int_Q \mathcal{L}u \cdot \omega dx dy dt + \int_\Omega \left(\ell u \cdot \omega_0 + \frac{d\ell u}{dx} \right) dx dy = 0, \quad (33)$$

where u runs over E and $W = (\omega, \omega_0) \in F$, implies that $W = 0$.

Putting $u \in D(L_0)$ in (33), we get

$$\int_Q \mathcal{L}u \cdot \omega dx dy dt = 0.$$

Hence Theorem 3.1 implies that $\omega = 0$. Thus (33) becomes

$$\int_\Omega \left(\ell u \cdot \omega_0 + \frac{d\ell u}{dx} \right) dx dy = 0, \quad \forall u \in D(L). \quad (34)$$

Since the range $R(\ell)$ of the trace operator ℓ is everywhere dense in $V^{1,0}(\Omega)$, then it follows from (34) that $\omega_0 = 0$. Hence $W = 0$. This completes the proof of Theorem 3.3. □

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