Georgian Mathematical Journal
Volume 9 (2002), Number 1, 137–148

THE BOCHNER MEAN SQUARE DEVIATION AND LAW OF LARGE NUMBERS FOR SQUARES OF RANDOM ELEMENTS IN BANACH LATTICES

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Dedicated to Professor N. Vakhania
on the occasion of his 70th birthday

Abstract. A Bochner mean square deviation for random elements of 2-convex Banach lattices is introduced and investigated. Results, analogous to the law of large numbers for squares of random elements are proved in some classes of Köthe function spaces.

2000 Mathematics Subject Classification: 60B12, 60B11.
Key words and phrases: Random variables, Banach lattice, law of large numbers.

1. Introduction

The notions of mathematical expectation and mean square deviation of a real-valued random variable (r.v.) ξ defined on the probability space $(\Omega, \mathfrak{B}, \mathbf{P})$, are well known:

$$\mathbf{E}\xi = \int_\Omega \xi(\omega) d\mathbf{P}\,, \quad \sigma\xi = (\mathbf{E}|\xi - \mathbf{E}\xi|^2)^{1/2}\,.$$

Let X be a random element (r.e.) with values in a Banach lattice of functions on a set T (we consider the sequence lattice as a space of functions of a natural argument). It is naturally to introduce the "pointwise" mathematical expectation and the mean square deviation of X as the functions $\mathbf{E}X(t)$ and $(\mathbf{E}|X(t) - \mathbf{E}X(t)|^2)^{1/2}$. Besides, there are notions of mathematical expectation of a r.e. X in an abstract Banach space B in the Pettis [12] and the Bochner [4] sense.

The pointwise mean square deviation (and, more general, the pointwise p-th moment) in Banach lattices of functions was considered in [9], where conditions of relative stability, the law of large numbers (LLN) for p-th degrees and the central limit theorem (CLT) are proved. The "weak" mean square deviation and its generalizations (the Pettis integral is used) in an arbitrary Banach lattice B was considered in [7]. In particular, in [7] the LLN for p-th degrees was proved.

To investigate some problems of probability theory it is necessary to introduce the notion of Bochner mean square deviation of a Banach-valued r.e. In Section

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2 we introduce and investigate this notion for some classes of Banach lattices. The proposed notion of the mean square deviation, which is similar to the Bochner mathematical expectation, is suitable for the so-called 2-convex Banach lattices (see Definition 2), in particular for L_p , $2 \le p < \infty$, and is not suitable for L_p , $1 \le p < 2$. It is analogous to the fact that the Bochner mathematical expectation is suitable for Banach spaces and is not suitable for many linear metric spaces, in particular for L_p , $0 < p < 1$. Section 3 is devoted to the law of large numbers for squares of a r.e. in some function Banach lattices. On the real line the LLN for squares is equivalent to the central limit theorem. This is a classical result. We show that a similar relation holds true for some Banach lattices too (see also [3] and [11]). We shall extensively use the definitions, notations and results of [6] concerning Banach lattices.

2. Bochner Mean Square Deviation

Let us recall the definition of the Bochner mathematical expectation (Bochner integral) in a Banach space B. A B-valued r.e. X (i.e., a strongly measurable mapping) defined on a probability space $(\Omega, \mathfrak{B}, \mathbf{P})$ is called *simple* if it takes only finite number of values. A r.e. X is said to have the *Bochner mathematical expectation* if there exists a sequence of simple elements X_n such that $\mathbf{E}\|X_n - X\| \to 0$ as $n \to \infty$. In this case the Bochner mathematical expectation $\mathbf{E}X := \lim_{n\to\infty} \mathbf{E}X_n$. The expression $\mathbf{E}X_n$ has an obvious meaning as a usual sum and by lim we mean the norm limit. For details about the Bochner integral see ([2], III.2). Note that for a strongly measurable r.e. X in a Banach space B the Bochner mathematical expectation $\mathbf{E}X$ exists if and only if $\mathbf{E}\|X\| < \infty$.

Let B be a Banach lattice, $x_1, \dots, x_n$ be a finite sequence in B. Then the expression $(\sum_{i=1}^{n} |x_i|^2)^{1/2}$ is defined and is an element of B (see [6], p. 42).

Let now X be a simple r.e. in B which takes values $x_1, \dots, x_n$ with probabilities $p_1, \dots, p_n$. We put

$$\Delta(X) = \left(\sum_{i=1}^{n} p_i |x_i - \mathbf{E}X|^2 \right)^{1/2}.$$

Definition 1. Let X be a (strongly measurable) r.e. in a Banach lattice B. We say that X has the *Bochner mean square deviation* if $\mathbf{E}\|X\|^2 < \infty$ and there exists an element $\mathfrak{S}X \in B$ such that for any sequence of a simple r.e. X_n in B satisfying

$$\mathbf{E}\|X_n - X\|^2 \to 0 \quad as \quad n \to \infty \tag{1}$$

we have $\|\Delta(X_n) - \mathfrak{S}X\| \to 0$ as $n \to \infty$.

Remark 1. If X is a simple r.e. and $\mathfrak{S}X$ exists, then $\mathfrak{S}X = \Delta(X)$.

In function Banach lattices

$$(\mathfrak{S}X)(t) = (\mathbf{E}|X(t)|^2)^{1/2} \tag{2}$$

if $\mathbf{E}X = 0$. More exactly, the following two propositions hold true.

Proposition 1. *Let X be a r.e. with values in a Banach lattice B of functions on some set T, where the functionals $f_t(x) = x(t)$, $t \in T$, are defined and continuous. For example, let B be a lattice of sequences or continuous functions. If X has the Bochner mean square deviation and $\mathbf{E}X = 0$, then* (2) *is true for any $t \in T$.*

Proof. Let X_n be a simple r.e. from Definition 1. Then (1) implies that for every $t \in T$

$$\begin{aligned}\mathbf{E}\,|\,|X_n(t)| - |X(t)|\,|^2 &= \mathbf{E}\,|\,|f_t(X_n)| - |f_t(X)|\,|^2 \\ &\le \mathbf{E}|f_t(X_n - X)|^2 \le \| f_t \|^2\, \mathbf{E} \| X_n - X \|^2 \to 0 \quad \text{as} \quad n \to \infty\,.\end{aligned}$$

Thus $\mathbf{E}|X_n(t)|^2 \to \mathbf{E}|X(t)|^2$. Since $\mathbf{E}X = 0$, we have $\mathbf{E}X(t) \equiv 0$. Then, by Definition 1, for every $t \in T$ $(\mathbf{E}|X_n(t)|^2)^{1/2} \to (\mathfrak{S}X)(t)$ as $n \to \infty$. Thus $(\mathfrak{S}X)(t) = (\mathbf{E}|X(t)|^2)^{1/2}$. □

Proposition 2. *Let B be a Banach lattice* (*of classes*) *of measurable functions on a set T with a finite measure μ, continuously imbedded in $L_2(\mu)$ and let X be a B-valued r.e. If X has the Bochner mean square deviation and $\mathbf{E}X = 0$, then* (2) *is true for almost every $t \in T$.*

Proof. Let X_n be a simple r.e. from Definition 1. Since B is continuously imbedded in $L_2(\mu)$, from (1) it follows that

$$\mathbf{E} \int_T |X_n(t) - X(t)|^2 d\mu \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3}$$

Definition 1 implies

$$\int_T |(\mathbf{E}|X_n(t)|^2)^{1/2} - (\mathfrak{S}X)(t)| d\mu \to 0 \quad \text{as} \quad n \to \infty\,. \tag{4}$$

Therefore there is a sequence $n_k \uparrow \infty$ such that for almost every $t \in T$

$$(\mathbf{E}|X_{n_k}(t)|^2)^{1/2} \to (\mathfrak{S}X)(t) \quad \text{as} \quad k \to \infty\,. \tag{5}$$

From (3)–(5) it follows that $\mathbf{E}|X(t)|^2$ exists, $(\mathbf{E}|X_{n_k}(t)|^2)^{1/2} \to (\mathbf{E}|X(t)|^2)^{1/2}$ as $k \to \infty$ and $(\mathfrak{S}X)(t) = (\mathbf{E}|X(t)|^2)^{1/2}$ for almost every t. □

Remark 2. When considering a r.e. in a function Banach lattice B, the following question naturally arises. Let X be a r.e. in B. Is $X(t)$ a r.v. for any t? It follows from the proof of Proposition 2 that in the conditions of this proposition $X(t)$ is a r.v. for almost every t. See also Remark 5.

Definition 2. Let $1 \le p < \infty$. A Banach lattice B is called *p-convex* if there exists a constant $D^{(p)} = D^{(p)}(B)$ such that for every n and for any elements $(x_i)_1^n \subset B$ we have

$$\Big\| \Big(\sum_{i=1}^n |x_i|^p \Big)^{1/p} \Big\| \le D^{(p)} \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}$$

and, similarly, B is called *p-concave* if for some constant $D_{(p)} = D_{(p)}(B)$ the converse inequality

$$\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p} \le D_{(p)} \Big\| \Big(\sum_{n=1}^{n} |x_i|^p\Big)^{1/p} \Big\|$$

holds.

We shall also use the notion of *p-concavification* $B_{(p)}$ for a Banach lattice B. Let us describe this notion (for details see [6], p. 54). As $B_{(p)}$ we take B with the original order and operations $x \oplus y = (x^p + y^p)^{1/p}$, $\alpha \odot x = \alpha^{1/p} x$. By α^p we mean $\operatorname{sign}(\alpha)|\alpha|^p$ and $(x^p + y^p)^{1/p}$ is the element of B which corresponds to the function $f(s,t) = \operatorname{sign}(s^p + t^p)|s^p + t^p|^{1/p}$ (see [6], p. 53). The variable

$$\|x\|_p = \inf \Big\{ \sum_{i=1}^{n} \|x_i\|^p : |x| = \sum_{i=1}^{n} \oplus |x_i| \ , \ x_i \in B \ , \ n = 1, 2, \dots \Big\}$$

is a lattice norm in $B_{(p)}$ if B is a p-convex space. Moreover, for every $x \in B$

$$\|x\|_p \le \|x\|^p \le (D^{(p)})^p \|x\|_p \ . \tag{6}$$

Lemma 1 (see [3]). *Let B be a 2-convex Banach lattice. The identity map $\Phi : B \to B_{(2)}$ is a homeomorphism. More exactly, for any elements $x, y \in B$*

$$\|x \ominus y\|_2 \le \|x - y\| \| |x| + |y| \| \quad \textit{and} \quad \|x - y\| \le \sqrt{2} D^{(2)} \|x \ominus y\|_2^{1/2}.$$

Proof. By (6), the elementary scalar inequality $|\operatorname{sign}(s)|s|^2 - \operatorname{sign}(t)|t|^2|^{1/2} \le |s - t|^{1/2}(|s| + |t|)^{1/2}$ and the Hölder type inequality ([6], p. 43) we have

$$\|x \ominus y\|_2 \le \|x \ominus y\|^2 = \|(x^2 - y^2)^{1/2}\|^2 \le \|x - y\| \| |x| + |y| \| \, .$$

For scalars the inequality $|t - s| \le \sqrt{2} |\operatorname{sign}(t) t^2 - \operatorname{sign}(s) s^2|^{1/2}$ is valid. Therefore, by ([6], p.42), the definition of $(x^2 - y^2)^{1/2}$ and (6), we have

$$\|x - y\| \le \sqrt{2} \|(x^2 - y^2)^{1/2}\| = \sqrt{2} \|x \ominus y\| \le \sqrt{2} D^{(2)} \|x \ominus y\|_2^{1/2} . \qquad \square$$

The following theorem, corollary and example show that Definition 1 is suitable for 2-convex Banach lattices and probably for them only.

Theorem 1. *Let X be a r.e. in a 2-convex separable Banach lattice B and let $\mathbf{E}_{(2)} X$ be the Bochner mathematical expectation of X in the lattice $B_{(2)}$. Then X has the Bochner mean square deviation if and only if there exists $\mathbf{E}_{(2)} X$. Moreover,*

$$\mathfrak{S} X = \mathbf{E}_{(2)} |X - \mathbf{E} X| \, . \tag{7}$$

Proof. The existence of mean square deviation implies the existence of $\mathbf{E}\|X\|^2$, i.e., (by (6)) $\mathbf{E}\|X\|_2$, i.e., $\mathbf{E}_{(2)} X$.

Conversely, the existence of $\mathbf{E}_{(2)} X$ implies the existence of a simple r.e. $(X_n)_1^\infty$ in $B_{(2)}$ such that $\mathbf{E}\|X_n \ominus X\|_2 \to 0$ as $n \to \infty$. By Lemma 1, $\|X_n - X\|^2 \le 2(D^{(2)})^2 \|X_n \ominus X\|_2$, thus

$$\mathbf{E}\|X_n - X\|^2 \to 0 \quad \text{and} \quad \mathbf{E} X_n \to \mathbf{E} X \quad \text{as } n \to \infty \ , \tag{8}$$

i.e., (1) is satisfied. For scalars the inequality $||t| - |s|| \le |t - s|$ is valid. Therefore, by ([6], p. 42), such an inequality is valid for the elements of a Banach lattice as well. Thus, by the first inequality of Lemma 1,

$$\begin{aligned}\mathbf{E}\||X_n - \mathbf{E}X_n| \ominus |X - \mathbf{E}X|\|_2 &\le \mathbf{E}\|(X_n - \mathbf{E}X_n) \ominus (X - \mathbf{E}X)\|_2 \\ &\le \mathbf{E}(\|X_n - \mathbf{E}X_n - (X - \mathbf{E}X)\| \cdot \||X_n - \mathbf{E}X_n| + |X - \mathbf{E}X|\|) \\ &\le (\mathbf{E}\|X_n - \mathbf{E}X_n - X + \mathbf{E}X\|^2)^{1/2} \cdot (\mathbf{E}\||X_n - \mathbf{E}X_n| + |X - \mathbf{E}X|\|^2)^{1/2}.\end{aligned}$$

Using (8) we obtain that $\mathbf{E}\||X_n - \mathbf{E}X_n| \ominus |X - \mathbf{E}X|\|_2 \to 0$ as $n \to \infty$. Now, by the triangle inequality, the sequence $\mathbf{E}_{(2)}|X_n - \mathbf{E}X_n|$ converges in $B_{(2)}$ to $\mathbf{E}_{(2)}|X - \mathbf{E}X|$. For simple elements (7) follows from Definition 1 and the definition of $B_{(2)}$. Then, by Lemma 1, the sequence $\mathfrak{S}X_n = \mathbf{E}_{(2)}|X_n - \mathbf{E}X_n|$ converges to $\mathbf{E}_{(2)}|X - \mathbf{E}X|$ as $n \to \infty$ in B. It implies (7) also for the r.e. X. $\square$

Corollary 1. *The mean square deviation of a r.e. X in a 2-convex separable Banach lattice exists if and only if $\mathbf{E}\|X\|^2 < \infty$.*

Example. In the space $l_p, 1 \le p < 2$, the r.e. X which is identically equal to zero, has not the Bochner mean square deviation.

Indeed, let X_n be a symmetric r.e. taking values $\pm n^{\frac{1}{2} - \frac{1}{p}} e_i$ with probabilities $\frac{1}{2n}$, $i = 1, \ldots, n$, where (e_i) is the standard basis of l_p . Then $\mathbf{E}\|X_n - X\|^2 = 2n(n^{\frac{1}{2} - \frac{1}{p}})^2 \frac{1}{2n} \to 0$ as $n \to \infty$ but $\sum_{i=1}^n (\frac{1}{n}(n^{1/2-1/p})^2)^{1/2} e_i \not\to 0$. Of course, for the elements $Y_n \equiv 0$ we have $\mathbf{E}\|Y_n - X\|^2 = 0$.

Remark 3. The following two statements are the corollaries of the results of [7] and Theorem 1. Recall first the definition. Let X be a (strongly measurable) weak second order r.e. in B. Then for any r.v. ξ with $\mathbf{E}\xi^2 < \infty$ the Pettis integral $\tilde{\mathbf{E}}(\xi X)$ exists (see [12]). Suppose now $\tilde{\mathbf{E}}X = 0$. The *Pettis mean square deviation* is defined by the formula

$$\tilde{\mathfrak{S}}X = \sup\Big\{\tilde{\mathbf{E}}(\xi X) : \xi \text{ is a r. v., } \mathbf{E}|\xi|^2 \le 1\Big\}.$$

1. Let X be a r.e. in a 2-convex separable Banach lattice with $\mathbf{E}X = 0$. Then the Pettis mean square deviation $\tilde{\mathfrak{S}}X$ exists if and only if there exists the Pettis integral $\tilde{\mathbf{E}}_{(2)}|X|$. Moreover,

$$\tilde{\mathfrak{S}}X = \tilde{\mathbf{E}}_{(2)}|X|\,.$$

2. Let X be a r.e. in a 2-convex separable Banach lattice with $\mathbf{E}X = 0$. If there exists the Bochner mean square deviation $\mathfrak{S}X$, then there exists the Pettis mean square deviation $\tilde{\mathfrak{S}}X$. Moreover,

$$\tilde{\mathfrak{S}}X = \mathfrak{S}X\,.$$

3. Law of Large Numbers for Squares

From now on we shall denote by B a *Köthe function space* ([6], p.28), i.e., the space (of classes) of locally integrable functions on a complete σ-finite measurable space (T, Σ, μ) for which the following conditions hold:

(1) If $|x(t)| \leqslant |y(t)|$ a.e. on T with x measurable and $y \in B$, then $x \in B$ and $\|x\| \leqslant \|y\|$.
(2) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to B.

Let $X_i = (X_i(t),\ t \in T)$ be a sequence of independent r.e. in a separable Köthe function space B, $\mathbf{E}X_i = 0$, and $Z_n := \left(\frac{1}{n}\sum\limits_{i=1}^{n} |X_i|^2\right)^{\frac{1}{2}}$. Then $Z_n(t) = \left(\frac{1}{n}\sum\limits_{i=1}^{n} |X_i(t)|^2\right)^{\frac{1}{2}}$ for almost every t. Put $\sigma_i(t) = (\mathbf{E}|X_i(t)|^2)^{\frac{1}{2}}$.

Definition 3. We say that (X_i) *satisfies the law of large numbers for squares* if

$$\left\| Z_n(t) - \left(\frac{1}{n}\sum_{i=1}^{n} |\sigma_i(t)|^2\right)^{\frac{1}{2}} \right\| \xrightarrow{\text{a.s.}} 0 \tag{9}$$

or

$$\left\| Z_n(t) - \left(\frac{1}{n}\sum_{i=1}^{n} |\sigma_i(t)|^2\right)^{\frac{1}{2}} \right\| \xrightarrow{\mathbf{P}} 0 \tag{10}$$

as $n \to \infty$, where $\xrightarrow{\text{a.s.}}$ $(\xrightarrow{\mathbf{P}})$ means the convergence almost surely (in probability).

Relation (9) is naturally called the *strong LLN* and (10) the *weak LLN*.

Remark 4. If (X_i) are independent copies of a r.e. X, then formulas (9), (10) are equivalent to

$$\|Z_n(t) - \sigma(t)\| \xrightarrow{\text{a.s.}} 0 \tag{11}$$

and

$$\|Z_n(t) - \sigma(t)\| \xrightarrow{\mathbf{P}} 0 \tag{12}$$

as $n \to \infty$, where $\sigma(t) := (\mathbf{E}|X(t)|^2)^{\frac{1}{2}}$.

If (11) or (12) is satisfied, then we say that X *satisfies the strong* (*resp. weak*) *LLN for squares.*

Remark 5. In connection with Definition 3 the question on the measurability of Z_n arises, i.e., whether this variable is a r.e. if X_i are r.e. The answer is affirmative at least in separable Banach lattices [8]. Obviously, the existence of $\mathbf{E}X_i$ implies that X_i are separable valued.

Remark 6. As in Proposition 2, a question naturally arises whether $X(t)$ is a r.v. for almost every t if X is a r.e. One can easily to check that this is so if $\mathbf{E}X$ exists. Indeed, for a simple r.e. this follows from the definition. Let X_n be simple r.e. such that $\mathbf{E}\|X_n - X\| \to 0$ as $n \to \infty$. Then $X_{n_k} \to X$ a.s. for some subsequence (n_k) . So a.s. $X_{n_k}(t) \to X(t)$ for almost every t and $X(t)$ is a r.v. for almost every t. In general, we take

$$X^{(m)} = \begin{cases} X & \text{if } \|X\| \le m\,, \\ 0 & \text{if } \|X\| > m\,. \end{cases}$$

Then a.s. $\|X^{(m)} - X\| \to 0$, hence

$$\mathbf{P}(X^{(m)}(t) \to X(t) \text{ for almost all } t) = 1\,.$$

If X is separable-valued, then $\mathbf{E}X^{(m)}$ exists, hence $X^{(m)}(t)$ is a r.e. for almost all t. So $X(t)$ is a r.v. for almost all t.

Theorem 2. *Let X be a r.e. in a separable q-concave Köthe function space B, $2 \leqslant q < \infty$, $\mathbf{E}X = 0$, $\{(\mathbf{E}|X(t)|^q)^{\frac{1}{q}}\,,\ t \in T\} \in B$ and let (X_i) be the independent copies of X. Then*

1. *X satisfies the CLT, i.e., the sequence $n^{-\frac{1}{2}}\sum_{i=1}^n X_i$ weakly converges to a Gaussian distribution in B.*
2. *X satisfies the weak LLN for squares* (12) *and for $q > 2$*

$$\lim_{n\to\infty} \mathbf{E}\|Z_n(t) - \sigma(t)\|^q = 0\,. \tag{13}$$

Proof. 1. Let us consider the space $B(L_q)$ of functions $X(\omega,t)$ with the norm $\|X\|_{B(L_q)} = \|(\mathbf{E}|X(\omega,t)|^q)^{\frac{1}{q}}\|$. Since B is a q-concave Köthe function space $(q < \infty)$, B has a σ-order continuous norm (see [6], p.83). Thus the cross-product $B \otimes L_q$ is dense in $B(L_q)$, i.e., for any $\varepsilon > 0$ there is $X^{(\varepsilon)} \in B \otimes L_q$ such that

$$\|X - X^{(\varepsilon)}\|_{B(L_q)} < \varepsilon\,, \tag{14}$$

moreover, we can choose $X^{(\varepsilon)}$ as a function of the form

$$X^{(\varepsilon)} = \left\{X^{(\varepsilon)}(t) = \sum_{k=1}^m \chi_{A_k}(t)\xi_k,\ \ t \in T\right\}, \tag{15}$$

where $\chi_{A_k}(t) \in B$ denotes the characteristic function of a set $A_k \in \Sigma$, $A_k \cap A_l = \varnothing$ for $k \ne l$, $\xi_k \in L_q(\Omega, \mathfrak{B}, \mathbf{P})$ ([1], p. 85).

If $\mathbf{E}X = 0$, we can assume $\mathbf{E}X^{(\varepsilon)} = 0$ too. Indeed, by (14)

$$\|\mathbf{E}X^{(\varepsilon)}\| = \|\mathbf{E}X - \mathbf{E}X^{(\varepsilon)}\| \le \|\mathbf{E}|X - X^{(\varepsilon)}|\| \le \|X - X^{(\varepsilon)}\|_{B(L_q)} < \varepsilon\,.$$

So, putting $\widetilde{X}^{(\varepsilon)} = X^{(\varepsilon)} - \mathbf{E}X^{(\varepsilon)}$ we have $\widetilde{X}^{(\varepsilon)} \in B \otimes L_q$, $\mathbf{E}\widetilde{X}^{(\varepsilon)} = 0$ and

$$\|X - \widetilde{X}^{(\varepsilon)}\|_{B(L_q)} = \|X - (X^{(\varepsilon)} - \mathbf{E}X^{(\varepsilon)})\|_{B(L_q)} \le \|X - X^{(\varepsilon)}\|_{B(L_q)} + \|\mathbf{E}X^{(\varepsilon)}\| < 2\varepsilon\,.$$

Hence we can take $\widetilde{X}^{(\varepsilon)}$ instead of $X^{(\varepsilon)}$.

Let $(X_i^{(\varepsilon)})$ be the independent copies of $X^{(\varepsilon)}$, $S_n = \sum_{i=1}^n X_i$ and let $S_n^{(\varepsilon)} = \sum_{i=1}^n X_i^{(\varepsilon)}$. The main step in the proof of Theorem 2 is connected with the estimation [8]

$$(\mathbf{E}\|Y(t)\|^q)^{\frac{1}{q}} \le D_{(q)}\|(\mathbf{E}|Y(t)|^q)^{\frac{1}{q}}\| \ , \tag{16}$$

which is valid for every r.e. $Y(t)$ in a q-concave Köthe function space $B,\ q < \infty$. From (16) we have

$$(\mathbf{E}\|S_n(t) - S_n^{(\varepsilon)}(t)\|^q)^{\frac{1}{q}} \le D_{(q)}\left\|\left(\mathbf{E}\Big|\sum_{i=1}^n (X_i(t) - X_i^{(\varepsilon)}(t))\Big|^q\right)^{\frac{1}{q}}\right\| . \tag{17}$$

To estimate the right side of (17) we use the well known Kadec inequality [5]: Let $q \geqslant 1,\ \xi_i$ be a K-unconditional sequence in $L_q(\Omega, \mathfrak{B}, \mathbf{P})$ and $p = \min(q, 2)$. Then

$$\left(\mathbf{E}\Big|\sum_{i=1}^n \xi_i\Big|^q\right)^{\frac{1}{q}} \le C_q K\left(\sum_{i=1}^n \left(\mathbf{E}|\xi_i|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} , \tag{18}$$

moreover, if $q \leqslant 2$, then $C_q = 1$.

Since (X_i) are independent and $\mathbf{E}X_i = 0$, the r.v. $(X_i(t), i \geqslant 1)$ are independent for almost every t and $\mathbf{E}X_i(t) = 0$ a.e. Obviously, every sequence of independent r.v. $\xi_i,\ \ \mathbf{E}\xi_i = 0$, is 2-unconditional in L_q. So, since $q \geqslant 2$, for almost every t

$$\left(\mathbf{E}\Big|\sum_{i=1}^n (X_i(t) - X_i^{(\varepsilon)}(t))\Big|^q\right)^{\frac{1}{q}} \le 2C_q\sqrt{n}\Big(\mathbf{E}|X(t) - X^{(\varepsilon)}(t)|^q\Big)^{\frac{1}{q}} . \tag{19}$$

Inequalities (19), (17) and (14) imply

$$\left(\mathbf{E}\left\|\frac{S_n - S_n^{(\varepsilon)}}{\sqrt{n}}\right\|^q\right)^{\frac{1}{q}} \le 2C_q D_{(q)}\|X - X^{(\varepsilon)}\|_{B(L_q)} \le 2C_q D_{(q)}\varepsilon .$$

Obviously, $X^{(\varepsilon)}$ satisfies the CLT. Since $\varepsilon > 0$ is arbitrary, by the known Pisier result (see [11], Theorem 4.1), the last inequality implies that X satisfies the CLT.

2. Let $q > 2$. The weak LLN (12) will be established if we prove (13). From an elementary scalar inequality

$$|s - t| \le |s^2 - t^2|^{\frac{1}{2}} , \quad s, t \ge 0$$

and (16) we obtain

$$(\mathbf{E}\|Z_n(t) - \sigma(t)\|^q)^{\frac{1}{q}} \le D_{(q)}\|(\mathbf{E}|Z_n(t) - \sigma(t)|^q)^{\frac{1}{q}}\| \le D_{(q)}\|(\mathbf{E}|Z_n^2(t) - \sigma^2(t)|^{\frac{q}{2}})^{\frac{1}{q}}\| .$$

To estimate the last term of this inequality we apply (18). Since (X_i) are independent, $X_i^2(t) - \sigma^2(t)$ are independent for almost every t and $\mathbf{E}(X_i^2(t) -$

$\sigma^2(t)) = 0$. So for $\tilde{p} = \min(q/2, 2)$ we have

$$\begin{aligned}\mathbf{E}|Z_n^2(t) - \sigma^2(t)|^{\frac{q}{2}} &\le C'_q \left(\sum_{i=1}^n \left(\mathbf{E} \left| \frac{X_i^2(t) - \sigma^2(t)}{n} \right|^{\frac{q}{2}} \right)^{\frac{2\tilde{p}}{q}} \right)^{\frac{q}{2\tilde{p}}} \\ &\le C'_q \mathbf{E}|X^2(t) - \sigma^2(t)|^{\frac{q}{2}} \cdot n^{1-\tilde{p}}.\end{aligned}$$

Since $\{(\mathbf{E}|X^2(t) - \sigma^2(t)|^{\frac{q}{2}})^{\frac{1}{q}}, t \in T\} \in B$, it implies (13).

The case $q = 2$ requires an individual consideration. Let $X^{(\varepsilon)}$ be defined by (15), and let inequality (14) be fulfilled. Put $\sigma^{(\varepsilon)}(t) = (\mathbf{E}|X^{(\varepsilon)}(t)|^2)^{\frac{1}{2}}$ and $Z_n^{(\varepsilon)}(t) = (\frac{1}{n}\sum_{i=1}^n |X_i^{(\varepsilon)}(t)|^2)^{\frac{1}{2}}$. By the triangle inequality we have

$$\begin{aligned}\|Z_n(t) - \sigma(t)\| &\le \|Z_n(t) - Z_n^{(\varepsilon)}(t)\| + \|Z_n^{(\varepsilon)}(t) - \sigma^{(\varepsilon)}(t)\| \\ &\quad + \|\sigma^{(\varepsilon)}(t) - \sigma(t)\| \, . \end{aligned} \tag{20}$$

For the first summand in the right side of (20)

$$\begin{aligned}\mathbf{E}\|Z_n(t) - Z_n^{(\varepsilon)}(t)\| &\le \mathbf{E}\left\| \left(\frac{1}{n} \sum_{i=1}^n |X_i(t) - X_i^{(\varepsilon)}(t)|^2 \right)^{\frac{1}{2}} \right\| \\ &\le \left(\mathbf{E}\left\| \left(\frac{1}{n} \sum_{i=1}^n |X_i(t) - X_i^{(\varepsilon)}(t)|^2 \right)^{\frac{1}{2}} \right\|^2 \right)^{\frac{1}{2}} \quad \text{(by(16))} \\ &\le D_{(2)} \|(\mathbf{E}|X(t) - X^{(\varepsilon)}(t)|^2)^{\frac{1}{2}}\| \\ &= D_{(2)} \|X - X^{(\varepsilon)}\|_{B(L_2)} < D_{(2)}\varepsilon \, . \end{aligned} \tag{21}$$

From (15) it follows that

$$|X^{(\varepsilon)}(t)|^2 = \sum_{k=1}^m \chi_{A_k}(t)|\xi_k|^2 \quad \text{and} \quad \sigma^{(\varepsilon)}(t) = \sum_{k=1}^m \chi_{A_k}(t)(\mathbf{E}|\xi_k|^2)^{\frac{1}{2}},$$

so a.s.

$$\|Z_n^{(\varepsilon)}(t) - \sigma^{(\varepsilon)}(t)\| \to 0 \quad \text{as} \quad n \to \infty \, . \tag{22}$$

By the triangle inequality

$$|\sigma^{(\varepsilon)}(t) - \sigma(t)| \le (\mathbf{E}|X^{(\varepsilon)}(t) - X(t)|^2)^{\frac{1}{2}}$$

for any t. This inequality and (14) imply

$$\|\sigma^{(\varepsilon)}(t) - \sigma(t)\| \le \|X^{(\varepsilon)}(t) - X(t)\|_{B(L_2)} < \varepsilon \, . \tag{23}$$

Now from (20)–(23) follows (12). □

Remark 7. The analysis of this proof shows that one can generalize part 2 of Theorem 2 as follows. Let (X_i) be a sequence of independent random elements in a separable q-concave Köthe function space B, $2 < q < \infty$, $\mathbf{E}X_i = 0$ and let there exist an element $u \in B$ such that for every $i \ge 1$

$$(\mathbf{E}|X_i(t)|^q)^{\frac{1}{q}} \le u(t) \, , \quad t \in T \, .$$

Then (X_i) satisfies the weak LLN (12).

To prove the next theorem we shall use following lemma.

Lemma 2. *Let* (η_i) *be a sequence of independent random variables,* $\mathbf{E}\eta_i = 0$, $1 < p \le 2$, $S_n = \sum_{i=1}^n \eta_i$. *Then*

$$\mathbf{E}\sup_{n\ge1} \frac{S_n}{n} \le 1 + 2p\zeta^2(p)m_p \ ,$$

where $\zeta(z) = \sum_{k=1}^\infty \frac{1}{k^z}$ *is the Riemann zeta-function and* $m_p = \sup_{i\ge1} \mathbf{E}|\eta_i|^p$.

Proof. We use the inequality (see [13], Theorem 14)

$$\mathbf{P}\Big(\max_{1\le k\le n} a_k S_k \ge s\Big) \le \frac{1}{\varphi(s)}\Big\{\varphi(a_n)\cdot\mathbf{E}\varphi(|S_n|) + \sum_{k=1}^{n-1}(\varphi(a_k)-\varphi(a_{k+1}))\mathbf{E}\varphi(|S_k|)\Big\} \ , \tag{24}$$

where $a_1 \ge a_2 \ge \ldots \ge a_n > 0$, $\varphi(s)$ is a nondecreasing and convex function defined on the positive semi-axis such that $\varphi(+0) = 0$ and $\varphi(s\cdot t) \le \varphi(s)\cdot\varphi(t)$. Put in (24) $a_k = \frac{1}{k}$ and $\varphi(s) = s^p$. Then

$$\mathbf{P}\Big(\max_{1\le k\le n} \frac{S_k}{k} \ge s\Big) \le \frac{1}{s^p}\Big\{\frac{\mathbf{E}|S_n|^p}{n^p} + \sum_{k=1}^{n-1}\frac{(k+1)^p - k^p}{k^p(k+1)^p}\mathbf{E}|S_k|^p\Big\} \ .$$

This formula, the elementary equality

$$(k+1)^p - k^p = p(k+\theta)^{p-1}, \quad 0 \le \theta \le 1 \ ,$$

and (18) give

$$\mathbf{P}\Big(\max_{1\le k\le n} \frac{S_k}{k} \ge s\Big) \le \frac{2}{s^p}\left(\frac{m_p}{n^{p-1}} + p\sum_{k=1}^{n-1}\frac{m_p}{k^{p-1}(k+1)}\right) \le \frac{2m_p}{s^p}\left(\frac{1}{n^{p-1}} + p\sum_{k=1}^{n-1}\frac{1}{k^p}\right).$$

Hence

$$\mathbf{P}\Big(\sup_{n\ge1} \frac{S_n}{n} \ge s\Big) \le 2p\zeta(p)\frac{m_p}{s^p} \ .$$

From the last inequality we have

$$\mathbf{E}\sup_{n\ge1}\frac{S_n}{n} \le \sum_{k=0}^\infty \mathbf{P}\Big(\sup_{n\ge1}\frac{S_n}{n} \ge k\Big) \le 1 + \sum_{k=1}^\infty 2p\zeta(p)\frac{m_p}{k^p} = 1 + 2p\zeta^2(p)m_p \ . \qquad \square$$

Theorem 3. *Let* (X_i) *be a sequence of independent random elements in a separable q-concave Köthe function space* B, $2 \le q < \infty$, *and* $\mathbf{E}X_i = 0$ *for every* i. *If there exist a number* $r > q$ *and an element* $u \in B$ *such that for every* $i \ge 1$

$$(\mathbf{E}|X_i(t)|^r)^{\frac{1}{r}} \le u(t), \quad t \in T \ , \tag{25}$$

then (X_i) *satisfies the strong LLN for squares (9).*

Proof. From the strong LLN for real-valued r.v. ([10], Ch. IX, §3) and (25) we have for any $t \in T$

$$Z_n(t) - \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{E}|X_i(t)|^2\right)^{\frac{1}{2}} \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty \ . \tag{26}$$

Now we shall show that a.s.

$$\sup_{n\geq 1}\left|Z_n(t) - \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{E}|X_i(t)|^2\right)^{\frac{1}{2}}\right| \in B \ . \tag{27}$$

Since under our conditions B has an order continuous norm, for B the abstract version of the Lebesgue theorem on majorized convergence ([1], p.72) holds true. Thus (26), (27) imply the strong LLN (9).

From (25) it follows that in order to prove (27) it is sufficient to show that a.s.

$$\left\{\sup_{n\geq 1} Z_n(t),\ t \in T\right\} \in B \ . \tag{28}$$

For this purpose apply first the well known inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} s_i\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} s_i^p \quad \text{for} \quad p \geq 1,\ s_i \geq 0 \ .$$

We have

$$\left|\sup_{n\geq 1} Z_n(t)\right|^q \leq \sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}|X_i(t)|^q \leq |u(t)|^q + |u(t)|^q \sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\eta_i(t) \ , \tag{29}$$

where

$$\eta_i(t) = \begin{cases} \frac{|X_i(t)|^q - \mathbf{E}|X_i(t)|^q}{|u(t)|^q} & \text{if } u(t) > 0\,, \\ 0 & \text{if } u(t) = 0\,. \end{cases} \tag{30}$$

Put in Lemma 2 $p = r/q$, $2p\zeta^2(p) = c_p$ and determine $\eta_i(t)$ by (30). Without loss of generality we can assume that $1 < p < 2$. From (29) and Lemma 2 we get for every $t \in T$

$$\mathbf{E}\left|\sup_{n\geq 1} Z_n(t)\right|^q \leq |u(t)|^q + |u(t)|^q(1 + c_p m_p(t)) \leq |u(t)|^q(2 + c_p m_p(t)) \ . \tag{31}$$

It is not difficult to verify that $m_p(t) = \sup_{n\geq 1}\mathbf{E}|\eta_n(t)|^p \leq 2$. This inequality and estimations (16), (31) allow us to obtain (28). Indeed,

$$\left(\mathbf{E}\left\|\sup_{n\geq 1} Z_n(t)\right\|^q\right)^{\frac{1}{q}} \leq D_{(q)}\left\|\left(\mathbf{E}\left|\sup_{n\geq 1} Z_n(t)\right|^q\right)^{\frac{1}{q}}\right\| \leq [2(1 + c_p)]^{\frac{1}{q}} D_{(q)}\|u(t)\| < \infty \ ,$$

which gives (28). □

Remark 8. Counterexamples to the strong LLN for real valued r.v. [10] show that in Theorem 3 we cannot replace the condition $r > q$ by $r \geq q$.

Acknowledgement

The authors wish to express their thanks to the referee for valuable remarks.

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(Received 14.07.2000; revised 3.12.2001)

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