Georgian Mathematical Journal
Volume 9 (2002), Number 1, 1–12

APPLICATION OF SINGULAR INTEGRAL EQUATIONS IN THE BOUNDARY VALUE PROBLEMS OF ELECTROELASTICITY

L. BITSADZE

Abstract. The purpose of this paper is to consider the three-dimensional versions of the theory of electroelasticity for a transversally esotropic body. Applying the potential method and the theory of singular integral equations, the normality of singular integral equations corresponding to the boundary value problems of electroelasticity are proved and the symbolic matrix is calculated. The uniqueness and existence theorem for the basic **BVP**s of electroelasticity are given.

2000 Mathematics Subject Classification: 74E10, 74F15
Key words and phrases: Electroelasticity, potential method, integral equations, symbolic matrix.

Introduction

In this paper we consider the three-dimensional versions of the theory of electroelasticity for a transversally isotropic body which is the simplest anisotropic one and for which we can do explicit computations.

The idea to apply singular integral equations in the boundary value problem (BVP) is due to Basheleishvili and Natroshvili [1]. In [1] the normality of singular integral operators and also existence and uniqueness theorems for the BVPs are proved by applying the potential method and the theory of singular integral equations. The present paper is an attempt to extend this result to boundary value problems of electroelasticity for a transversally isotropic electroelastic body.

1. Some Previous Results

The following notations are used throughout the paper. Let E_3 be the 3-dimensional real Euclidean space, $D^+ \in E_3$ be a finite domain bounded by the surface S. Suppose that S belongs to the Lyapunov class $L_2(\alpha)$ of order $\alpha > 0$ (see, e.g., [1]), $D^- = E_3 \backslash \bar{D}^+$, where $\bar{D}^+ = D^+ \cup S, x = x(x_1, x_2, x_3) \in E_3, u(x) = u(u_1, u_2, u_3, u_4)$ is a four-dimensional vector function.

Basic Equations of Electroelasticity. A basic equation of statics of a transversally isotropic electroelastic body can be written in the form [2, 3]

$$C(\partial x)U(x) = 0, \tag{1}$$

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

where $C(\partial x) = \|C_{km}\|_{4,4}$,

$$C_{11} = c_{11}\frac{\partial^2}{\partial x_1^2} + c_{66}\frac{\partial^2}{\partial x_2^2} + c_{44}\frac{\partial^2}{\partial x_3^2},\ C_{12} = (c_{11} - c_{66})\frac{\partial^2}{\partial x_1 \partial x_2},$$
$$C_{j3} = (c_{13} + c_{44})\frac{\partial^2}{\partial x_j \partial x_3},\ j = 1,2,\ C_{j4} = (e_{13} + e_{15})\frac{\partial^2}{\partial x_j \partial x_3},\ j = 1,2,$$
$$C_{22} = c_{11}\frac{\partial^2}{\partial x_2^2} + c_{66}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2},\ C_{34} = e_{15}\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) + e_{33}\frac{\partial^2}{\partial x_3^2},$$
$$C_{33} = c_{44}\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) + c_{33}\frac{\partial^2}{\partial x_3^2},\ C_{44} = -\epsilon_{11}\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) - \epsilon_{33}\frac{\partial^2}{\partial x_3^2},$$
$$C_{kj} = C_{jk}, U = U(u_1, u_2, u_3, u_4),$$

u_1, u_2, u_3 are the displacement vector components, u_4 is an electrostatic potential, $s_{kj}, e_{kj}, \epsilon_{kj}$ are the elastic, piezoelastic and dielectric constants, respectively.

Definition 1. A vector-function $U(x)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second derivatives in $D^+(D^-)$, and $U(x)$ itself and its first derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i. e., $u \in C^2(D^+) \cap C^1(\bar{D}^+)$. For the domain D^- the conditions at infinity are added:

$$u_k = O(|x|^{-1}),\ \frac{\partial u_k}{\partial x_j} = O(|x|^{-2}),\ |x|^2 = x_1^2 + x_2^2 + x_3^2,\ k = 1,2,3,4,\ j = 1,2,3. \quad (2)$$

The Electroelastic Stress Vector. Now let us write an expression for the components of the electromechanical stress vector. Denoting the stress vector by $T(\partial x, n)U$, we have [4]

$$T(\partial x, n)U = ((Tu)_1, (Tu)_2, (Tu)_3, (Tu)_4),$$
$$(TU)_k = \sum_{j=i}^{3} \tau_{kj} n_j, k = 1,2,3. (TU)_4 = (n, D),$$

where $T(\partial x, n) = \|T_{kj}(\partial x, n)\|_{4.4}$ is the stress tensor with the elements

$$T_{11}(\partial x, n) = c_{11}n_1\frac{\partial}{\partial x_1} + s_{66}n_2\frac{\partial}{\partial x_2} + c_{44}n_3\frac{\partial}{\partial x_3},$$
$$T_{12}(\partial x, n) = (c_{11} - 2c_{66})n_1\frac{\partial}{\partial x_2} + c_{66}n_2\frac{\partial}{\partial x_1},\ T_{13}(\partial x, n) = c_{13}n_1\frac{\partial}{\partial x_3} + c_{44}n_3\frac{\partial}{\partial x_1},$$
$$T_{14}(\partial x, n) = e_{13}n_1\frac{\partial}{\partial x_3} + e_{15}n_3\frac{\partial}{\partial x_1},\ T_{21}(\partial x, n) = c_{66}n_1\frac{\partial}{\partial x_2} + (c_{11} - 2c_{66})n_2\frac{\partial}{\partial x_1},$$
$$T_{22}(\partial x, n) = c_{66}n_1\frac{\partial}{\partial x_1} + s_{11}n_2\frac{\partial}{\partial x_2} + c_{44}n_3\frac{\partial}{\partial x_3},\ T_{23}(\partial x, n) = c_{13}n_2\frac{\partial}{\partial x_3} + c_{44}n_3\frac{\partial}{\partial x_2},$$
$$T_{24}(\partial x, n) = e_{13}n_2\frac{\partial}{\partial x_3} + e_{15}n_3\frac{\partial}{\partial x_2},\ T_{31}(\partial x, n) = c_{44}n_1\frac{\partial}{\partial x_3} + c_{13}n_3\frac{\partial}{\partial x_1},$$

$$T_{32}(\partial x,n)=c_{44}n_2\frac{\partial}{\partial x_3}+c_{13}n_3\frac{\partial}{\partial x_2},\ T_{33}(\partial x,n)=c_{44}\Big(n_1\frac{\partial}{\partial x_1}+n_2\frac{\partial}{\partial x_2}\Big)+c_{33}n_3\frac{\partial}{\partial x_3},$$
$$T_{34}(\partial x,n)=e_{15}\Big(n_1\frac{\partial}{\partial x_1}+n_2\frac{\partial}{\partial x_2}\Big)+e_{33}n_3\frac{\partial}{\partial x_3}=T_{43},$$
$$T_{41}(\partial x,n)=e_{15}n_1\frac{\partial}{\partial x_3}+e_{13}n_3\frac{\partial}{\partial x_1},\ \ T_{42}(\partial x,n)=e_{15}n_2\frac{\partial}{\partial x_3}+e_{13}n_3\frac{\partial}{\partial x_2},$$
$$T_{44}(\partial x,n)=-\epsilon_{11}\Big(n_1\frac{\partial}{\partial x_1}+n_2\frac{\partial}{\partial x_2}\Big)-\epsilon_{33}n_3\frac{\partial}{\partial x_3}. \tag{3}$$

Throughout this paper $n(x)$ denotes the exterior to D^+ unit normal vector at the point $x\in S$. $D=(D_1,D_2,D_3)$ is an induction vector, τ_{kj} are the stress tensor components

$$\tau_{11}=c_{11}\frac{\partial u_1}{\partial x_1}+(c_{11}-2c_{66})\frac{\partial u_2}{\partial x_2}+c_{13}\frac{\partial u_3}{\partial x_3}+e_{13}\frac{\partial u_4}{\partial x_3},$$
$$\tau_{22}=(c_{11}-2c_{66})\frac{\partial u_1}{\partial x_1}+c_{11}\frac{\partial u_2}{\partial x_2}+c_{13}\frac{\partial u_3}{\partial x_3}+e_{13}\frac{\partial u_4}{\partial x_3},$$
$$\tau_{33}=c_{13}\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)+c_{33}\frac{\partial u_3}{\partial x_3}+e_{33}\frac{\partial u_4}{\partial x_3},\tau_{12}=c_{66}\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big),$$
$$\tau_{j3}=c_{44}\Big(\frac{\partial u_j}{\partial x_3}+\frac{\partial u_3}{\partial x_j}\Big)+e_{15}\frac{\partial u_4}{\partial x_j},j=1,2.\tau_{kj}=\tau_{jk},$$
$$D_j=-\epsilon_{11}\frac{\partial u_4}{\partial x_j}+e_{15}\Big(\frac{\partial u_j}{\partial x_3}+\frac{\partial u_3}{\partial x_j}\Big),j=1,2.,$$
$$D_3=-\epsilon_{33}\frac{\partial u_4}{\partial x_3}+e_{13}\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)+e_{33}\frac{\partial u_3}{\partial x_3}.$$

The Basic BVPs. For equation (1) we pose the following BVPs. Find in $D^+(D^-)$ a regular solution $U(x)$ to equation (1), satisfying on the boundary S one of the following boundary conditions:

Problem $(1)^+$. $U^+=f(y),y\in S$ (D^- is vacuum),

Problem $(1)^-$. $(U)^-=f(y),y\in S$ (D^+ is vacuum),

Problem $(2)^+$. $(TU)^+=f(y),y\in S$ (D^- is vacuum),

Problem $(2)^-$. $(TU)^-=f(y),y\in S$ (D^+ is vacuum),

where $()^\pm$ denotes the limiting value from $D^\pm$, and $f(y)$ is a given function on S.

The Uniqueness Theorems. In this subsection we investigate the question of the uniqueness of solutions of the above-mentioned problems.

Let the first BVP have in the domain D^+ two regular solutions $u^{(1)}$ and $u^{(2)}$. We write $u=u^{(1)}-u^{(2)}$. Evidently, the vector $u(x)$ satisfies (1) and the boundary condition $u^+=0$ on S. Note that, if u is a regular solution of the equation (1), we have the following Green's formula

$$\int\limits_{D^+}E(u,u)d\sigma=\int\limits_S u^+(Tu)^+ds,\ \int\limits_{D^-}E(u,u)d\sigma=-\int\limits_S u^-(Tu)^-ds, \tag{4}$$

where $E(u,u)$ is a potential energy:

$$E(u,u)=\frac{1}{c_{11}}\Big[c_{11}\frac{\partial u_1}{\partial x_1}+(c_{11}-2c_{66})\frac{\partial u_2}{\partial x_2}+c_{13}\frac{\partial u_3}{\partial x_3}\Big]^2$$
$$+\frac{4c_{11}c_{66}^2}{c_{11}^2c_{66}(c_{11}-c_{66})}\Big[(c_{11}-c_{66})\frac{\partial u_2}{\partial x_2}+\frac{c_{13}}{2}\frac{\partial u_3}{\partial x_3}\Big]^2$$
$$+\frac{c_{33}(c_{11}-c_{66})-c_{13}^2}{c_{11}-c_{66}}\Big(\frac{\partial u_3}{\partial x_3}\Big)^2+4c_{44}\Big[\Big(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\Big)^2+\Big(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\Big)^2\Big]$$
$$+4c_{66}\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)^2+\epsilon_{11}\Big[\Big(\frac{\partial u_4}{\partial x_1}\Big)^2+\Big(\frac{\partial u_4}{\partial x_1}\Big)^2\Big]+\epsilon_{33}\Big(\frac{\partial u_4}{\partial x_3}\Big)^2. \tag{5}$$

For the positive definiteness of the potential energy the inequalities

$$c_{11}>0,\quad c_{44}>0,\quad c_{66}>0,\quad \epsilon_{11}>0,\quad \epsilon_{33}>0,$$
$$c_{33}(c_{11}-c_{66})-c_{13}^2>0,\quad c_{11}-c_{66}>0$$

are necessary and sufficient.

Using (4) and taking into account the fact that the potential energy is positive definite, we conclude that $u=const, x\in D^+$. Since $u^+=0$, we have $u=0, x\in D^+$. Thus the first BVP has, in the domain D^+, at most one regular solution.

The vectors $u^{(1)}$ and $u^{(2)}$ in the domain D^- must satisfy condition (2). In this case the regular vector $u=u^{(1)}-u^{(2)}$ satisfies (1) and thus formulae (4) are valid and $u(x)=const, x\in D^-$. But $u(x)$ satisfies $(u)^-=0$ on the boundary, which implies that $u=0, x\in D^-$. Thus the first BVP has, in the domain D^-, at most one regular solution.

Let $(TU)^+=0$ on S. Then applying (4) to a regular solution we have

$$(u_1,u_2,u_3)=a+[b,x],\quad u_4=const,\quad x\in D^+,$$

where a,b are arbitrary real constants.

The vector u in the domain D^- must satisfy condition (2). In that case, from (4) we obtain $u=0$.

Therefore we shall formulate the final results.

Theorem 1. *Problems* $(1)^\pm$ *and* $(2)^-$ *have at most one regular solution.*

Theorem 2. *A regular solution of BVP* $(2)^+$ *is not unique in the domain* D^+. *Two regular solutions may differ by a rigid displacement.*

Matrix of Fundamental Solutions. The basic matrix of fundamental solutions of equation (1) has the form [4]

$$\Gamma(x-y)=-\sum_{k=1}^{4}\|\Gamma_{pq}^{(k)}\|_{4,4}, \tag{6}$$

where

$$\Gamma_{jj}=\frac{\lambda_k}{r_k}+\alpha_k\frac{\partial^2\Phi_k}{\partial x_j^2},\quad j=1,2;\quad \Gamma_{12}^{(k)}=\alpha_k\frac{\partial^2\Phi_k}{\partial x_1\partial x_2}\,,$$

$$\Gamma_{j3}^{(k)}=\gamma_k\frac{\partial^2\Phi_k}{\partial x_j\partial x_3},\ \Gamma_{j4}^{(k)}=\beta_k\frac{\partial^2\Phi_k}{\partial x_j\partial x_3},\ j=1,2,\ \ \Gamma_{34}=\frac{\eta_k}{r_k},\ \Gamma_{33}=\frac{\delta_k}{r_k},\ \Gamma_{44}=\frac{m_k}{r_k},$$
$$\Phi_k=(x_3-y_3)\ln(x_3-y_3+r_k)-r_k,\quad k=1,2,3,4,$$
$$r_k^2=a_k[(x_1-y_1)^2+(x_2-y_2)^2]+(x_3-y_3)^2,\quad \Gamma_{pq}^{(k)}=\Gamma_{qp}^{(k)},$$
$$\lambda_k=-\delta_{1k}c_{66}^{-1},\quad \alpha_1=-c_{44}^{-1},$$
$$\alpha_k=\frac{d_k}{a_k}\Big[(c_{33}-c_{44}a_k)(\epsilon_{33}-\epsilon_{11}a_k)+(e_{33}-e_{15}a_k)^2\Big],\quad k=2,3,4,$$
$$\gamma_k=d_k\Big[(\epsilon_{33}-\epsilon_{11}a_k)(c_{13}+c_{44})+(e_{13}+e_{15})(e_{33}-e_{15}a_k)\Big],\quad k=2,3,4,\quad \gamma_1=0,$$
$$\beta_k=d_k\Big[(c_{13}+c_{44})(e_{33}-e_{15}a_k)-(e_{13}+e_{15})(c_{33}-c_{44}a_k)\Big],\quad k=2,3,4,\quad \beta_1=0,$$
$$\delta_k=d_k\Big[(e_{13}+e_{15})^2a_k-(c_{44}-c_{11}a_k)(\epsilon_{33}-\epsilon_{11}a_k)\Big],\quad k=2,3,4,\quad \delta_1=0,$$
$$\eta_k=d_k\Big[a_k(c_{11}a_k-c_{44})(e_{33}-e_{15}a_k)$$
$$-a_k(e_{13}+e_{15})(c_{13}+c_{44})\Big],\quad k=2,3,4,\quad \eta_1=0,$$
$$m_k=d_k\Big[(c_{44}-c_{11}a_k)(c_{33}-c_{44}a_k)+a_k(c_{13}+c_{44})^2\Big],\quad k=2,3,4,\quad m_1=0,$$
$$d_k^{-1}=(-1)^kb_0(a_3-a_2)(a_4-a_k),\quad k=2,3,\quad d_4^{-1}=b_0(a_2-a_4)(a_3-a_4),$$
$$b_0=c_{11}(\epsilon_{11}c_{44}+e_{15}^2)>0,\quad \sum_{k=1}^4\beta_k=\sum_{k=1}^4\alpha_k=\sum_{k=1}^4\gamma_k=0,\quad a_1=c_{44}c_{66}^{-1},$$

$a_k, k=2,3,4$, are the positive roots of the characteristic equation

$$b_0a^3-b_1a^2+b_2a-b_3=0,$$

where

$$\begin{aligned}
b_1&=c_{11}\alpha_0-\epsilon_{11}(c_{13}+c_{44})^2+c_{44}(e_{13}+e_{15})^2\\
&\quad-2e_{15}(c_{13}+c_{44})(e_{13}+e_{15})+c_{44}b_0,\\
b_2&=c_{11}b_3+c_{44}\alpha_0-\epsilon_{33}(c_{13}+c_{44})^2+c_{33}(e_{13}+e_{15})^2\\
&\quad-2e_{33}(c_{13}+c_{44})(e_{13}+e_{15}),\\
\alpha_0&=c_{33}\epsilon_{11}+c_{44}\epsilon_{33}+2e_{15}e_{33}>0,\\
b_3&=c_{44}(c_{33}\epsilon_{33}+e_{33}^2)>0.
\end{aligned}$$

Singular Matrix of Solutions. Using the basic fundamental matrix (6) we shall construct the so-called singular matrix of solutions. By applying the operator $T(\partial x,n)$ to the matrix $\Gamma(x-y)$ we get the matrix

$$[T(\partial y,n)\Gamma(y-x)]^*=-\sum_{k=1}^4\|A_{pq}^{(k)}\|_{4,4}, \tag{7}$$

which is obtained from $T(\partial x,n)\Gamma(x-y)$ by transposition of the columns and rows and of the variables x and y. We can easily prove that every column of matrix (7) is a solution of system (1) with respect to the point x if $x \neq y$. The elements $A_{pq}^{(k)}$ are the following:

$$A_{11}^{(k)} = -\delta_{1k}\frac{\partial_k}{\partial n}\frac{1}{r_k} + a_0^{(k)}\frac{\partial}{\partial s_2}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3} - 2c_{66}\alpha_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1\partial x_2},$$
$$A_{21}^{(k)} = \delta_{1k}\frac{\partial}{\partial s_3}\frac{1}{r_k} + a_0^{(k)}\frac{\partial}{\partial s_2}\frac{\partial^2\Phi_k}{\partial x_2\partial x_3} - 2c_{66}\alpha_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_2^2},$$
$$A_{31}^{(k)} = b_0^{(k)}\frac{\partial}{\partial s_2}\frac{1}{r_k} - 2c_{66}\gamma_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3},\ A_{41}^{(k)} = c_0^{(k)}\frac{\partial}{\partial s_2}\frac{1}{r_k} - 2c_{66}\beta_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_2\partial x_3},$$
$$A_{12}^{(k)} = -\delta_{1k}\frac{\partial}{\partial s_3}\frac{1}{r_k} + 2c_{66}\alpha_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1^2} - a_0^{(k)}\frac{\partial}{\partial s_1}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3},$$
$$A_{22}^{(k)} = -\delta_{1k}\frac{\partial_k}{\partial n}\frac{1}{r_k} - a_0^{(k)}\frac{\partial}{\partial s_1}\frac{\partial^2\Phi_k}{\partial x_2\partial x_3} + 2c_{66}\alpha_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1\partial x_2},$$
$$A_{32}^{(k)} = 2c_{66}\gamma_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3} - b_0^{(k)}\frac{\partial}{\partial s_1}\frac{1}{r_k},\ A_{42}^{(k)} = 2c_{66}\beta_k\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3} - c_0^{(k)}\frac{\partial}{\partial s_1}\frac{1}{r_k},$$
$$A_{13}^{(k)} = a_0^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3} - c_{44}\lambda_k\frac{\partial}{\partial s_2}\frac{1}{r_k},\ A_{23}^{(k)} = a_0^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_2\partial x_3} + c_{44}\lambda_k\frac{\partial}{\partial s_1}\frac{1}{r_k},$$
$$A_{14}^{(k)} = l_0^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_1\partial x_3} - e_{15}\lambda_k\frac{\partial}{\partial s_2}\frac{1}{r_k},\ A_{24}^{(k)} = l_0^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_2\partial x_3} + e_{15}\lambda_k\frac{\partial}{\partial s_1}\frac{1}{r_k},$$
$$A_{33}^{(k)} = (e_{15}\eta_k + c_{44}\delta_k + c_{44}\gamma_k)\frac{\partial_k}{\partial n}\frac{1}{r_k},\ A_{43}^{(k)} = (e_{15}m_k + c_{44}\beta_k + c_{44}\eta_k)\frac{\partial_k}{\partial n}\frac{1}{r_k},$$
$$A_{34}^{(k)} = (e_{15}\delta_k - \epsilon_{11}\eta_k + e_{15}\gamma_k)\frac{\partial_k}{\partial n}\frac{1}{r_k},\ A_{44}^{(k)} = (e_{15}\eta_k - \epsilon_{11}m_k + e_{15}\beta_k)\frac{\partial_k}{\partial n}\frac{1}{r_k},$$
$$\frac{\partial_k}{\partial n} = n_1\frac{\partial}{\partial y_1} + n_1\frac{\partial}{\partial y_2} + n_3 a_k\frac{\partial}{\partial y_3},$$
$$\frac{\partial}{\partial s_1} = n_2\frac{\partial}{\partial y_3} - n_3\frac{\partial}{\partial y_2},\ \frac{\partial}{\partial s_2} = n_3\frac{\partial}{\partial y_1} - n_1\frac{\partial}{\partial y_3},$$
$$\frac{\partial}{\partial s_3} = n_1\frac{\partial}{\partial y_2} - n_2\frac{\partial}{\partial y_1}. a_0^{(k)} = c_{44}(\alpha_k + \gamma_k) + e_{15}\beta_k,$$
$$b_0^{(k)} = c_{44}\delta_k + e_{15}\eta_k + c_{44}\gamma_k,$$
$$c_0^{(k)} = c_{44}\eta_k + e_{15}m_k + c_{44}\beta_k,\ l_0^{(k)} = -\epsilon_{11}\beta_k + e_{15}\alpha_k + e_{15}\gamma_k,$$
$$\sum_1^4 a_0^{(k)} = \sum_1^4 l_0^{(k)} = \sum_1^4 c_0^{(k)} = 0,\ \sum_1^4 b_0^{(k)} = -1, a_0^{(1)} = -1.$$

Note that all elements $A_{pq}^{(k)}$ have a singularity of type $|x|^{-2}$.

First we introduce the following definitions:

Definition 2. The vector defined by the equality

$$V(x) = \frac{1}{2\pi}\iint_S \Gamma(x-y)h(y)ds,$$

where h is a real vector density, is called a simple-layer potential.

Definition 3. The vector defined by the equality

$$W(x)=\frac{1}{2\pi}\iint\limits_S [T(\partial_y,n)\Gamma(y-x)]^* g(y)ds,$$

is called a double-layer potential.

These potentials are solutions of system (1) both in the domain D^+ and in D^-.

Theorem 3. *If $S\in L_1(\alpha)$ and $g\in C^{1,\beta}(S)$, $0<\beta<\alpha\le 1$, then the vector $W(x)$ is a regular function in $D^\pm$. When the point x tends to any point of the boundary z (from the inside or from the outside) we have the discontinuity formula*

$$W^\pm=\mp g(z)+\frac{1}{2\pi}\iint\limits_S [T(\partial_y,n)\Gamma(y-x)]^* g(y)ds,\quad z\in S. \tag{8}$$

Theorem 4. *If $S\in L_1(\alpha)$ and $h\in C^{0,\beta}(S)$ $0<\beta<\alpha\le 1$, then the vector $V(x)$ is a regular function in $D^\pm$ and*

$$[T(\partial_y,n)V]^\pm=\pm h(z)+\frac{1}{2\pi}\iint\limits_S T(\partial_y,n)\Gamma(y-x)h(y)ds,\quad z\in S. \tag{9}$$

Integral Equations of BVPs. A solution of the BVP $(1)^\pm$ is sought in the form of a double-layer potential, while the solution of BVP $(2)^\pm$ is sought in the form of a simple-layer potential. Then for determining the unknown real vector functions g and h, we obtain the following integral equations of second kind:

$$\mp g(z)+\frac{1}{2\pi}\iint\limits_S [T(\partial_y,n)\Gamma(y-x)]^* g(y)ds=f(z),\quad z\in S, \tag{10}$$

$$\pm h(z)+\frac{1}{2\pi}\iint\limits_S T(\partial_y,n)\Gamma(y-x)h(y)ds=f(z),\quad z\in S. \tag{11}$$

2. Calculation of a Symbolic Matrix

Consider the operator

$$K^{1+}(g)(z)=-g(z)+\frac{1}{2\pi}\iint\limits_S [T(\partial_y,n)\Gamma(y-x)]^* g(y)ds. \tag{12}$$

We follow the results obtained in [1]. According to [1] we have (see [1] for details)

$$\frac{\partial}{\partial s_j}\sum_{k=1}^{4}\alpha_k\frac{\partial^2\Phi_k}{\partial x_1\partial x_2}=-2\pi i\xi_j\Big(\sum_{k=1}^{4}\alpha_k\frac{\partial^2\Phi_k}{\partial x_1\partial x_2}\Big)_{x-y=\xi}$$

$$\approx -2\pi i\xi_1\xi_2\xi_j\sum_{k=1}^{4}\frac{\alpha_k a_k^2}{\rho_k(\rho_k+\xi_3)^2}, \tag{13}$$

where

$$\xi_j=\alpha_{j2}\cos\theta-\alpha_{j1}\sin\theta,\quad j=1,2,3,\quad \rho_k^2=a_k(\xi_1^2+\xi_2^2)+\xi_3^2,\quad k=1,2,3,4.$$

Direct calculations lead to the equality

$$\sum_{k=1}^{4}\frac{\alpha_k a_k^2}{\rho_k(\rho_k+\xi_3)^2}=\sum_{k=1}^{3}\frac{\alpha_k(a_k-a_4)(a_4\rho_k+a_k\rho_4)}{\rho_4\rho_k(\rho_k+\rho_4)^2}. \tag{14}$$

Now we can construct symbolic matrix for the operator K^{1+}, We have

$$\sigma^{1+}(z,\theta)=\begin{pmatrix} -1+i\xi_1\xi_2\xi_3A_1, & -i\xi_3(B_1+\xi_1^2A_1), & -i\xi_2C_1, & -i\xi_2A_2\\ i\xi_3(B_1+\xi_2^2A_1), & -(1+i\xi_1\xi_2\xi_3A_1), & i\xi_1C_1, & i\xi_1A_2\\ i\xi_2D_1, & -i\xi_1D_1, & -1 & 0\\ i\xi_2E_1, & -i\xi_1E_1 & 0 & -1\end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} &A_1=\sum_{k=1}^{3}\frac{(a_k-a_4)}{\rho_4\rho_k(\rho_k+\rho_4)}\Big[a_0^{(k)}-2\alpha_k c_{66}\frac{(a_4\rho_k+a_k\rho_4)}{\rho_k+\rho_4}\Big],\\ &B_1=\frac{1}{\rho_1}+2c_{66}\sum_{k=1}^{3}\frac{\alpha_k(a_k-a_4)}{(\rho_k+\rho_4)},\quad D_1=\sum_{k=1}^{4}\frac{b_0^{(k)}}{\rho_k}+2c_{66}\xi_3^2\sum_{k!=2}^{3}\frac{\gamma_k(a_4-a_k)}{\rho_4\rho_k(\rho_k+\rho_4)},\\ &E_1=\sum_{k=1}^{4}\frac{c_0^{(k)}}{\rho_k}-2c_{66}\xi_3^2\sum_{k=2}^{3}\frac{\beta_k(a_k-a_4)}{\rho_4\rho_k(\rho_k+\rho_4)},\\ &C_1=-\frac{a_1}{\rho_1}-\sum_{k=1}^{3}a_0^{(k)}\frac{(a_k-a_4)}{\rho_k+\rho_4},\quad A_2=-\frac{e_{15}}{c_{66}\rho_1}-\sum_{k=1}^{3}l_0^{(k)}\frac{(a_k-a_4)}{\rho_k+\rho_4}. \end{aligned} \tag{16}$$

From (15) we have

$$\det\sigma^{1+}(z,\theta)=1-\xi_3^2B_1^2-(\xi_1^2+\xi_2^2)^2(C_1D_1+A_2E_1+A_1B_1\xi_3^2). \tag{17}$$

Since $\xi_1^2+\xi_2^2+\xi_3^2=1$, it is possible to take $\xi_1^2+\xi_2^2=\cos\theta, \xi_3^2=\sin\theta$ and we get

$$\det\sigma^{1+}(z,\theta)=1-B_1^2\sin^2\theta-(C_1D_1+A_2E_1+A_1B_1\sin^2\theta)cos^2\theta. \tag{18}$$

3. On Some Auxiliary Formulas

A general representation of a regular solution of equation (1) in the half-space $x_1>0$ have the form

$$u(x)=-\frac{1}{4\pi}\iint_{-\infty}^{+\infty}[T(\partial_y,n)\Gamma(y-x)]^*u^+(y)ds-\Gamma(y-x)(TU)^+]dy_2\,dy_3, \tag{19}$$

where $\Gamma(y-x))$ and $(T(\partial_y,n)\Gamma(y-x))^*$ are given by the formulas (6), (7), if we substitute $n=n(1,0,0)$.

Taking into account the identities [1]

$$
\begin{aligned}
&\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{1}{r_k}\exp\Big(i\sum_{j=2}^{3}p_jy_j\Big)dy_2\,dy_3=\frac{\exp(-x_1\sqrt{p_2^2+a_kp_3^2})}{\sqrt{p_2^2+a_kp_3^2)}}\Big(i\sum_{j=2}^{3}p_jx_j\Big),\\
&\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\Phi_k\exp\Big(i\sum_{j=2}^{3}p_jy_j\Big)dy_2\,dy_3=-\frac{\exp(-x_1\sqrt{p_2^2+a_kp_3^2})}{p_3^2\sqrt{p_2^2+a_kp_3^2)}}\Big(i\sum_{j=2}^{3}p_jx_j\Big),
\end{aligned}
\tag{20}
$$

where

$$
r_k^2=a_k[x_1^2+(x_2-y_2)^2]+(x_3-y_3)^2,\quad \Phi_k=(x_3-y_3)\ln(x_3-y_3+r_k)-r_k,
$$

from (19), after performing the Fourier transform and substituting $x_1=0$, we obtain

$$
\rho A(\widehat{u})^+ + B(\widehat{Tu})^+ = 0, \tag{21}
$$

where $(\widehat{u})^+$ and $(\widehat{Tu})^+$ are the Fourier transforms of functions $(u)^+$ and $(Tu)^+$, $\rho^2=p_2^2+p_3^2$,

$$
A=\begin{pmatrix}
1 & -i\bar{B}_1\cos\theta, & i\bar{C}_1\sin\theta, & i\bar{A}_2\sin\theta\\
i(B_1+\bar{A}_1\sin^2\theta)\cos\theta, & 1 & 0 & 0\\
-i\bar{D}_1\sin\theta, & 0 & 1 & 0\\
-i\bar{E}_1\sin\theta, & 0 & 0 & 1
\end{pmatrix}, \tag{22}
$$

$$
B=\sum_{k=1}^{4}\begin{pmatrix}
-\dfrac{\lambda_k}{R_k}+\dfrac{\alpha_kR_k}{\sin^2\theta} & 0 & 0 & 0\\
0 & -\dfrac{\lambda_k}{R_k}-\dfrac{\alpha_k\coth^2\theta}{R_k} & -\gamma_k\dfrac{\coth\theta}{R_k} & -\beta_k\dfrac{\coth\theta}{R_k}\\
0 & -\gamma_k\dfrac{\coth\theta}{R_k} & -\dfrac{\delta_k}{R_k} & -\dfrac{\eta_k}{R_k}\\
0 & -\beta_k\dfrac{\coth\theta}{R_k} & -\dfrac{\eta_k}{R_k} & -\dfrac{m_k}{R_k}
\end{pmatrix}, \tag{23}
$$

where

$$
\begin{aligned}
&\bar{A}_1=\sum_{k=1}^{3}\frac{(a_k-a_4)}{\bar\rho_4\bar\rho_k(\bar\rho_k+\bar\rho_4)}\Big[a_0^{(k)}-2\alpha_kc_{66}\frac{(a_4\bar\rho_k+a_k\bar\rho_4)}{\bar\rho_k+\bar\rho_4}\Big],\\
&\bar{B}_1=\frac{1}{\bar\rho_1}+2c_{66}\sum_{k=1}^{3}\frac{\alpha_k(a_k-a_4)}{(\bar\rho_k+\bar\rho_4)},\quad \bar{D}_1=\sum_{k=1}^{4}\frac{b_0^{(k)}}{\bar\rho_k}+2c_{66}\xi_3^2\sum_{k=2}^{3}\frac{\gamma_k(a_4-a_k)}{\bar\rho_4\bar\rho_k(\bar\rho_k+\bar\rho_4)},\\
&\bar{E}_1=\sum_{k=1}^{4}\frac{c_0^{(k)}}{\bar\rho_k}-2c_{66}\xi_3^2\sum_{k=2}^{3}\frac{\beta_k(a_k-a_4)}{\bar\rho_4\bar\rho_k(\bar\rho_k+\bar\rho_4)},\quad \bar{C}_1=-\frac{a_1}{\bar\rho_1}-\sum_{k=1}^{3}a_0^{(k)}\frac{(a_k-a_4)}{\bar\rho_k+\bar\rho_4},\\
&\bar{A}_2=-\frac{e_{15}}{c_{66}\bar\rho_1}-\sum_{k=1}^{3}l_0^{(k)}\frac{(a_k-a_4)}{\bar\rho_k+\bar\rho_4},\quad \bar\rho_k^2=p_2^2+a_kp_3^2,\\
&R_k^2=\cos^2\theta+a_k\sin^2\theta.
\end{aligned}
\tag{24}
$$

After certain simplifications we obtain

$$\begin{aligned}&\det A = 1 - \bar{B}_1^2\cos^2\theta - (\bar{C}_1\bar{D}_1 + \bar{A}_2\bar{E}_1 + p_2^2\bar{B}_1\bar{A}_1)\sin^2\theta,\\ &\det B = -\frac{c_{11}}{b_0}\,\alpha_1^2\,\frac{R_1\Delta_{11}^2}{R_1R_3R_4},\end{aligned}\tag{25}$$

where

$$\begin{aligned}\Delta_{11} = {}& \frac{R_1+R_2+R_3+R_4}{\Delta_0}\Big[\frac{b_3}{b_0}\sin^4\theta + a_1R_2R_3R_4(R_2+R_3+R_4)\\ &+ a_1\cos^2\theta(R_3^2 + a_4\sin^2\theta)\Big]\\ &+ \frac{a_1}{R_1}\,\frac{R_2R_3R_4(R_3+R_4) + R_1(R_1+R_2)\cos^2\theta}{(R_1+R_2)(R_3+R_4)(R_1+R_3)(R_1+R_4)}\\ &+ \frac{[(R_3+R_2)R_4+\cos^2\theta+R_2R_3]b_0+c_{11}(c_{33}\epsilon_{11}+c_{44}\epsilon_{33}+2e_{15}e_{33})a_k}{c_{11}b_0(R_3+R_4)(R_4+R_2)(R_3+R_2)}c_{44}>0,\\ \Delta_0 = {}& (R_1+R_2)(R_1+R_3)(R_1+R_4)(R_3+R_2)(R_4+R_2)(R_3+R_4)\\ &> 0,\ a_1 = c_{44}c_{66}^{-1}.\end{aligned}$$

If $\theta=\phi+\frac{\pi}{2}$, then $\bar{B}_1(\phi+\frac{\pi}{2})=B_1$, $\bar{A}_k(\phi+\frac{\pi}{2})=A_k$, $k=1,2$, $\bar{C}_1(\phi+\frac{\pi}{2})=C_1$, $\bar{D}_1(\phi+\frac{\pi}{2})=D_1$, $\bar{E}_1(\phi+\frac{\pi}{2})=E_1$, and $\det\bar{A}(\phi+\frac{\pi}{2})=\det\sigma^{1+}$.

Since $\det B\neq 0$, from (21) we get

$$(\widehat{Tu})^+ = -B^{-1}A(\hat{u})^+\rho.\tag{26}$$

We rewrite (5) as the bilinear form with respect to the vectors v and $\bar{v}$ ($\bar{v}$ is the complex conjugate of vector v), we rewrite

$$\begin{aligned}E(v,\bar{v}) = {}& \sum_{j=1}^{2}\frac{\partial v_4}{\partial x_j}\Big[-\epsilon_{11}\frac{\partial\bar{v}_4}{\partial x_j} + e_{15}\gamma_{j3}(\bar{v})\Big] + \frac{\partial v_4}{\partial x_3}\Big[-\epsilon_{33}\frac{\partial\bar{v}_4}{\partial x_3} + e_{13}(\bar{\gamma}_{11}+\bar{\gamma}_{22})\Big]\\ &+ c_{11}(|\gamma_{11}^2| + |\gamma_{22}^2|) + (c_{11}-2c_{66})(\gamma_{11}\bar{\gamma}_{22}+\gamma_{22}\bar{\gamma}_{11}) + c_{13}\gamma_{12}\bar{\gamma}_{33}\\ &+ e_{13}\frac{\partial\bar{v}_4}{\partial x_3}(\gamma_{11}+\gamma_{22}) + c_{66}(|\gamma_{13}^2| + |\gamma_{23}^2|) + e_{15}\Big[\frac{\partial\bar{v}_4}{\partial x_1}\gamma_{13}\\ &+ \frac{\partial\bar{v}_4}{\partial x_2}\gamma_{23}\Big] + \gamma_{33}\Big[c_{13}(\bar{\gamma}_{11}+\bar{\gamma}_{22}) + c_{33}\bar{\gamma}_{33} + e_{33}\frac{\partial v_4}{\partial x_3}\Big] > 0,\end{aligned}\tag{27}$$

where

$$\gamma_{j3} = \frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3},\ \ j=1,2,\quad \gamma_{12} = \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2},\quad \gamma_{jj} = \frac{\partial v_j}{\partial x_j},\ \ j=1,2,3.$$

Let us assume that

$$v(x) = M(x_1,p_2,p_3)\exp\Big(-i\sum_{j=2}^{3}p_jx_j\Big),\quad M=(M_1,M_2,M_3,M_4).\tag{28}$$

Then (26) implies

$$
\begin{aligned}
E(v,\bar v) &= -\epsilon_{11}\Big|\frac{dM_4}{dx_1}\Big|^2 - \epsilon_{33}p_3^2|M_4|^2 - \epsilon_{11}p_2^2|M_4|^2 + c_{11}\Big(p_2^2|M_2|^2 + \Big|\frac{dM_1}{dx_1}\Big|^2\Big) \\
&+ c_{66}\Big(\Big|\frac{dM_2}{dx_1}\Big|^2 + p_2^2|M_1|^2\Big) + c_{44}\Big[\Big|\frac{dM_3}{\partial x_1}\Big|^2 + p_3^2(|M_1|^2 + |M_2|^2) + p_2^2|M_3|^2\Big] \\
&+ c_{33}p_3^2|M_3|^2 + (c_{13}+c_{44})p_2p_3(\bar M_2 M_3 + \bar M_3 M_2) \\
&+ (e_{13}+e_{15})p_2p_3(\bar M_2 M_4 + \bar M_4 M_2) + ip_2c_{12}\frac{d}{dx_1}(\bar M_2 M_1 - \bar M_1 M_2) \\
&+ (c_{11}-c_{66})ip_2\Big(\bar M_1\frac{dM_2}{dx_1} - M_1\frac{d\bar M_2}{dx_1}\Big) + (e_{33}p_3^2 + e_{15}p_2^2)(\bar M_4 M_3 + \bar M_3 M_4) \\
&+ e_{13}ip_3\frac{d}{dx_1}(\bar M_4 M_1 - \bar M_1 M_4) + e_{15}\Big(\frac{d\bar M_4}{dx_1}\frac{dM_3}{dx_1} + \frac{dM_4}{dx_1}\frac{d\bar M_3}{dx_1}\Big) \\
&+ (e_{13}+e_{15})ip_3\Big(\frac{dM_4}{dx_1}\bar M_1 - \frac{d\bar M_4}{dx_1}M_1\Big) \\
&+ (c_{13}+c_{44})ip_3\Big(\frac{dM_3}{dx_1}\bar M_1 - \frac{d\bar M_3}{dx_1}M_1\Big) + ip_3c_{13}\Big(\frac{dM_1}{dx_1}\bar M_3 - \frac{d\bar M_1}{dx_1}M_3\Big).
\end{aligned}
$$

Let

$$
u(x) = \frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} M(x_1,p_2,p_3)\exp\Big(-i\sum_{j=2}^{3}P_jx_j\Big)dp_2\,dp_3.
$$

In order that the vector $u(x)$ be a solution of system (1) it is necessary and sufficient that

$$
\begin{gathered}
C\Big(\frac{d}{dx_1}, -ip_2, -ip_3\Big)M(x_1,p_2,p_3) = 0, \quad x_1 > 0, \\
\Big[\frac{d^jM_k}{dx_1^j}\Big]_{x_1=\infty} = 0, \quad j = 0,1, \quad k = 1,\dots,4.
\end{gathered}
$$

Consider the expression $\int_0^h\sum_{k=1}^4 \bar M_k[C(\frac{d}{dx_1}, -ip_2, -ip_3)M]_k dx_1$, where h is an arbitrary positive number. After some transformation we obtain

$$
\int\limits_0^\infty E(v,\bar v)dx_1 = -\hat{\bar u}^+(\widehat{\bar T u})^+, \tag{29}
$$

where

$$
\begin{aligned}
(\widehat{Tu})_1 &= -c_{12}ip_2M_2 + c_{11}\frac{dM_1}{dx_1} - ip_3(e_{13}M_4 + c_{13}M_3), \\
(\widehat{Tu})_2 &= -c_{66}ip_2M_1 + c_{66}\frac{dM_2}{dx_1}, \\
(\widehat{Tu})_3 &= -c_{44}ip_3M_1 + c_{44}\frac{dM_3}{dx_1} + e_{15}\frac{dM_4}{dx_1},
\end{aligned}
$$

$$(\widehat{Tu})_4 = -e_{15} i p_3 M_1 + e_{15} \frac{dM_3}{dx_1} - \epsilon_{11} \frac{dM_4}{dx_1}.$$

Substituting (25) into (28), we get

$$\int\limits_0^\infty E(v, \bar{v}) dx_1 = \rho (\widehat{\bar{u}})^+ B^{-1} A (\widehat{\bar{u}})^+. \tag{30}$$

Taking into account the fact that the energy $E(v, \bar{v})$ is positive definite, from (29) we conclude that the matrix $B^{-1}A$ is positive definite.

In particular from (29) we have

$$\det(B^{-1}A) = \det B^{-1} \det A > 0$$

which gives $\det A < 0$. Hence $\det \sigma^{1+} \neq 0$.

Analogously, we obtain $\det \sigma^{1-} \neq 0$, $\det \sigma^{\pm} \neq 0$.

References

1. M. O. Basheleishvili and D. G. Natroshvili, Application of singular integral equations in boundary value problems of the statics of a transversally isotropic body. (Russian) *Differential and Integral Equations. Boundary Value Problems.* (*Russian*) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **10**(1979) 11–31.
2. V. Z. Parton and B. A. Kudryavtsev, Electromagnetoelasticity of piezoelectric and electroconductive bodies. (Russian) *Nauka, Moscow*, 1988.
3. A. F. Ulitko, About oscillation theory for a piezoceramic body. (Russian) *Thermal Stresses in Structure Elements.* (Russian) No. 15, 90–99, *Naukova Dumka, Kiev*, 1975.
4. L. P. Bitsadze, Some solutions of the electroelasticity equations. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **16**(1985), 19–33.
5. N. I. Muskhelishvili, Singular integral equations. Boundary value problems of the theory of functions and some of their applications in mathematical physics. (Russian) 3rd ed. *Nauka, Moscow*, 1968; English translation from 2nd Russian ed. (1946): *P. Noordhoff, Groningen*, 1953, corrected reprint *Dover Publications, Inc. , N. Y.*, 1992.

(Received 22.09.2000)

Author's address:
I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 380043
Georgia
E-mail: lbits@viam.hepi.edu.ge