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SOME REMARKS ON FRACTALS GENERATED BY A SEQUENCE OF FINITE SYSTEMS OF CONTRACTIONS

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Abstract. We generalize some results shown by J. E. Hutchinson in [7].

Let $\mathfrak{F}_n = \{f_1^{(n)}, f_2^{(n)}, \dots, f_{m_n}^{(n)}\}$ be finite systems of contractions on a complete metric space; then, under some conditions on $(\mathfrak{F}_n)$, there exists a unique non-empty compact set K such that the sequence defined by $((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C))$ converges to K in the Hausdorff metric for every non-empty closed and bounded set C.

If the metric space is also separable and for every n, $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)}$ there are real numbers strictly between 0 and 1, satisfying the condition $\sum_{j=1}^{m_n} l_j^{(n)} = 1$, then there exists a unique probability Radon measure μ_K such that the sequence

$$\nu_n = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l_{i_k}^{(k)} \Big) \big(f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \cdots \circ f_{i_n}^{(n)} \big)_\sharp \nu$$

weakly converges to μ_K for every probability Borel regular measure ν with bounded support (where by $f_\sharp \nu$ we denote the image measure of ν under a contraction f). Moreover, K is the support of μ_K.

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1. Introduction

Let (X, d_X) be a complete separable metric space and let $f_1, f_2, \dots, f_M : X \to X$ be contractions. In [7] it is proved that there exists a unique non-empty closed and bounded subset K of X invariant with respect to $\mathfrak{F} = \{f_1, f_2, \dots, f_M\}$ i.e., such that

$$K = \mathfrak{F}(K) = \bigcup_{j=1}^{M} \overline{f_j(K)}. \tag{1}$$

Moreover, K is compact and if $C_0 \neq \varnothing$ is closed and bounded, then the sequence (C_n) defined by $C_n = \mathfrak{F}(C_{n-1})$ converges to K in the Hausdorff metric.

Let $r = \{r_1, r_2, \dots, r_M\}$ be a family of M real numbers in $]0,1[$ with $\sum_{j=1}^{M} r_j = 1$. Then there exists a unique Borel regular (outer) measure μ in

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X with compact support and of total mass 1 such that μ is invariant with respect to $(\mathfrak{F}, r)$, i.e.,

$$\mu(A) = \sum_{j=1}^{M} r_j \mu(f_j^{-1}(A)) \quad \text{for every Borel set } A \subseteq X. \tag{2}$$

Furthermore, the support of μ is the fractal K.

We consider the case in which the system $\mathfrak{F}$ is replaced by a sequence $(\mathfrak{F}_n)$ of finite systems of contractions, i.e., $\mathfrak{F}_n = \{f_1^{(n)}, f_2^{(n)}, \dots, f_{m_n}^{(n)}\}$ with $m_n \geq 2$. Obviously, we cannot write an expression like (1), but we can still construct a sequence of closed and bounded subsets of X and ask if such a sequence is convergent with respect to the Hausdorff metric. More precisely, if the sequence $(\mathfrak{F}_n)$ satisfies the following two conditions:

- there exists a bounded set $Q \subseteq X$ such that $\bigcup_{j=1}^{m_n} f_j^{(n)}(Q) \subseteq Q$ for any $n \in \mathbb{N}$;
- $\lim_n \prod_{k=1}^{n} \rho^{(k)} = 0$; here $\rho^{(k)}$ is the greatest of the Lipschitz constants of the contractions $f_1^{(k)}, f_2^{(k)}, \dots, f_{m_k}^{(k)}$;

then there exists a unique non-empty closed and bounded set $K \subseteq X$ such that the sequence $((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \dots \circ \mathfrak{F}_n)(C_0))$ converges to K in the Hausdorff metric, for every non empty closed and bounded subset C_0 of X. Moreover K is compact.

As an interesting example, given $d \in]0,1[$, we construct a d-dimensional compact subset of the real line by considering a sequence $(\mathfrak{F}_n)$ of finite systems of contractive similitudes $f_1^{(n)}, f_2^{(n)}, \dots, f_{m_n}^{(n)}$ with Lipschitz constants $\rho^{(n)}$ (depending only on n) such that $m_n(\rho^{(n)})^d = 1$. We will study the entropy numbers related to this set.

In Section 4 we consider a generalization of the invariant measure found in [7].

As before, we cannot write an expression like (2). Let X be a complete separable metric space and let for every n, $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)} \in]0,1[$ be so that $\sum_{j=1}^{m_n} l_j^{(n)} = 1$, then there exists a unique Radon probability measure μ_K so that for every Radon probability measure ν on X, with bounded support, the sequence of measures defined by

$$\nu_n(A) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \left(\prod_{k=1}^{n} l_{i_k}^{(k)} \right) \nu\Big(\big(f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \dots \circ f_{i_n}^{(n)}\big)^{-1}(A) \Big),$$

for Borel sets $A \subseteq X$, weakly converges to μ_K.

Moreover, the support of μ_K is the fractal K.

2. Notation and preliminary results

In this note (X, d_X) will always be a complete metric space. Additional requirements for X will be specified when necessary.

$\mathbb{N} = \{1, 2, \dots\}$ is the set of all positive integer numbers.

The closed and open balls in X will be indicated by the symbols $B_X(x_0, r)$ and $D_X(x_0, r)$:

$$B_X(x_0,r)=\Big\{x \in X \mid d_X(x,x_0) \le r\Big\},\ \ D_X(x_0,r)=\Big\{x \in X \mid d_X(x,x_0) < r\Big\}.$$

The diameter of a subset A of X is indicated by $|A|$: $|A| = \sup\limits_{x,y\in A} d_X(x,y)$ and its number of elements is indicated by $\sharp A$.

If X is separable and $s \ge 0$, $\mathcal{H}^s(A)$ stands for the s-dimensional Hausdorff measure of A and $\dim A$ for its Hausdorff dimension.

If $X = \mathbb{R}^N$ then we will use the Euclidean metric $d_{\mathbb{R}^N}(x,y) = \|x-y\|_2 = \sqrt{\sum_{i=1}^N (\xi_i - \eta_i)^2}$, where $x = (\xi_1, \xi_2, \ldots, \xi_N)$ and $y = (\eta_1, \eta_2, \ldots, \eta_N)$.

Lemma 2.1. *Let $E \subseteq \mathbb{R}^N$ and let $d > 0$. If $f : E \to \mathbb{R}^N$ is a mapping and $c > 0$ is a constant such that $\|f(x)-f(y)\|_2 \le c\|x-y\|_2$ for every $x, y \in E$, then $\mathcal{H}^d(f(E)) \le c^d \mathcal{H}^d(E)$.*

Proof. See [4], Chapter 2, Proposition 2.2. □

2.1. d-sets in $\mathbb{R}^N$.

Definition. Let Γ be a closed non-empty subset of $\mathbb{R}^N$ and let $d \in]0,N]$. A positive Borel outer measure μ with support Γ is called a d-measure on Γ if there exist $c_1, c_2 \in]0,+\infty[$ such that for every $x_0 \in \Gamma$ and for every $r \in]0,1]$

$$c_1 r^d \le \mu\Big(B_{\mathbb{R}^N}(x_0,r)\Big) \le c_2 r^d \tag{1}$$

holds.

Remark 1. One can replace the condition $r \in]0,1]$ in the above definition by the condition $r \in]0,r_0]$; obviously, the constants c_1 and c_2 will be replaced by some constants $c_1(r_0) > 0$ and $c_2(r_0) > 0$ depending on r_0.

Definition. A closed non-empty subset Γ of $\mathbb{R}^N$ is called a d-set if there exists a d-measure on Γ.

It can be proved that if Γ is a d-set, μ_1 and μ_2 are d-measures on Γ. Then there exist constants $a, b \in]0,+\infty[$ such that $a\mu_1(A) \le \mu_2(A) \le b\mu_1(A)\ \forall A \subseteq \mathbb{R}^N$. Moreover, the restriction to Γ of the Hausdorff measure $\mathcal{H}^d$ is a d-measure on Γ; so every d-set has its canonical d-measure and therefore d is unique.

For the proof of these facts see [9], Chapter 2 or [11], Chapter 1.

2.2. Entropy numbers. Let Ω be a bounded subset of X. The n-th entropy number of Ω is defined by

$$\varepsilon_n(\Omega) = \inf\Big\{\varepsilon > 0 \mid \exists\, x_1, x_2, \ldots, x_n \in X \text{ such that } \Omega \subseteq \bigcup_{i=1}^n B_X(x_i,\varepsilon)\Big\}.$$

The sequence $(\varepsilon_n(\Omega))$ is monotonically decreasing and tends to zero if and only if Ω is precompact.

See [1] for a complete treatment.

2.3. The Hausdorff metric.

Definition. If $x_0 \in X$ and $A \subseteq X$, we define the distance between x_0 and A by

$$d_X(x_0, A) = \inf_{x \in A} d_X(x_0, x).$$

Remark 2. For every $x_0 \in X$ and $A \subseteq X$,we have $d_X(x_0, A) = d_X(x_0, \overline{A})$.

Definition. Let $\mathfrak{B}$ be the class of all non-empty closed bounded subsets of X.

The Hausdorff metric D on $\mathfrak{B}$ is defined by

$$D(A, B) = \sup\Big\{d_X(x, B), d_X(y, A) \mid x \in A,\ y \in B\Big\}.$$

Remark 3. D is a metric on $\mathfrak{B}$. Moreover, for every $A, B \in \mathfrak{B}$

$$D(A, B) = \inf\Big\{\varepsilon > 0 \mid A \subseteq \bigcup_{y \in B} D_X(y, \varepsilon) \text{ and } B \subseteq \bigcup_{x \in A} D_X(x, \varepsilon)\Big\}.$$

Lemma 2.2. *Let* $f : X \to X$ *be a Lipschitz function and let* $\rho = \inf\Big\{c > 0 \mid d_X(f(x), f(y)) \le c d_X(x, y) \ \ \forall x, y \in X\Big\}$ *be its Lipschitz constant. Then*

$$D(\overline{f(A)}, \overline{f(B)}) \le \rho D(A, B) \ \ \forall A, B \in \mathfrak{B}. \tag{2}$$

Proof. By remark 2 $D(\overline{f(A)}, \overline{f(B)}) = \sup\Big\{d_X(u, f(B)), d_X(v, f(A)) \mid u \in f(A),\ v \in f(B)\Big\}$.

By the Lipschitz condition on f it follows that $d_X(f(x), f(B)) \le \rho d_X(x, B)$ and $d_X(f(y), f(A)) \le \rho d_X(y, A)$, $\forall x \in A \ \ \forall y \in B$ and then the (2). $\square$

Lemma 2.3. *Let* $\{A_j \mid j \in J\}$, $\{B_j \mid j \in J\}$ *be two families of elements of* $\mathfrak{B}$. *Then*

$$D\Big(\overline{\bigcup_{j \in J} A_j}, \overline{\bigcup_{j \in J} B_j}\Big) \le \sup_{j \in J} D(A_j, B_j)$$

provided that $\bigcup_{j \in J} A_j$ *and* $\bigcup_{j \in J} B_j$ *are bounded.*

Proof. Let $c > \sup_{j \in J} D(A_j, B_j)$: for all $j \in J$ $D(A_j, B_j) < c$ and then

$$A_j \subseteq \bigcup_{y_j \in B_j} D_X(y_j, c) \subseteq \bigcup_{i \in J} \bigcup_{y_i \in B_i} D_X(y_i, c) = \bigcup_{y \in \bigcup_{i \in J} B_i} D_X(y, c).$$

It follows that

$$\bigcup_{j \in J} A_j \subseteq \bigcup_{y \in \bigcup_{i \in J} B_i} D_X(y, c).$$

In the same way we obtain

$$\bigcup_{j \in J} B_j \subseteq \bigcup_{x \in \bigcup_{j \in J} A_j} D_X(x, c).$$

Then

$$D\Big(\overline{\bigcup_{j \in J} A_j}, \overline{\bigcup_{j \in J} B_j}\Big) < c. \qquad \square$$

2.4. Sequences of indices and product spaces. From now on, (m_n) is a fixed sequence of integer numbers, with $m_n \geq 2$ for all $n \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$ we fix $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)} \in]0,1[$ so that $\sum_{j=1}^{m_n} l_j^{(n)} = 1$.

Definition. For $n \in \mathbb{N}$ we set $I_n = \{1, 2, \dots, m_n\}$ and $I = \prod_{n=1}^{+\infty} I_n$. Each I_n is equipped with the discrete topology. On I we consider the function $d_I : I \times I \to \mathbb{R}$,

$$d_I(k,h) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \frac{|k_j - h_j|}{1 + |k_j - h_j|},$$

where $k = (k_j)$, $h = (h_j)$, $k_j, h_j \in I_j$ for all $j \in \mathbb{N}$.

Remark 4. It is well known that d_I is a metric on I; moreover d_I induces the product topology on I.

It follows that (I, d_I) is a compact metric space and then it is complete and separable (see, for example, [2], Chapters 2, 5 and 6).

Definition. Given $i_1, i_2, \dots, i_n \in \mathbb{N}$, we define the natural projection $\pi_{i_1, i_2, \dots, i_n} : I \to \prod_{j=1}^{n} I_{i_j}$ by

$$\pi_{i_1, i_2, \dots, i_n}(k) = (k_{i_1}, k_{i_2}, \dots, k_{i_n}).$$

Remark 5. For every $i_j \in \mathbb{N}$, π_{i_j} is a continuous function by the definition of product topology. Then $\pi_{i_1, i_2, \dots, i_n}$ is continuous.

Definition. Let X and Y be metric spaces, μ an outer measure on X and $f : X \to Y$ a function.

The image of μ under f is defined by

$$f_\sharp \mu(A) = \mu(f^{-1}(A)) \quad \forall A \subseteq Y.$$

For the proof of the following two theorems see [10], Chapter 1, Theorems 1.18 and 1.19.

Theorem 2.1. *Let X and Y be separable metric spaces. If $f : X \to Y$ is continuous and μ is a Radon measure on X with compact support, then $f_\sharp \mu$ is a Radon measure. Moreover, if $C \subseteq X$ is the support of μ, then $f(C)$ is the support of $f_\sharp \mu$.*

Definition. Let X and Y be separable metric spaces. A mapping $f : X \to Y$ is a Borel mapping if $f^{-1}(U)$ is a Borel set for every open set $U \subseteq Y$.

Let $A \subseteq X$ be a Borel set. A function $g : A \to [-\infty, +\infty]$ is a Borel function if the set $\{x \in A \mid f(x) < c\}$ is a Borel set for every $c \in \mathbb{R}$.

Theorem 2.2. *Let X and Y be separable metric spaces and suppose that $f : X \to Y$ is a Borel mapping, μ is a Borel measure on X and g is a non-negative Borel function on Y. Then*

$$\int_Y g \, d f_\sharp \mu = \int_X (g \circ f) d\mu.$$

Definition. For every $n \in \mathbb{N}$ we define a measure τ_n on I_n by

$$\tau_n(A) = \sum_{j \in A} l_j^{(n)} \quad \forall A \subseteq I_n.$$

Remark 6. For every $n \in \mathbb{N}$, τ_n is a Radon measure and $\tau_n(I_n) = 1$.

Remark 7. From the definition of product measure of two measures it follows that

$$(\tau_1 \times \tau_2 \times \cdots \times \tau_n)(A) = \sum_{(k_1,k_2,\ldots,k_n) \in A} \prod_{j=1}^{n} l_{k_j}^{(j)} \quad \forall n \in \mathbb{N} \ \ \forall A \subseteq \prod_{j=1}^{n} I_j.$$

In order to define the product measure on I, we need the following theorem.

Theorem 2.3. *Let $\{X_\alpha \mid \alpha \in A\}$ be a family of compact Hausdorff spaces and let, for each $\alpha \in A$, μ_α be a Radon measure on X_α, with $\mu_\alpha(X_\alpha) = 1$.*
Then there exists a unique Radon measure μ on $\prod_{\alpha \in A} X_\alpha$ such that $\mu(\prod_{\alpha \in A} X_\alpha) = 1$ and $\mu_{\alpha_1} \times \mu_{\alpha_2} \times \cdots \times \mu_{\alpha_n} = \pi_{\alpha_1,\alpha_2,\ldots,\alpha_n \sharp}\mu$ for any distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$.

Proof. See [5], Chapter 9, Theorem 9.19. □

Remark 8. By the previous theorem there is a unique Radon measure τ on I such that $\tau(I) = 1$ and $\tau_{i_1} \times \tau_{i_2} \times \cdots \times \tau_{i_n} = \pi_{i_1,i_2,\ldots,i_n \sharp}\tau$ for any distinct $i_1, i_2, \ldots, i_n \in \mathbb{N}$.

3. Limit sets

3.1. Basic notation. From now on we will use the following notation:

- for any $n \in \mathbb{N}$ $i \in I_n$, $f_i^{(n)} : X \to X$ is a contraction;
  - $\rho_i^{(n)} = \inf\{c > 0 \mid d_X(f_i^{(n)}(x), f_i^{(n)}(y)) \le c d_X(x,y) \ \ \forall x, y \in X\}$ ($\rho_i^{(n)}$ is the Lipschitz constant of $f_i^{(n)}$),
  - $\rho^{(n)} = \max\{\rho_1^{(n)}, \rho_2^{(n)}, \ldots, \rho_{m_n}^{(n)}\}$,
  - $\rho = \sup_{n \in \mathbb{N}} \rho^{(n)}$,
  - $x_i^{(n)} \in X$ is the fixed point of $f_i^{(n)}$;
- for every $n \in \mathbb{N}$
  - $\mathfrak{F}_n = \{f_1^{(n)}, f_2^{(n)}, \ldots, f_{m_n}^{(n)}\}$,
  - for every $A \subseteq X$ $\mathfrak{F}_n(A) = \overline{\bigcup_{i=1}^{m_n} f_i^{(n)}(A)}$,
  - for every $A \subseteq X$ $(\mathfrak{F}_n \circ \mathfrak{F}_{n+1})(A) = \mathfrak{F}_n(\mathfrak{F}_{n+1}(A))$;
- for every $i \in I$, $n, k \in \mathbb{N}$ with $k \le n$:
  - $f_{i_1 i_2 \cdots i_n} = f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \cdots \circ f_{i_n}^{(n)}$ and $x_{i_1 i_2 \cdots i_n}$ is its fixed point,
  - $f_{i_k i_{k+1} \cdots i_n}^{(k)} = f_{i_k}^{(k)} \circ f_{i_{k+1}}^{(k+1)} \circ \cdots \circ f_{i_n}^{(n)}$ and $x_{i_k i_{k+1} \cdots i_n}^{(k)}$ is its fixed point;
- $F = \{x_i^{(n)} \mid n \in \mathbb{N} \ i \in I_n\}$ is the set of all fixed points of the contractions $f_i^{(n)}$.

3.2. Existence and uniqueness.

Lemma 3.1. *Let* (g_n) *be a sequence of contraction maps on* X*, each of them with the Lipschitz constant* ρ_n*. Let us suppose that the following two conditions hold:*

(1) *there exists a non-empty closed and bounded set* $Q \subseteq X$ *such that* $g_n(Q) \subseteq Q$ *for every* $n \in \mathbb{N}$*;*
(2) $\lim_n \prod_{k=1}^n \rho_k = 0$.

Then there exists a unique $x \in X$ *so that*

$$\lim_n (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0) = x \quad for\ every \quad x_0 \in X.$$

Moreover, $x \in Q$*.*

Proof. It is easy to prove, by induction, that for every $x, y \in X$ and $n \in \mathbb{N}$

$$d_X((g_1 \circ g_2 \circ \cdots \circ g_n)(x), (g_1 \circ g_2 \circ \cdots \circ g_n)(y)) \leq (\prod_{k=1}^n \rho_k) d_X(x, y).$$

Now, let $x_0 \in Q$ and $\varepsilon > 0$; by the second hypothesis there exists $n_\varepsilon \in \mathbb{N}$ so that $\prod_{k=1}^n \rho_k < \varepsilon/|Q|$ $\forall n \in \mathbb{N}$ with $n > n_\varepsilon$.

Then, for all $m, n \in \mathbb{N}$ with $m > n > n_\varepsilon$ we have

$$\begin{aligned} &d_X\Big((g_1 \circ g_2 \circ \cdots \circ g_n)(x_0), (g_1 \circ g_2 \circ \cdots \circ g_m)(x_0)\Big) \\ &\leq \Big(\prod_{k=1}^n \rho_k\Big) d_X\Big(x_0, (g_{n+1} \circ g_{n+2} \circ \cdots \circ g_m)(x_0)\Big) < \varepsilon. \end{aligned}$$

Since X is complete, there exists $x \in X$ such that $\lim_n (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0) = x$.

Now we prove that x does not depend on x_0.

Let $y_0 \in X$ and $n \in \mathbb{N}$:

$$\begin{aligned} d_X\Big(x, (g_1 \circ g_2 \circ \cdots \circ g_n)(y_0)\Big) & d_X\Big(x, (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0)\Big) \\ &+ d_X\Big((g_1 \circ g_2 \circ \cdots \circ g_n)(x_0), (g_1 \circ g_2 \circ \cdots \circ g_n)(y_0)\Big) \\ &\leq d_X\Big(x, (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0)\Big) + \Big(\prod_{k=1}^n \rho_k\Big) d_X(x_0, y_0) \end{aligned}$$

and by letting $n \to +\infty$ we obtain $x = \lim_n (g_1 \circ g_2 \circ \cdots \circ g_n)(y_0)$. $\square$

Definition. Let $x_0 \in X$ be fixed; we define $\forall n \in \mathbb{N}$ a function $p_n : \prod_{j=1}^n I_j \to X$ by

$$p_n(i_1, i_2, \ldots, i_n) = f_{i_1 i_2 \cdots i_n}(x_0).$$

Remark 1. Obviously, p_n depends on x_0 and it is a continuous function on $\prod_{j=1}^n I_j$.

From now on we will suppose that the following two hypotheses are valid:

(1) there exists a non-empty closed bounded set $Q \subseteq X$ such that $\mathfrak{F}_n(Q) \subseteq Q$ for all $n \in \mathbb{N}$;

(2) $\lim_n \prod_{k=1}^{n} \rho^{(k)} = 0$.

Remark 2. The hypotheses 1 and 2 above are implied by the following:

3. F is bounded;
4. $\rho < 1$.

Indeed, it is obvious that 4 $\Rightarrow$ 2; moreover, let

$$Q = \bigcap_{n=1}^{+\infty} \bigcap_{i=1}^{m_n} B_X\Big(x_i^{(n)}, \frac{|F|}{1-\rho}\Big);$$

we prove that Q satisfies 1.

Q is closed and bounded; moreover $F \subseteq Q$.

Let $n, k \in \mathbb{N}$, $i \in I_n$ and $j \in I_k$; we have $\forall x \in Q$

$$\begin{aligned} d_X(f_i^{(n)}(x), x_j^{(k)}) &\le d_X(f_i^{(n)}(x), x_i^{(n)}) + d_X(x_i^{(n)}, x_j^{(k)}) \\ &\le d_X(f_i^{(n)}(x), f_i^{(n)}(x_i^{(n)})) + |F| \le \rho_i^{(n)} d_X(x, x_i^{(n)}) + |F| \\ &\le \rho \frac{|F|}{1-\rho} + |F| = \frac{|F|}{1-\rho} \end{aligned}$$

and then

$$f_i^{(n)}(x) \in B_X\Big(x_j^{(k)}, \frac{|F|}{1-\rho}\Big) \quad \forall k \in \mathbb{N} \quad \forall j \in I_k.$$

Definition. Let $x_0 \in X$ be fixed; we define a function $p : I \to X$ by

$$p(k) = \lim_n f_{k_1 k_2 \cdots k_n}(x_0).$$

Remark 3. By Lemma 3.1 the function p is well defined and does not depend on $x_0 \in X$.

If we take $x_0 \in Q$, then we would see that $p(I) \subseteq Q$. We will always suppose $x_0 \in Q$.

Definition. We denote the set $p(I)$ by K.

Proposition 3.1. *The function p is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ be so that $\prod_{i=1}^{n_\varepsilon} \rho^{(i)} < \varepsilon/|Q|$. Let $\delta = 2^{-n_\varepsilon - 1}$; for every $k, h \in I$, $d_I(k, h) < \delta$ implies $k_j = h_j$ $\forall j \in \mathbb{N}$ with $j \le n_\varepsilon$ and then, if we suppose $x_0 \in Q$,

$$\begin{aligned} &d_X(p(k), p(h)) \\ &\quad = \lim_j d_X\Big(f_{k_1 k_2 \cdots k_{n_\varepsilon}}(f^{(n_\varepsilon+1)}_{k_{n_\varepsilon+1} k_{n_\varepsilon+2} \cdots k_j}(x_0)), f_{k_1 k_2 \cdots k_{n_\varepsilon}}(f^{(n_\varepsilon+1)}_{h_{n_\varepsilon+1} h_{n_\varepsilon+2} \cdots h_j}(x_0))) \\ &\quad \le \Big(\prod_{i=1}^{n_\varepsilon} \rho^{(i)}\Big) \lim_j d_X\Big(f^{(n_\varepsilon+1)}_{k_{n_\varepsilon+1} k_{n_\varepsilon+2} \cdots k_j}(x_0), f^{(n_\varepsilon+1)}_{h_{n_\varepsilon+1} h_{n_\varepsilon+2} \cdots h_j}(x_0)\Big) < \frac{\varepsilon}{|Q|}|Q| = \varepsilon. \quad \square \end{aligned}$$

Corollary 3.2. *K is compact and $K \subseteq Q$.*

Proposition 3.3. *For every $C \in \mathfrak{B}$ $\lim_n D((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), K) = 0$.*

Proof. Let $x_0 \in Q$, $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ be so that $\prod_{j=1}^{n_\varepsilon} \rho^{(j)} < \varepsilon/|Q|$. For every $n \in \mathbb{N}$ with $n > n_\varepsilon$ we have $D(p_n(\pi_{1,2,\ldots,n}(I)), K) \le \varepsilon$.

Indeed, for every $k \in I$, we have $p(k) \in K$ and

$$\begin{aligned} d_X\Big(p_n(\pi_{1,2,\ldots,n}(k)), p(k)\Big) &\le \Big(\prod_{j=1}^{n} \rho^{(j)}\Big) d_X\Big(x_0, \lim_i f^{(n+1)}_{k_{n+1}k_{n+2}\cdots k_i}(x_0)\Big) \\ &< \Big(\prod_{j=1}^{n_\varepsilon} \rho^{(j)}\Big)|Q| < \varepsilon \end{aligned}$$

from which $d_X(p_n(\pi_{1,2,\ldots,n}(k)), K) < \varepsilon$. On the other hand, $\forall x \in K$ there exists $k \in I$ so that $p(k) = x$ and then $d_X(p_n(\pi_{1,2,\ldots,n}(I)), x) < \varepsilon$.

But

$$p_n(\pi_{1,2,\ldots,n}(I)) = \bigcup_{k_1=1}^{m_1} \bigcup_{k_2=1}^{m_2} \cdots \bigcup_{k_n=1}^{m_n} \{f_{k_1k_2\cdots k_n}(x_0)\} = (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\}).$$

Now, let $C \in \mathfrak{B}$: by Lemmas 2.3 and 2.2 it follows that $\forall n \in \mathbb{N}$

$$D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\})\Big) \le \Big(\prod_{j=1}^{n} \rho^{(j)}\Big) D(C, \{x_0\});$$

then, if $n > n_\varepsilon$,

$$\begin{aligned} &D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), K\Big) \\ &\quad\le D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\})\Big) \\ &\quad+ D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\}), K\Big) < \varepsilon\Big(1 + \frac{D(C, \{x_0\})}{|Q|}\Big). \qquad \square \end{aligned}$$

3.3. Some properties of K. We follow the notation of the previous paragraphs.

Definition. Given a finite family of contraction maps $\mathcal{G} = \{g_1, g_2, \ldots, g_m\}$ and a subset A of X, we say that A is invariant with respect to $\mathcal{G}$ if $\mathcal{G}(A) = A$.

Remark 4. If $\mathfrak{F}_n = \mathfrak{F} = \{f_1, f_2, \ldots, f_m\}$ for every $n \in \mathbb{N}$, then K is invariant with respect to $\mathfrak{F}$.

If there exists $k \in \mathbb{N}$ such that $\mathfrak{F}_{n+k} = \mathfrak{F}_n$ for all $n \in \mathbb{N}$, then K is invariant with respect to $\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_k$.

Remark 5. If there exists $n \in \mathbb{N}$ such that $\mathfrak{F}_{n+k} = \mathfrak{F}_n$ for all $k \in \mathbb{N}$, then there is a unique non-empty compact set $H \subseteq X$ which is invariant with respect to $\mathfrak{F}_n$. Moreover,

$$K = \bigcup_{i_1=1}^{m_1} \bigcup_{i_2=1}^{m_2} \cdots \bigcup_{i_n=1}^{m_n} f_{i_1i_2\cdots i_n}(H).$$

Lemma 3.2. *Let $A = \bigcup_{j=1}^{N} A_j \subseteq X$. If A is connected, then $|A| \le \sum_{j=1}^{N} |A_j|$.*

Proof. We consider the case $N = 2$, the general statement follows by induction.

Let $A = B \bigcup C$ be a connected subset of X and let $x, y \in A$. If $x, y \in B$ or $x, y \in C$; then $d_X(x, y) \leq |B| + |C|$.

If $x \in B$ and $y \in C$, then $\forall z \in B$, $\forall w \in C$,

$$d_X(x, y) \leq d_X(x, z) + d_X(z, w) + d_X(w, y) \leq |B| + d_X(z, w) + |C|.$$

It follows that

$$|A| \leq |B| + |C| + \inf\Big\{d_X(z, w) \mid z \in B,\ w \in C\Big\}.$$

Let us suppose that $\inf\{d_X(z, w) \mid z \in B,\ w \in C\} = \varepsilon > 0$; then $B' = \bigcup_{x\in B} D_X(x, \varepsilon/4)$ and $C' = \bigcup_{y\in C} D_X(y, \varepsilon/4)$ are disjoint open sets whose union contains $A \bigcup B$. $\square$

Corollary 3.4. *If* $\lim_n \prod_{k=1}^n \sum_{i=1}^{m_k} \rho_i^{(k)} = 0$, *then* K *is totally disconnected.*

Proof. Let $x, y \in K$ with $x \neq y$ and let $n_{xy} \in \mathbb{N}$ be such that $\prod_{k=1}^n \sum_{i=1}^{m_k} \rho_i^{(k)} < d_X(x, y)/|Q|$ $\forall n \in \mathbb{N}$ with $n > n_{xy}$. Then, if $n > n_{xy}$, we have

$$\begin{aligned} K &= \lim_k (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_k)(Q) \\ &= (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)\Big(\lim_k (\mathfrak{F}_{n+1} \circ \mathfrak{F}_{n+2} \circ \cdots \circ \mathfrak{F}_k)(Q)\Big) \end{aligned}$$

because, by Lemmas 2.2 and 2.3, $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ are contraction maps with respect to the Hausdorff metric and then they are continuous. It follows that

$$K \subseteq (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(Q).$$

Now, let $A \subseteq K$ be connected and such that $x, y \in A$: we have

$$A \subseteq K \subseteq \bigcup_{i_1=1}^{m_1} \bigcup_{i_2=1}^{m_2} \cdots \bigcup_{i_n=1}^{m_n} \overline{f_{i_1 i_2 \cdots i_n}(Q)}$$

and by Lemma 3.2

$$\begin{aligned} |A| &\leq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} |f_{i_1 i_2 \cdots i_n}(Q)| \\ &\leq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} \rho_{i_k}^{(k)}\Big)|Q| = |Q| \prod_{k=1}^{n} \sum_{i=1}^{m_k} \rho_i^{(k)} < d_X(x, y) \leq |A|. \qquad \square \end{aligned}$$

3.4. Examples.

3.4.1. *Cantor sets in* $\mathbb{R}$. Let X be the set of real numbers with the Euclidean distance; we construct a generalized version of the Cantor set. To do this, we suppose that we are given a real number $d \in]0,1[$ and we construct two sequences (t_n) and (d_n) in the following way:

- $t_0 = 1$ and $m_n t_n^d = t_{n-1}^d$;
- $d_n = \frac{t_{n-1} - m_n t_n}{m_n - 1}$.

Then the set K of the previous paragraph is obtained by setting

$$f_i^{(n)}(x) = \frac{t_n x + (i-1)(t_n + d_n)}{t_{n-1}} = m_n^{-\frac{1}{d}} x + \frac{i-1}{m_n - 1}(1 - m_n^{-\frac{1}{d}}) \ \ \forall n \in \mathbb{N} \ \ \forall i \in I_n.$$

Now we show some properties of the fractal set K so obtained. In the definition of the functions p_n we assume $x_0 = 0$.

Remark 6. The functions p_n are given by

$$p_n(i_1, i_2, \dots, i_n) = \sum_{j=1}^{n} (i_j - 1)(t_j + d_j). \tag{1}$$

We prove this by induction: let $n \in \mathbb{N}$, we have $f_{i_n}^{(n)}(0) = \frac{1}{t_{n-1}}(i_n - 1)(t_n + d_n)$. Let us suppose that for $k \in \mathbb{N}$ $2 \le k \le n$,

$$f_{i_k i_{k+1} \cdots i_n}^{(k)}(0) = \frac{1}{t_{k-1}} \sum_{j=k}^{n} (i_j - 1)(t_j + d_j), \tag{2}$$

then

$$f_{i_{k-1} i_k \cdots i_n}^{(k-1)}(0) = f_{i_{k-1}}^{(k-1)}(f_{i_k i_{k+1} \cdots i_n}^{(k)}(0)) = \frac{1}{t_{k-2}} \sum_{j=k-1}^{n} (i_j - 1)(t_j + d_j).$$

Then (2) holds for any $k \le n$ and in particular

$$p_n(i_1, i_2, \dots, i_n) = f_{i_1 i_2 \cdots i_n}(0) = \sum_{j=1}^{n} (i_j - 1)(t_j + d_j).$$

It follows that

$$p(i) = \sum_{j=1}^{+\infty} (i_j - 1)(t_j + d_j) \ \ \forall i \in I \tag{3}$$

and

$$K = \left\{ \sum_{j=1}^{+\infty} (i_j - 1)(t_j + d_j) \mid i_j \in I_j \ \ \forall j \in \mathbb{N} \right\}. \tag{4}$$

Example 1. If $m_n = 2$ $\forall n \in \mathbb{N}$ and $d = \log_3 2$, then K is the classical Cantor set.

Indeed, in this case $t_n = d_n = 3^{-n}$, $f_1^{(n)}(x) = x/3$ and $f_2^{(n)}(x) = (x + 2)/3$ $\forall n \in \mathbb{N}$ and by Remark 4 K is invariant with respect to $\mathfrak{F} = \{f_1^{(1)}, f_2^{(1)}\}$.

It may also be noted that (4) becomes

$$K=\Big\{\sum_{j=1}^{+\infty}\frac{c_j}{3^j}\mid c_j\in\{0,2\}\ \ \forall j\in\mathbb{N}\Big\}.$$

Remark 7. It is easy to prove, by induction, that

$$\forall n\in\mathbb{N}\ \ t_n=\Big(\prod_{j=1}^{n}m_j\Big)^{-\frac{1}{d}}\ \text{and}\ d_n=\frac{1-m_n^{1-\frac{1}{d}}}{m_n-1}\Big(\prod_{j=1}^{n-1}m_j\Big)^{-\frac{1}{d}}.$$

Remark 8. From Remark 6 we have $K\subseteq[0,1]$ and $0,1\in K$.

Indeed, if we set $k,h\in I$, $k_j=1$, $h_j=m_j$ for every $j\in\mathbb{N}$, then by (3) , $p(k)=0$, $p(h)=\sum_{j=1}^{+\infty}(t_{j-1}-t_j)=t_0=1$ and for all $i\in I$ $0\le p(i)\le p(h)=1$.

It may also be noted that $0\le f_j^{(n)}(x)\le 1$ $\forall x\in[0,1]$ and for every $n,j\in\mathbb{N}$, $j\le m_n$; then we can set $Q=[0,1]$.

Remark 9. Given $n\in\mathbb{N}$, the $\prod_{j=1}^{n}m_j$ intervals of the form $[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n]$ are pairwise disjoint and $|p_n(i_1,i_2,\dots,i_n)-(p_n(k_1,k_2,\dots,k_n)+t_n)|\ge d_n$ for any different $(i_1,i_2,\dots,i_n),(k_1,k_2,\dots,k_n)\in\prod_{j=1}^{n}I_j$.

Moreover, for all $(i_1,i_2,\dots,i_{n+1})\in\prod_{j=1}^{n+1}I_j$ the interval $[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n]$ contains $[p_n(i_1,i_2,\dots,i_{n+1}),p_n(i_1,i_2,\dots,i_{n+1})+t_{n+1}]$ and

$$K=\bigcap_{n=1}^{+\infty}\Big(\bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_n=1}^{m_n}\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big).$$

Now we are going to prove that K is a d-set if and only if the sequence (m_n) is bounded.

Theorem 3.5. *We have* $\mathcal{H}^d(K)=1$ *and so* $\dim K=d$.

Proof. See [3] Chapter 1, Theorem 1.15. □

Lemma 3.3. *For every* $n\in\mathbb{N}$ *and* $(i_1,i_2,\dots,i_n)\in\prod_{j=1}^{n}I_j$, *we have*

$$\mathcal{H}^d\Big(K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big)=t_n^d.$$

Proof. Let $n\in\mathbb{N}$ and $(i_1,i_2,\dots,i_n)\in\prod_{j=1}^{n}I_j$; we define

$$f:K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\to K\bigcap[0,t_n],$$
$$f(x)=x-p_n(i_1,i_2,\dots,i_n).$$

It is clear that $0\le f(x)\le t_n$; moreover, $f(x)\in K$ by (3) and (1). Then f is well defined.

The function f is one-to-one because it is injective and $\forall y\in K\bigcap[0,t_n]$, $y=f(y+p_n(i_1,i_2,\dots,i_n))$. Moreover, f is an isometry and then, by Lemma 2.1, $\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)\le\mathcal{H}^d\Big(K\bigcap[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n]\Big)$. By applying the same arguments to f^{-1} we obtain the opposite inequality.

Finally,

$$1=\mathcal{H}^d(K)=\mathcal{H}^d\Big(\bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_n=1}^{m_n}\Big(K\bigcap\Big[p_n(i_1,i_2,\ldots,i_n),p_n(i_1,i_2,\ldots,i_n)+t_n\Big]\Big)\Big)$$
$$=\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\mathcal{H}^d\Big(K\bigcap\Big[p_n(i_1,i_2,\ldots,i_n),p_n(i_1,i_2,\ldots,i_n)+t_n\Big]\Big)$$
$$=\Big(\prod_{j=1}^{n}m_j\Big)\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)$$

and then

$$\mathcal{H}^d\Big(K\bigcap\Big[p_n(i_1,i_2,\ldots,i_n),p_n(i_1,i_2,\ldots,i_n)+t_n\Big]\Big)=\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)$$
$$=\Big(\prod_{j=1}^{n}m_j\Big)^{-1}=t_n^d. \qquad \square$$

Proposition 3.6. *The set K is a d-set if and only if the sequence (m_n) is bounded.*

Proof. Let $x_0\in K$ and $r\in]0,1]$; by Remark 9, for every $k\in\mathbb{N}$ there exist $(i_1,i_2,\ldots,i_k)\in\prod_{j=1}^k I_j$ such that $x_0\in[p_k(i_1,i_2,\ldots,i_k),p_k(i_1,i_2,\ldots,i_k)+t_k]$. Let

$$n=\min\Big\{k\in\mathbb{N}\mid \exists(i_1,i_2,\ldots,i_k)\in\prod_{j=1}^{k}I_j \text{ so that}$$
$$x_0\in\Big[p_k(i_1,i_2,\ldots,i_k),p_k(i_1,i_2,\ldots,i_k)+t_k\Big]\subseteq[x_0-r,x_0+r]\Big\}$$

(n is well defined because $\lim_k t_k=0$ by Remark 7). We prove that

$$\frac{1}{m_n}r^d\le\mathcal{H}^d\Big(k\bigcap[x_0-r,x_0+r]\Big)\le 2^{1+d}m_nr^d; \tag{5}$$

it will follow that K is a d-set if the sequence (m_n) is bounded.

By Lemma 3.3 $\mathcal{H}^d\Big(K\bigcap[x_0-r,x_0+r]\Big)\ge t_n^d=t_{n-1}^d/m_n$; if $t_{n-1}\le r$, we would have $x_0\in[p_{n-1}(i_1,i_2,\ldots,i_{n-1}),p_{n-1}(i_1,i_2,\ldots,i_{n-1})+t_{n-1}]\subseteq[x_0-r,x_0+r]$ and this is absurd; then the first inequality follows.

Let us prove the second inequality: we have $[p_n(i_1,i_2,\ldots,i_n),p_n(i_1,i_2,\ldots,i_n)+t_n]\subseteq[x_0-r,x_0+r]\bigcap[p_{n-1}(i_1,i_2,\ldots,i_{n-1}),p_{n-1}(i_1,i_2,\ldots,i_{n-1})+t_{n-1}]$ and for every $j\in\mathbb{N}$ the following implications hold:

- if $1\le j<i_{n-1}-1$ then $[x_0-r,x_0+r]\bigcap[p_{n-1}(i_1,i_2,\ldots,i_{n-2},j),p_{n-1}(i_1,i_2,\ldots,i_{n-2},j)+t_{n-1}]=\varnothing$ because otherwise we would have $[p_{n-1}(i_1,i_2,\ldots,i_{n-2},i_{n-1}-1),p_{n-1}(i_1,i_2,\ldots,i_{n-2},i_{n-1}-1)+t_{n-1}]\subseteq[x_0-r,x_0+r]$;

- if $i_{n-1}+1<j\le m_{n-1}$ then $[x_0-r,x_0+r]\bigcap[p_{n-1}(i_1,i_2,\dots,i_{n-2},j),p_{n-1}(i_1,i_2,\dots,i_{n-2},j)+t_{n-1}]=\varnothing$ because otherwise we would have $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1)+t_{n-1}]\subseteq[x_0-r,x_0+r]$.

Moreover, at least one of the intervals $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}-1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}-1)+t_{n-1}]$ and $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1)+t_{n-1}]$ does not intersect $[x_0-r,x_0+r]$ because otherwise we would have $[p_{n-1}(i_1,i_2,\dots,i_{n-1}),p_{n-1}(i_1,i_2,\dots,i_{n-1})+t_{n-1}]\subseteq[x_0-r,x_0+r]$.

Then $\mathcal{H}^d\Big(K\bigcap[x_0-r,x_0+r]\Big)<2t_{n-1}^d=2m_nt_n^d\le 2^{1+d}m_nr^d$.

Now we suppose that the sequence (m_n) is not bounded; by taking $x_0=0$ and $r_n=t_n+d_n$ for every $n\in\mathbb{N}$, we have

$$\frac{\mathcal{H}^d\Big(K\bigcap[x_0-r_n,x_0+r_n]\Big)}{r_n^d}=\frac{\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)}{r_n^d}=\frac{t_n^d}{(t_n+d_n)^d}=\Big(\frac{m_n-1}{m_n^{\frac{1}{d}}-1}\Big)^d.$$

Let (m_{n_k}) be a subsequence of (m_n) such that $\lim_k m_{n_k}=+\infty$; then

$$\lim_k\frac{\mathcal{H}^d\Big(K\bigcap[x_0-r_{n_k},x_0+r_{n_k}]\Big)}{r_{n_k}^d}=0$$

because $d<1$. $\square$

Remark 10. If in the above proposition we suppose $t_k\le d_k$ for all $k\in\mathbb{N}$, then (5) becomes

$$\frac{1}{m_n}r^d\le\mathcal{H}^d\Big(k\bigcap[x_0-r,x_0+r]\Big)\le 2^dm_nr^d. \tag{6}$$

We recall that by Remark 7 $t_k\le d_k$ if $d\le\log_{(2m_k-1)}m_k$. If $d\le\log_3 2$, then $t_k\le d_k$ independently of m_k.

Example 2. For the classical Cantor set, (6) gives

$$\frac{1}{2}r^d\le\mathcal{H}^d\Big(k\bigcap[x_0-r,x_0+r]\Big)\le 2^{1+d}r^d\quad\forall x_0\in K\quad\forall r\in]0,1].$$

Now we estimate the entropy numbers of K under the assumption that $t_k\le d_k$ $\forall k\in\mathbb{N}$.

To avoid tedious notation, we set for every $k\in\mathbb{N}$,

$$C_k=\bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_n=1}^{m_n}\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big].$$

By Remark 9 $K=\bigcap_{k=1}^{+\infty}C_k$.

Let $k\in\mathbb{N}$ and $n_k=\prod_{j=1}^k m_j$; then $\varepsilon_{n_k}(K)\le\varepsilon_{n_k}(C_k)\le t_k/2$.

Since the extreme points of the intervals of C_k are in K, it follows that

$$\varepsilon_{n_k}(K)=\frac{1}{2}t_k=\frac{1}{2}n_k^{-\frac{1}{d}}. \tag{7}$$

Let $h \in \mathbb{N}$ be a divisor of m_{k+1}: we compute $\varepsilon_{hn_k}(K)$.

The set C_{k+1} is a disjoint union of n_{k+1} closed intervals with amplitude t_{k+1} and mutual distance greater than or equal to t_{k+1}. Since hn_k divides $n_{k+1} = m_{k+1}n_k$, we can cover all the m_{k+1} closed intervals of C_{k+1} that are included into a single interval of C_k with h closed intervals of the form

$$\Big[p_{k+1}\Big(i_1, i_2, \ldots, i_k, \frac{lm_{k+1}}{h} + 1\Big), p_{k+1}\Big(i_1, i_2, \ldots, i_k, \frac{(l+1)m_{k+1}}{h}\Big) + t_{k+1}\Big]$$
$$0 \le l < h$$

and, as before, the extreme points of these intervals are in K; so

$$\varepsilon_{hn_k}(K) = \frac{1}{2}\Big(\frac{m_{k+1}}{h}t_{k+1} + \Big(\frac{m_{k+1}}{h} - 1\Big)d_{k+1}\Big)$$

and by Remark 7 we have

$$\varepsilon_{hn_k}(K) = \frac{(h-1)m_{k+1}^{1-\frac{1}{d}} + m_{k+1} - h}{2h^{1-\frac{1}{d}}(m_{k+1} - 1)}(hn_k)^{-\frac{1}{d}}. \tag{8}$$

If h is not a divisor of m_{k+1}, we have $\varepsilon_{hn_k}(K) \le t_k/(2h)$ and then

$$\varepsilon_{hn_k}(K) \le \frac{1}{2}\, h^{\frac{1}{d}-1}(hn_k)^{-\frac{1}{d}}. \tag{9}$$

Let $\lfloor m_{k+1}/h \rfloor = \max\{n \in \mathbb{N} \mid n \le m_{k+1}/h\}$; then we may not cover K by using hn_k intervals of length $(t_{k+1} + d_{k+1})\lfloor m_{k+1}/h \rfloor$ because at least one of the intervals of C_{k+1} will not be covered; so it must be

$$\varepsilon_{hn_k}(K) > \frac{1}{2}\,(t_{k+1} + d_{k+1})\Big\lfloor \frac{m_{k+1}}{h} \Big\rfloor,$$

i.e.,

$$\varepsilon_{hn_k}(K) > \frac{h^{\frac{1}{d}}(1 - m_{k+1}^{-\frac{1}{d}})}{2(m_{k+1} - 1)}\Big\lfloor \frac{m_{k+1}}{h} \Big\rfloor (hn_k)^{-\frac{1}{d}}. \tag{10}$$

Finally, if $l \in \mathbb{N}$ $l < n_k$, then

$$\varepsilon_{hn_k+l}(K) = \varepsilon_{hn_k}(K) \tag{11}$$

because the additional l intervals can not be equally distributed between the connected components of C_k.

Example 3. Let K be the classical Cantor set; by (7) and (11) we have

$$\varepsilon_{2^k+l}(K) = 2^{-\frac{k}{d}-1} = \frac{1}{2}3^{-k}.$$

It follows that the entropy numbers of the classical Cantor set have the following asymptotic behaviour:

$$\frac{1}{2}n^{-\frac{1}{d}} \le \varepsilon_n(K) < \frac{3}{2}n^{-\frac{1}{d}} \quad \forall n \in \mathbb{N}$$

(see also [6], Example 2.2).

Remark 11. If the sequence (m_n) is bounded, then K is a d-set by Proposition 3.6 and we can apply Proposition 3.1 of [6] to see that there exist $a, b \in]0, +\infty[$ such that

$$an^{-\frac{1}{d}} \le \varepsilon_n(K) \le bn^{-\frac{1}{d}} \quad \forall n \in \mathbb{N} .$$

Moreover, by Corollary 2.7 of [6], the box dimension of K is d.

Remark 12. Let us suppose that the sequence (m_n) is not bounded and $t_k \le d_k \ \ \forall k \in \mathbb{N}$. Let (m_{n_k}) be a subsequence of (m_n) such that $\lim_k m_{n_k} = +\infty$; we set $p_k = \prod_{j=1}^{n_k-1} m_j$ for every $k \in \mathbb{N}$; then, by (7) we have

$$\varepsilon_{p_k}(K) = \frac{1}{2} p_k^{-\frac{1}{d}}.$$

If m_{n_k} is even, then, by setting $2h_k = m_{n_k}$, we have from (8)

$$\varepsilon_{h_k p_k}(K)(h_k p_k)^{\frac{1}{d}} = \frac{2^{1-\frac{1}{d}}(h_k - 1)h_k + h_k^{1+\frac{1}{d}}}{2h_k(2h_k - 1)}. \tag{12}$$

If m_{n_k} is odd, then, we set $2h_k = m_{n_k} + 1$ and by (10) it follows

$$\varepsilon_{h_k p_k}(K)(h_k p_k)^{\frac{1}{d}} > \frac{h_k^{\frac{1}{d}}(1 - m_{n_k}^{-\frac{1}{d}})}{2(m_{n_k} - 1)} \left\lfloor \frac{m_{n_k}}{h_k} \right\rfloor,$$

but $\lfloor m_{n_k}/h_k \rfloor = \lfloor 2 - h_k^{-1} \rfloor = 1$; then

$$\varepsilon_{h_k p_k}(K)(h_k p_k)^{\frac{1}{d}} > \frac{h_k^{\frac{1}{d}}\left(1 - (2h_k - 1)^{-\frac{1}{d}}\right)}{4(h_k - 1)}. \tag{13}$$

Since $\lim_k h_k = +\infty$, it follows from (12) and (13) that

$$\limsup_n \varepsilon_n(K) n^{\frac{1}{d}} = +\infty.$$

3.4.2. *Sierpiński gaskets.* Let X be the plane $\mathbb{R}^2$ with the Euclidean distance; a generalized version of the Sierpiński gasket may be constructed in the following way.

Let (k_n) be a sequence of integer numbers, with $k_n \ge 2$ for all $n \in \mathbb{N}$; the sequence (m_n) is given by

$$m_n = \sum_{j=0}^{k_n} j = \frac{1}{2} k_n(k_n + 1) \quad \forall n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, $i \in I_n$ the contraction $f_i^{(n)}$ is given by

$$f_i^{(n)}(x, y) = \left(\frac{x}{k_n}, \frac{y}{k_n}\right) + (a_i^{(n)}, b_i^{(n)}),$$

where

$$a_i^{(n)} = \frac{k_n - (h_i^{(n)} + 1)^2 + 2(i - 1)}{2k_n}, \ b_i^{(n)} = \frac{\sqrt{3}(k_n - h_i^{(n)} - 1)}{2k_n}$$

and $0 \leq h_i^{(n)} < k_n$ is so that

$$\sum_{j=0}^{h_i^{(n)}} j < i \leq \sum_{j=0}^{h_i^{(n)}+1} j.$$

As before we set $K = p(I)$.

Remark 13. If $k_n = 2$ for all $n \in \mathbb{N}$, then K is the Sierpiński gasket. Indeed, in this case, we have $m_n = 3$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} f_1^{(n)}(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big) + \Big(\frac{1}{4}, \frac{\sqrt{3}}{4}\Big), \\ f_2^{(n)}(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big), \\ f_3^{(n)}(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big) + \Big(\frac{1}{2}, 0\Big). \end{aligned} \tag{14}$$

Remark 14. Even for the points of the Sierpiński gasket we can give a representation by means of series.

For the sake of simplicity we only consider the case in which $k_n = 2$ $\forall n \in \mathbb{N}$. Let us consider the functions $f, g : \{1,2,3\} \to \mathbb{Z}$,

$$f(i) = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ 2 & i = 3 \end{cases}, \quad g(i) = \begin{cases} 1 & i = 1 \\ 0 & i > 1 \end{cases}. \tag{15}$$

Let $x_0 = (0,0)$, then for every $n \in \mathbb{N}$ and $i_1, i_2, \dots, i_n \in \{1,2,3\}$ we have

$$p_n(i_1, i_2, \dots, i_n) = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{2^j} \Big(f(i_j), \sqrt{3} g(i_j)\Big). \tag{16}$$

As in Remark 6, even (16) is proven by induction. Let $n \in \mathbb{N}$, by (14) and (15) we have

$$f_{i_n}^{(n)}(0,0) = \frac{1}{4} \Big(f(i_n), \sqrt{3} g(i_n)\Big).$$

Let us suppose that for $k \in \mathbb{N}$ $k \leq n$,

$$f_{i_{k+1} i_{k+2} \cdots i_n}^{(k+1)}(0,0) = \frac{1}{2} \sum_{j=k+1}^{n} \frac{1}{2^{j-k}} \Big(f(i_j), \sqrt{3} g(i_j)\Big),$$

then

$$\begin{aligned} f_{i_k i_{k+1} \cdots i_n}^{(k)}(0,0) &= \frac{1}{4} \sum_{j=k+1}^{n} \frac{1}{2^{j-k}} \Big(f(i_j), \sqrt{3} g(i_j)\Big) + \frac{1}{4} \Big(f(i_k), \sqrt{3} g(i_k)\Big) \\ &= \frac{1}{2} \sum_{j=k}^{n} \frac{1}{2^{j-k+1}} \Big(f(i_j), \sqrt{3} g(i_j)\Big). \end{aligned}$$

In particular, for $k = 1$ we have (16).

It follows that

$$p(i) = \frac{1}{2}\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big(f(i_j), \sqrt{3}g(i_j)\Big) \ \ \forall i \in I$$

and

$$K = \Big\{\frac{1}{2}\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big(f(i_j), \sqrt{3}g(i_j)\Big) \mid i_j \in \{1,2,3\} \ \ \forall j \in \mathbb{N}\Big\}.$$

4. A measure on K

In this section X is a complete separable metric space.

We recall (§ 2.4) that for every $n \in \mathbb{N}$, $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)}$ are real numbers in $]0,1[$ so that $\sum_{j=1}^{m_n} l_j^{(n)} = 1$, τ_n is a measure on I_n defined by $\tau_n(A) = \sum_{j\in A} l_j^{(n)}$ $\forall A \subseteq I_n$ and τ is a unique Radon measure on I such that $\tau(I) = 1$ and $\tau_{i_1} \times \tau_{i_2} \times \cdots \times \tau_{i_n} = \pi_{i_1,i_2,\dots,i_n\sharp}\tau$ for any distinct $i_1, i_2, \dots, i_n \in \mathbb{N}$.

Definition. We set $\mu_K = p_\sharp\tau$.

Remark 1. K is the support of μ_K and $\mu_K(K) = 1$.

Definition. Let (ν_n) be a sequence of Radon measures on X. We say that the sequence (ν_n) converges weakly to a Radon measure ν if

$$\lim_n \int_X f d\nu_n = \int_X f d\nu \ \ \forall f \in C_c(X).$$

We denote this fact by writing $\nu_n \rightharpoonup \nu$.

Remark 2. We recall that if ν is a Borel regular measure on X and for every $x \in X$ there is $r > 0$ such that $\nu(B_X(x,r)) < +\infty$, then ν is a Radon measure on X. For the proof see [8], chapter 5, theorem V.5.3.

Proposition 4.1. *Let ν be a Borel regular measure on X, with bounded support and such that $\nu(X) = 1$. For every $n \in \mathbb{N}$ we set*

$$\nu_n = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l_{i_k}^{(k)}\Big) f_{i_1 i_2\cdots i_n\sharp}\nu.$$

Then $\nu_n \rightharpoonup \mu_K$.

Proof. Let ν be a Borel regular measure on X with bounded support and such that $\nu(X) = 1$; let C be the support of ν and let $f \in C_c(X)$. We prove that

$$\lim_n \int_X f d\nu_n = \int_X f d\mu_K.$$

Let $\varepsilon > 0$ and $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ $\forall x, y \in X$ with $d_X(x,y) < \delta$; we consider $n'_\varepsilon \in \mathbb{N}$ such that

$$\Big(\prod_{k=1}^{n'_\varepsilon}\rho^{(k)}\Big)\sup_{x\in C} d_X(x,x_0) < \delta. \tag{1}$$

Moreover, let $n''_\varepsilon \in \mathbb{N}$ be such that $|Q|\prod_{j=1}^{n''_\varepsilon}\rho^{(j)} < \delta$; for every $n \in \mathbb{N}$ with $n > n''_\varepsilon$ we have

$$\begin{aligned} d_X(p_n(\pi_{1,2,\ldots,n}(i)), p(i)) &= \lim_k d_X\Big(f_{i_1 i_2\cdots i_n}(x_0), f_{i_1 i_2\cdots i_n}(f^{(n+1)}_{i_{n+1} i_{n+2}\cdots i_k}(x_0))\Big) \\ &\le |Q|\prod_{j=1}^{n}\rho^{(j)} < \delta \quad \forall i \in I \quad \forall n \in \mathbb{N} \quad n > n''_\varepsilon \end{aligned}$$

and then

$$\Big|f(p_n(\pi_{1,2,\ldots,n}(i))) - f(p(i))\Big| < \varepsilon \quad \forall i \in I \quad \forall n \in \mathbb{N} \quad n > n''_\varepsilon. \tag{2}$$

Then, if $n > \max\{n'_\varepsilon, n''_\varepsilon\}$, we have

$$\begin{aligned} &\Big|\int_X f d\nu_n - \int_X f d\mu_K\Big| \\ &= \Big|\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l^{(k)}_{i_k}\Big)\int_X f df_{i_1 i_2\cdots i_n\sharp}\nu - \int_X f dp_\sharp\tau\Big| \\ &\le \Big|\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l^{(k)}_{i_k}\Big)\Big(\int_X f(f_{i_1 i_2\cdots i_n}(x))d\nu(x) - f(f_{i_1 i_2\cdots i_n}(x_0))\Big)\Big| \\ &+ \Big|\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l^{(k)}_{i_k}\Big) f(f_{i_1 i_2\cdots i_n}(x_0)) - \int_I f(p(i))d\tau(i)\Big| \\ &\le \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l^{(k)}_{i_k}\Big)\Big|\int_X f(f_{i_1 i_2\cdots i_n}(x))d\nu(x) - \int_X f(f_{i_1 i_2\cdots i_n}(x_0))d\nu(x)\Big| \\ &+ \Big|\int_{\prod_{j=1}^{n} I_j} f(p_n(i_1, i_2, \ldots, i_n))d(\tau_1\times\tau_2\times\cdots\times\tau_n) - \int_I f(p(i))d\tau(i)\Big| \\ &\le \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l^{(k)}_{i_k}\Big)\int_X \Big|f(f_{i_1 i_2\cdots i_n}(x)) - f(f_{i_1 i_2\cdots i_n}(x_0))\Big|d\nu(x) \\ &+ \Big|\int_I f(p_n(\pi_{1,2,\ldots,n}(i)))d\tau(i) - \int_I f(p(i))d\tau(i)\Big| \\ &\le \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l^{(k)}_{i_k}\Big)\int_C \Big|f(f_{i_1 i_2\cdots i_n}(x)) - f(f_{i_1 i_2\cdots i_n}(x_0))\Big|d\nu(x) \\ &+ \int_I \Big|f(p_n(\pi_{1,2,\ldots,n}(i))) - f(p(i))\Big|d\tau(i). \end{aligned}$$

By (1)

$$d_X\Big(f_{i_1 i_2\cdots i_n}(x), f_{i_1 i_2\cdots i_n}(x_0)\Big) < \delta \quad \forall x \in C$$

and then

$$\Big|f(f_{i_1 i_2\cdots i_n}(x)) - f(f_{i_1 i_2\cdots i_n}(x_0))\Big| < \varepsilon \quad \forall x \in C. \tag{3}$$

Then, by (3) and (2) it follows

$$\left|\int_X f d\nu_n - \int_X f d\mu_K\right| < \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\left(\prod_{k=1}^{n} l_{i_k}^{(k)}\right)\varepsilon\nu(C)+\varepsilon\tau(I)=2\varepsilon$$

because

$$\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\left(\prod_{k=1}^{n} l_{i_k}^{(k)}\right)=1. \qquad \square$$

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