Georgian Mathematical Journal
Volume 8 (2001), Number 4, 733–752

SOME REMARKS ON FRACTALS GENERATED BY A SEQUENCE OF FINITE SYSTEMS OF CONTRACTIONS

GIORGIO FOLLO

Abstract. We generalize some results shown by J. E. Hutchinson in [7].

Let $\mathfrak{F}_n = \{f_1^{(n)}, f_2^{(n)}, \dots, f_{m_n}^{(n)}\}$ be finite systems of contractions on a complete metric space; then, under some conditions on $(\mathfrak{F}_n)$, there exists a unique non-empty compact set K such that the sequence defined by $((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \dots \circ \mathfrak{F}_n)(C))$ converges to K in the Hausdorff metric for every non-empty closed and bounded set C.

If the metric space is also separable and for every n, $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)}$ there are real numbers strictly between 0 and 1, satisfying the condition $\sum_{j=1}^{m_n} l_j^{(n)} = 1$, then there exists a unique probability Radon measure μ_K such that the sequence

$$\nu_n = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l_{i_k}^{(k)} \Big) \big(f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \dots \circ f_{i_n}^{(n)}\big)_\sharp \nu$$

weakly converges to μ_K for every probability Borel regular measure ν with bounded support (where by $f_\sharp \nu$ we denote the image measure of ν under a contraction f). Moreover, K is the support of μ_K.

2000 Mathematics Subject Classification: Primary: 28A80; Secondary: 28A33.
Key words and phrases: Fractals, construction of measures.

1. Introduction

Let (X, d_X) be a complete separable metric space and let $f_1, f_2, \dots, f_M : X \to X$ be contractions. In [7] it is proved that there exists a unique non-empty closed and bounded subset K of X invariant with respect to $\mathfrak{F} = \{f_1, f_2, \dots, f_M\}$ i.e., such that

$$K = \mathfrak{F}(K) = \bigcup_{j=1}^{M} \overline{f_j(K)}. \tag{1}$$

Moreover, K is compact and if $C_0 \neq \varnothing$ is closed and bounded, then the sequence (C_n) defined by $C_n = \mathfrak{F}(C_{n-1})$ converges to K in the Hausdorff metric.

Let $r = \{r_1, r_2, \dots, r_M\}$ be a family of M real numbers in $]0,1[$ with $\sum_{j=1}^{M} r_j = 1$. Then there exists a unique Borel regular (outer) measure μ in

ISSN 1072-947X / $8.00 /

X with compact support and of total mass 1 such that μ is invariant with respect to $(\mathfrak{F}, r)$, i.e.,

$$\mu(A) = \sum_{j=1}^{M} r_j \mu(f_j^{-1}(A)) \quad \text{for every Borel set } A \subseteq X. \tag{2}$$

Furthermore, the support of μ is the fractal K.

We consider the case in which the system $\mathfrak{F}$ is replaced by a sequence $(\mathfrak{F}_n)$ of finite systems of contractions, i.e., $\mathfrak{F}_n = \{f_1^{(n)}, f_2^{(n)}, \dots, f_{m_n}^{(n)}\}$ with $m_n \geq 2$. Obviously, we cannot write an expression like (1), but we can still construct a sequence of closed and bounded subsets of X and ask if such a sequence is convergent with respect to the Hausdorff metric. More precisely, if the sequence $(\mathfrak{F}_n)$ satisfies the following two conditions:

- there exists a bounded set $Q \subseteq X$ such that $\bigcup_{j=1}^{m_n} f_j^{(n)}(Q) \subseteq Q$ for any $n \in \mathbb{N}$;
- $\lim_n \prod_{k=1}^{n} \rho^{(k)} = 0$; here $\rho^{(k)}$ is the greatest of the Lipschitz constants of the contractions $f_1^{(k)}, f_2^{(k)}, \dots, f_{m_k}^{(k)}$;

then there exists a unique non-empty closed and bounded set $K \subseteq X$ such that the sequence $((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C_0))$ converges to K in the Hausdorff metric, for every non empty closed and bounded subset C_0 of X. Moreover K is compact.

As an interesting example, given $d \in]0, 1[$, we construct a d-dimensional compact subset of the real line by considering a sequence $(\mathfrak{F}_n)$ of finite systems of contractive similitudes $f_1^{(n)}, f_2^{(n)}, \dots, f_{m_n}^{(n)}$ with Lipschitz constants $\rho^{(n)}$ (depending only on n) such that $m_n(\rho^{(n)})^d = 1$. We will study the entropy numbers related to this set.

In Section 4 we consider a generalization of the invariant measure found in [7].

As before, we cannot write an expression like (2). Let X be a complete separable metric space and let for every n, $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)} \in]0, 1[$ be so that $\sum_{j=1}^{m_n} l_j^{(n)} = 1$, then there exists a unique Radon probability measure μ_K so that for every Radon probability measure ν on X, with bounded support, the sequence of measures defined by

$$\nu_n(A) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l_{i_k}^{(k)} \Big) \nu\Big((f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \cdots \circ f_{i_n}^{(n)} \Big)^{-1}(A)),$$

for Borel sets $A \subseteq X$, weakly converges to μ_K.

Moreover, the support of μ_K is the fractal K.

2. Notation and preliminary results

In this note (X, d_X) will always be a complete metric space. Additional requirements for X will be specified when necessary.

$\mathbb{N} = \{1, 2, \dots\}$ is the set of all positive integer numbers.

The closed and open balls in X will be indicated by the symbols $B_X(x_0, r)$ and $D_X(x_0, r)$:

$$B_X(x_0,r)=\Big\{x \in X \mid d_X(x,x_0) \le r\Big\}, \quad D_X(x_0,r)=\Big\{x \in X \mid d_X(x,x_0) < r\Big\}.$$

The diameter of a subset A of X is indicated by $|A|$: $|A| = \sup_{x,y\in A} d_X(x,y)$ and its number of elements is indicated by $\sharp A$.

If X is separable and $s \ge 0$, $\mathcal{H}^s(A)$ stands for the s-dimensional Hausdorff measure of A and $\dim A$ for its Hausdorff dimension.

If $X = \mathbb{R}^N$ then we will use the Euclidean metric $d_{\mathbb{R}^N}(x,y) = \|x-y\|_2 = \sqrt{\sum_{i=1}^N (\xi_i - \eta_i)^2}$, where $x = (\xi_1, \xi_2, \dots, \xi_N)$ and $y = (\eta_1, \eta_2, \dots, \eta_N)$.

Lemma 2.1. *Let $E \subseteq \mathbb{R}^N$ and let $d > 0$. If $f : E \to \mathbb{R}^N$ is a mapping and $c > 0$ is a constant such that $\|f(x) - f(y)\|_2 \le c\|x-y\|_2$ for every $x, y \in E$, then $\mathcal{H}^d(f(E)) \le c^d \mathcal{H}^d(E)$.*

Proof. See [4], Chapter 2, Proposition 2.2. □

2.1. d-sets in $\mathbb{R}^N$.

Definition. Let Γ be a closed non-empty subset of $\mathbb{R}^N$ and let $d \in]0, N]$. A positive Borel outer measure μ with support Γ is called a d-measure on Γ if there exist $c_1, c_2 \in]0, +\infty[$ such that for every $x_0 \in \Gamma$ and for every $r \in]0, 1]$

$$c_1 r^d \le \mu\Big(B_{\mathbb{R}^N}(x_0, r)\Big) \le c_2 r^d \tag{1}$$

holds.

Remark 1. One can replace the condition $r \in]0,1]$ in the above definition by the condition $r \in]0, r_0]$; obviously, the constants c_1 and c_2 will be replaced by some constants $c_1(r_0) > 0$ and $c_2(r_0) > 0$ depending on r_0.

Definition. A closed non-empty subset Γ of $\mathbb{R}^N$ is called a d-set if there exists a d-measure on Γ.

It can be proved that if Γ is a d-set, μ_1 and μ_2 are d-measures on Γ. Then there exist constants $a, b \in]0, +\infty[$ such that $a\mu_1(A) \le \mu_2(A) \le b\mu_1(A)$ $\forall A \subseteq \mathbb{R}^N$. Moreover, the restriction to Γ of the Hausdorff measure $\mathcal{H}^d$ is a d-measure on Γ; so every d-set has its canonical d-measure and therefore d is unique.

For the proof of these facts see [9], Chapter 2 or [11], Chapter 1.

2.2. Entropy numbers. Let Ω be a bounded subset of X. The n-th entropy number of Ω is defined by

$$\varepsilon_n(\Omega) = \inf\Big\{\varepsilon > 0 \mid \exists\, x_1, x_2, \dots, x_n \in X \text{ such that } \Omega \subseteq \bigcup_{i=1}^n B_X(x_i, \varepsilon)\Big\}.$$

The sequence $(\varepsilon_n(\Omega))$ is monotonically decreasing and tends to zero if and only if Ω is precompact.

See [1] for a complete treatment.

2.3. The Hausdorff metric.

Definition. If $x_0 \in X$ and $A \subseteq X$, we define the distance between x_0 and A by

$$d_X(x_0, A) = \inf_{x \in A} d_X(x_0, x).$$

Remark 2. For every $x_0 \in X$ and $A \subseteq X$,we have $d_X(x_0, A) = d_X(x_0, \overline{A})$.

Definition. Let $\mathfrak{B}$ be the class of all non-empty closed bounded subsets of X.

The Hausdorff metric D on $\mathfrak{B}$ is defined by

$$D(A, B) = \sup\Big\{d_X(x, B), d_X(y, A) \mid x \in A,\ y \in B\Big\}.$$

Remark 3. D is a metric on $\mathfrak{B}$. Moreover, for every $A, B \in \mathfrak{B}$

$$D(A, B) = \inf\Big\{\varepsilon > 0 \mid A \subseteq \bigcup_{y \in B} D_X(y, \varepsilon) \text{ and } B \subseteq \bigcup_{x \in A} D_X(x, \varepsilon)\Big\}.$$

Lemma 2.2. *Let* $f : X \to X$ *be a Lipschitz function and let* $\rho = \inf\Big\{c > 0 \mid d_X(f(x), f(y)) \le c d_X(x, y) \ \ \forall x, y \in X\Big\}$ *be its Lipschitz constant. Then*

$$D(\overline{f(A)}, \overline{f(B)}) \le \rho D(A, B) \ \ \forall A, B \in \mathfrak{B}. \tag{2}$$

Proof. By remark 2 $D(\overline{f(A)}, \overline{f(B)}) = \sup\Big\{d_X(u, f(B)), d_X(v, f(A)) \mid u \in f(A),\ v \in f(B)\Big\}$.

By the Lipschitz condition on f it follows that $d_X(f(x), f(B)) \le \rho d_X(x, B)$ and $d_X(f(y), f(A)) \le \rho d_X(y, A)$, $\forall x \in A \ \ \forall y \in B$ and then the (2). □

Lemma 2.3. *Let* $\{A_j \mid j \in J\}$, $\{B_j \mid j \in J\}$ *be two families of elements of* $\mathfrak{B}$. *Then*

$$D\Big(\overline{\bigcup_{j \in J} A_j}, \overline{\bigcup_{j \in J} B_j}\Big) \le \sup_{j \in J} D(A_j, B_j)$$

provided that $\bigcup_{j \in J} A_j$ *and* $\bigcup_{j \in J} B_j$ *are bounded.*

Proof. Let $c > \sup_{j \in J} D(A_j, B_j)$: for all $j \in J$ $D(A_j, B_j) < c$ and then

$$A_j \subseteq \bigcup_{y_j \in B_j} D_X(y_j, c) \subseteq \bigcup_{i \in J} \bigcup_{y_i \in B_i} D_X(y_i, c) = \bigcup_{y \in \bigcup_{i \in J} B_i} D_X(y, c).$$

It follows that

$$\bigcup_{j \in J} A_j \subseteq \bigcup_{y \in \bigcup_{i \in J} B_i} D_X(y, c).$$

In the same way we obtain

$$\bigcup_{j \in J} B_j \subseteq \bigcup_{x \in \bigcup_{j \in J} A_j} D_X(x, c).$$

Then

$$D\Big(\overline{\bigcup_{j \in J} A_j}, \overline{\bigcup_{j \in J} B_j}\Big) < c. \qquad \square$$

2.4. Sequences of indices and product spaces. From now on, (m_n) is a fixed sequence of integer numbers, with $m_n \geq 2$ for all $n \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$ we fix $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)} \in]0,1[$ so that $\sum_{j=1}^{m_n} l_j^{(n)} = 1$.

Definition. For $n \in \mathbb{N}$ we set $I_n = \{1, 2, \dots, m_n\}$ and $I = \prod_{n=1}^{+\infty} I_n$. Each I_n is equipped with the discrete topology. On I we consider the function $d_I : I \times I \to \mathbb{R}$,

$$d_I(k,h) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \frac{|k_j - h_j|}{1 + |k_j - h_j|},$$

where $k = (k_j)$, $h = (h_j)$, $k_j, h_j \in I_j$ for all $j \in \mathbb{N}$.

Remark 4. It is well known that d_I is a metric on I; moreover d_I induces the product topology on I.

It follows that (I, d_I) is a compact metric space and then it is complete and separable (see, for example, [2], Chapters 2, 5 and 6).

Definition. Given $i_1, i_2, \dots, i_n \in \mathbb{N}$, we define the natural projection $\pi_{i_1, i_2, \dots, i_n} : I \to \prod_{j=1}^{n} I_{i_j}$ by

$$\pi_{i_1, i_2, \dots, i_n}(k) = (k_{i_1}, k_{i_2}, \dots, k_{i_n}).$$

Remark 5. For every $i_j \in \mathbb{N}$, π_{i_j} is a continuous function by the definition of product topology. Then $\pi_{i_1, i_2, \dots, i_n}$ is continuous.

Definition. Let X and Y be metric spaces, μ an outer measure on X and $f : X \to Y$ a function.

The image of μ under f is defined by

$$f_\sharp \mu(A) = \mu(f^{-1}(A)) \ \ \forall A \subseteq Y.$$

For the proof of the following two theorems see [10], Chapter 1, Theorems 1.18 and 1.19.

Theorem 2.1. *Let X and Y be separable metric spaces. If $f : X \to Y$ is continuous and μ is a Radon measure on X with compact support, then $f_\sharp \mu$ is a Radon measure. Moreover, if $C \subseteq X$ is the support of μ, then $f(C)$ is the support of $f_\sharp \mu$.*

Definition. Let X and Y be separable metric spaces. A mapping $f : X \to Y$ is a Borel mapping if $f^{-1}(U)$ is a Borel set for every open set $U \subseteq Y$.

Let $A \subseteq X$ be a Borel set. A function $g : A \to [-\infty, +\infty]$ is a Borel function if the set $\{x \in A \mid f(x) < c\}$ is a Borel set for every $c \in \mathbb{R}$.

Theorem 2.2. *Let X and Y be separable metric spaces and suppose that $f : X \to Y$ is a Borel mapping, μ is a Borel measure on X and g is a non-negative Borel function on Y. Then*

$$\int_Y g \, df_\sharp \mu = \int_X (g \circ f) d\mu.$$

Definition. For every $n \in \mathbb{N}$ we define a measure τ_n on I_n by

$$\tau_n(A) = \sum_{j \in A} l_j^{(n)} \quad \forall A \subseteq I_n.$$

Remark 6. For every $n \in \mathbb{N}$, τ_n is a Radon measure and $\tau_n(I_n) = 1$.

Remark 7. From the definition of product measure of two measures it follows that

$$(\tau_1 \times \tau_2 \times \cdots \times \tau_n)(A) = \sum_{(k_1,k_2,\ldots,k_n)\in A} \prod_{j=1}^{n} l_{k_j}^{(j)} \quad \forall n \in \mathbb{N} \quad \forall A \subseteq \prod_{j=1}^{n} I_j.$$

In order to define the product measure on I, we need the following theorem.

Theorem 2.3. *Let $\{X_\alpha \mid \alpha \in A\}$ be a family of compact Hausdorff spaces and let, for each $\alpha \in A$, μ_α be a Radon measure on X_α, with $\mu_\alpha(X_\alpha) = 1$.*

Then there exists a unique Radon measure μ on $\prod_{\alpha\in A} X_\alpha$ such that $\mu(\prod_{\alpha\in A} X_\alpha) = 1$ and $\mu_{\alpha_1} \times \mu_{\alpha_2} \times \cdots \times \mu_{\alpha_n} = \pi_{\alpha_1,\alpha_2,\ldots,\alpha_n \sharp}\mu$ for any distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$.

Proof. See [5], Chapter 9, Theorem 9.19. □

Remark 8. By the previous theorem there is a unique Radon measure τ on I such that $\tau(I) = 1$ and $\tau_{i_1} \times \tau_{i_2} \times \cdots \times \tau_{i_n} = \pi_{i_1,i_2,\ldots,i_n \sharp}\tau$ for any distinct $i_1, i_2, \ldots, i_n \in \mathbb{N}$.

3. Limit sets

3.1. Basic notation. From now on we will use the following notation:

- for any $n \in \mathbb{N}$ $i \in I_n$, $f_i^{(n)} : X \to X$ is a contraction;
  - $\rho_i^{(n)} = \inf\{c > 0 \mid d_X(f_i^{(n)}(x), f_i^{(n)}(y)) \le c d_X(x, y) \quad \forall x, y \in X\}$ ($\rho_i^{(n)}$ is the Lipschitz constant of $f_i^{(n)}$),
  - $\rho^{(n)} = \max\{\rho_1^{(n)}, \rho_2^{(n)}, \ldots, \rho_{m_n}^{(n)}\}$,
  - $\rho = \sup_{n\in\mathbb{N}} \rho^{(n)}$,
  - $x_i^{(n)} \in X$ is the fixed point of $f_i^{(n)}$;
- for every $n \in \mathbb{N}$
  - $\mathfrak{F}_n = \{f_1^{(n)}, f_2^{(n)}, \ldots, f_{m_n}^{(n)}\}$,
  - for every $A \subseteq X$ $\mathfrak{F}_n(A) = \bigcup_{i=1}^{m_n} \overline{f_i^{(n)}(A)}$,
  - for every $A \subseteq X$ $(\mathfrak{F}_n \circ \mathfrak{F}_{n+1})(A) = \mathfrak{F}_n(\mathfrak{F}_{n+1}(A))$;
- for every $i \in I$, $n, k \in \mathbb{N}$ with $k \le n$:
  - $f_{i_1 i_2 \cdots i_n} = f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \cdots \circ f_{i_n}^{(n)}$ and $x_{i_1 i_2 \cdots i_n}$ is its fixed point,
  - $f_{i_k i_{k+1} \cdots i_n}^{(k)} = f_{i_k}^{(k)} \circ f_{i_{k+1}}^{(k+1)} \circ \cdots \circ f_{i_n}^{(n)}$ and $x_{i_k i_{k+1} \cdots i_n}^{(k)}$ is its fixed point;
- $F = \{x_i^{(n)} \mid n \in \mathbb{N} \ i \in I_n\}$ is the set of all fixed points of the contractions $f_i^{(n)}$.

3.2. Existence and uniqueness.

Lemma 3.1. *Let* (g_n) *be a sequence of contraction maps on* X*, each of them with the Lipschitz constant* ρ_n*. Let us suppose that the following two conditions hold:*

(1) *there exists a non-empty closed and bounded set* $Q \subseteq X$ *such that* $g_n(Q) \subseteq Q$ *for every* $n \in \mathbb{N}$*;*
(2) $\lim_n \prod_{k=1}^n \rho_k = 0$.

Then there exists a unique $x \in X$ *so that*

$$\lim_n (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0) = x \quad \textit{for every} \quad x_0 \in X.$$

Moreover, $x \in Q$.

Proof. It is easy to prove, by induction, that for every $x, y \in X$ and $n \in \mathbb{N}$

$$d_X((g_1 \circ g_2 \circ \cdots \circ g_n)(x), (g_1 \circ g_2 \circ \cdots \circ g_n)(y)) \leq (\prod_{k=1}^n \rho_k) d_X(x, y).$$

Now, let $x_0 \in Q$ and $\varepsilon > 0$; by the second hypothesis there exists $n_\varepsilon \in \mathbb{N}$ so that $\prod_{k=1}^n \rho_k < \varepsilon/|Q|$ $\forall n \in \mathbb{N}$ with $n > n_\varepsilon$.

Then, for all $m, n \in \mathbb{N}$ with $m > n > n_\varepsilon$ we have

$$\begin{aligned} &d_X\Big((g_1 \circ g_2 \circ \cdots \circ g_n)(x_0), (g_1 \circ g_2 \circ \cdots \circ g_m)(x_0)\Big) \\ &\leq \Big(\prod_{k=1}^n \rho_k\Big) d_X\Big(x_0, (g_{n+1} \circ g_{n+2} \circ \cdots \circ g_m)(x_0)\Big) < \varepsilon. \end{aligned}$$

Since X is complete, there exists $x \in X$ such that $\lim_n (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0) = x$.

Now we prove that x does not depend on x_0.

Let $y_0 \in X$ and $n \in \mathbb{N}$:

$$\begin{aligned} &d_X\Big(x, (g_1 \circ g_2 \circ \cdots \circ g_n)(y_0)\Big) d_X\Big(x, (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0)\Big) \\ &\quad + d_X\Big((g_1 \circ g_2 \circ \cdots \circ g_n)(x_0), (g_1 \circ g_2 \circ \cdots \circ g_n)(y_0)\Big) \\ &\leq d_X\Big(x, (g_1 \circ g_2 \circ \cdots \circ g_n)(x_0)\Big) + \Big(\prod_{k=1}^n \rho_k\Big) d_X(x_0, y_0) \end{aligned}$$

and by letting $n \to +\infty$ we obtain $x = \lim_n (g_1 \circ g_2 \circ \cdots \circ g_n)(y_0)$. □

Definition. Let $x_0 \in X$ be fixed; we define $\forall n \in \mathbb{N}$ a function $p_n : \prod_{j=1}^n I_j \to X$ by

$$p_n(i_1, i_2, \ldots, i_n) = f_{i_1 i_2 \cdots i_n}(x_0).$$

Remark 1. Obviously, p_n depends on x_0 and it is a continuous function on $\prod_{j=1}^n I_j$.

From now on we will suppose that the following two hypotheses are valid:

(1) there exists a non-empty closed bounded set $Q \subseteq X$ such that $\mathfrak{F}_n(Q) \subseteq Q$ for all $n \in \mathbb{N}$;

(2) $\lim_n \prod_{k=1}^n \rho^{(k)} = 0$.

Remark 2. The hypotheses 1 and 2 above are implied by the following:

3. F is bounded;
4. $\rho < 1$.

Indeed, it is obvious that $4 \Rightarrow 2$; moreover, let

$$Q = \bigcap_{n=1}^{+\infty} \bigcap_{i=1}^{m_n} B_X\Big(x_i^{(n)}, \frac{|F|}{1-\rho}\Big);$$

we prove that Q satisfies 1.

Q is closed and bounded; moreover $F \subseteq Q$.

Let $n, k \in \mathbb{N}$, $i \in I_n$ and $j \in I_k$; we have $\forall x \in Q$

$$\begin{aligned} d_X(f_i^{(n)}(x), x_j^{(k)}) &\le d_X(f_i^{(n)}(x), x_i^{(n)}) + d_X(x_i^{(n)}, x_j^{(k)}) \\ &\le d_X(f_i^{(n)}(x), f_i^{(n)}(x_i^{(n)})) + |F| \le \rho_i^{(n)} d_X(x, x_i^{(n)}) + |F| \\ &\le \rho \frac{|F|}{1-\rho} + |F| = \frac{|F|}{1-\rho} \end{aligned}$$

and then

$$f_i^{(n)}(x) \in B_X\Big(x_j^{(k)}, \frac{|F|}{1-\rho}\Big) \quad \forall k \in \mathbb{N} \ \ \forall j \in I_k.$$

Definition. Let $x_0 \in X$ be fixed; we define a function $p : I \to X$ by

$$p(k) = \lim_n f_{k_1 k_2 \cdots k_n}(x_0).$$

Remark 3. By Lemma 3.1 the function p is well defined and does not depend on $x_0 \in X$.

If we take $x_0 \in Q$, then we would see that $p(I) \subseteq Q$. We will always suppose $x_0 \in Q$.

Definition. We denote the set $p(I)$ by K.

Proposition 3.1. *The function p is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ be so that $\prod_{i=1}^{n_\varepsilon} \rho^{(i)} < \varepsilon/|Q|$. Let $\delta = 2^{-n_\varepsilon - 1}$; for every $k, h \in I$, $d_I(k,h) < \delta$ implies $k_j = h_j$ $\forall j \in \mathbb{N}$ with $j \le n_\varepsilon$ and then, if we suppose $x_0 \in Q$,

$$\begin{aligned} &d_X(p(k), p(h)) \\ &\quad = \lim_j d_X\Big(f_{k_1 k_2 \cdots k_{n_\varepsilon}}(f^{(n_\varepsilon+1)}_{k_{n_\varepsilon+1} k_{n_\varepsilon+2} \cdots k_j}(x_0)), f_{k_1 k_2 \cdots k_{n_\varepsilon}}(f^{(n_\varepsilon+1)}_{h_{n_\varepsilon+1} h_{n_\varepsilon+2} \cdots h_j}(x_0))\Big) \\ &\quad \le \Big(\prod_{i=1}^{n_\varepsilon} \rho^{(i)}\Big) \lim_j d_X\Big(f^{(n_\varepsilon+1)}_{k_{n_\varepsilon+1} k_{n_\varepsilon+2} \cdots k_j}(x_0), f^{(n_\varepsilon+1)}_{h_{n_\varepsilon+1} h_{n_\varepsilon+2} \cdots h_j}(x_0)\Big) < \frac{\varepsilon}{|Q|}|Q| = \varepsilon. \quad \square \end{aligned}$$

Corollary 3.2. *K is compact and $K \subseteq Q$.*

Proposition 3.3. *For every $C \in \mathfrak{B}$ $\lim_n D((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), K) = 0$.*

Proof. Let $x_0 \in Q$, $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ be so that $\prod_{j=1}^{n_\varepsilon} \rho^{(j)} < \varepsilon/|Q|$. For every $n \in \mathbb{N}$ with $n > n_\varepsilon$ we have $D(p_n(\pi_{1,2,\dots,n}(I)), K) \le \varepsilon$.

Indeed, for every $k \in I$, we have $p(k) \in K$ and

$$\begin{aligned} d_X\Big(p_n(\pi_{1,2,\dots,n}(k)), p(k)\Big) &\le \Big(\prod_{j=1}^{n} \rho^{(j)}\Big) d_X\Big(x_0, \lim_i f^{(n+1)}_{k_{n+1}k_{n+2}\cdots k_i}(x_0)\Big) \\ &< \Big(\prod_{j=1}^{n_\varepsilon} \rho^{(j)}\Big)|Q| < \varepsilon \end{aligned}$$

from which $d_X(p_n(\pi_{1,2,\dots,n}(k)), K) < \varepsilon$. On the other hand, $\forall x \in K$ there exists $k \in I$ so that $p(k) = x$ and then $d_X(p_n(\pi_{1,2,\dots,n}(I)), x) < \varepsilon$.

But

$$p_n(\pi_{1,2,\dots,n}(I)) = \bigcup_{k_1=1}^{m_1} \bigcup_{k_2=1}^{m_2} \cdots \bigcup_{k_n=1}^{m_n} \{f_{k_1k_2\cdots k_n}(x_0)\} = (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\}).$$

Now, let $C \in \mathfrak{B}$: by Lemmas 2.3 and 2.2 it follows that $\forall n \in \mathbb{N}$

$$D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\})\Big) \le \Big(\prod_{j=1}^{n} \rho^{(j)}\Big) D(C, \{x_0\});$$

then, if $n > n_\varepsilon$,

$$\begin{aligned} &D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), K\Big) \\ &\quad \le D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(C), (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\})\Big) \\ &\quad + D\Big((\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(\{x_0\}), K\Big) < \varepsilon\Big(1 + \frac{D(C, \{x_0\})}{|Q|}\Big). \qquad \square \end{aligned}$$

3.3. Some properties of K. We follow the notation of the previous paragraphs.

Definition. Given a finite family of contraction maps $\mathcal{G} = \{g_1, g_2, \dots, g_m\}$ and a subset A of X, we say that A is invariant with respect to $\mathcal{G}$ if $\mathcal{G}(A) = A$.

Remark 4. If $\mathfrak{F}_n = \mathfrak{F} = \{f_1, f_2, \dots, f_m\}$ for every $n \in \mathbb{N}$, then K is invariant with respect to $\mathfrak{F}$.

If there exists $k \in \mathbb{N}$ such that $\mathfrak{F}_{n+k} = \mathfrak{F}_n$ for all $n \in \mathbb{N}$, then K is invariant with respect to $\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_k$.

Remark 5. If there exists $n \in \mathbb{N}$ such that $\mathfrak{F}_{n+k} = \mathfrak{F}_n$ for all $k \in \mathbb{N}$, then there is a unique non-empty compact set $H \subseteq X$ which is invariant with respect to $\mathfrak{F}_n$. Moreover,

$$K = \bigcup_{i_1=1}^{m_1} \bigcup_{i_2=1}^{m_2} \cdots \bigcup_{i_n=1}^{m_n} f_{i_1i_2\cdots i_n}(H).$$

Lemma 3.2. *Let* $A = \bigcup_{j=1}^{N} A_j \subseteq X$. *If* A *is connected, then* $|A| \le \sum_{j=1}^{N} |A_j|$.

Proof. We consider the case $N=2$, the general statement follows by induction.

Let $A = B \bigcup C$ be a connected subset of X and let $x, y \in A$. If $x, y \in B$ or $x, y \in C$; then $d_X(x,y) \le |B| + |C|$.

If $x \in B$ and $y \in C$, then $\forall z \in B$, $\forall w \in C$,

$$d_X(x,y) \le d_X(x,z) + d_X(z,w) + d_X(w,y) \le |B| + d_X(z,w) + |C|.$$

It follows that

$$|A| \le |B| + |C| + \inf\Big\{ d_X(z,w) \mid z \in B,\ w \in C \Big\}.$$

Let us suppose that $\inf\{d_X(z,w) \mid z \in B,\ w \in C\} = \varepsilon > 0$; then $B' = \bigcup_{x\in B} D_X(x, \varepsilon/4)$ and $C' = \bigcup_{y \in C} D_X(y, \varepsilon/4)$ are disjoint open sets whose union contains $A \bigcup B$. □

Corollary 3.4. *If* $\lim_n \prod_{k=1}^n \sum_{i=1}^{m_k} \rho_i^{(k)} = 0$, *then* K *is totally disconnected.*

Proof. Let $x, y \in K$ with $x \ne y$ and let $n_{xy} \in \mathbb{N}$ be such that $\prod_{k=1}^n \sum_{i=1}^{m_k} \rho_i^{(k)} < d_X(x,y)/|Q|$ $\forall n \in \mathbb{N}$ with $n > n_{xy}$. Then, if $n > n_{xy}$, we have

$$\begin{aligned} K &= \lim_k (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_k)(Q) \\ &= (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)\Big(\lim_k (\mathfrak{F}_{n+1} \circ \mathfrak{F}_{n+2} \circ \cdots \circ \mathfrak{F}_k)(Q)\Big) \end{aligned}$$

because, by Lemmas 2.2 and 2.3, $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ are contraction maps with respect to the Hausdorff metric and then they are continuous. It follows that

$$K \subseteq (\mathfrak{F}_1 \circ \mathfrak{F}_2 \circ \cdots \circ \mathfrak{F}_n)(Q).$$

Now, let $A \subseteq K$ be connected and such that $x, y \in A$: we have

$$A \subseteq K \subseteq \bigcup_{i_1=1}^{m_1} \bigcup_{i_2=1}^{m_2} \cdots \bigcup_{i_n=1}^{m_n} \overline{f_{i_1 i_2 \cdots i_n}(Q)}$$

and by Lemma 3.2

$$\begin{aligned} |A| &\le \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} |f_{i_1 i_2 \cdots i_n}(Q)| \\ &\le \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^n \rho_{i_k}^{(k)} \Big) |Q| = |Q| \prod_{k=1}^n \sum_{i=1}^{m_k} \rho_i^{(k)} < d_X(x,y) \le |A|. \qquad \square \end{aligned}$$

3.4. Examples.

3.4.1. *Cantor sets in* $\mathbb{R}$. Let X be the set of real numbers with the Euclidean distance; we construct a generalized version of the Cantor set. To do this, we suppose that we are given a real number $d \in]0,1[$ and we construct two sequences (t_n) and (d_n) in the following way:

- $t_0 = 1$ and $m_n t_n^d = t_{n-1}^d$;
- $d_n = \frac{t_{n-1} - m_n t_n}{m_n - 1}$.

Then the set K of the previous paragraph is obtained by setting

$$f_i^{(n)}(x) = \frac{t_n x + (i-1)(t_n + d_n)}{t_{n-1}} = m_n^{-\frac{1}{d}} x + \frac{i-1}{m_n - 1}(1 - m_n^{-\frac{1}{d}}) \;\; \forall n \in \mathbb{N} \;\; \forall i \in I_n.$$

Now we show some properties of the fractal set K so obtained. In the definition of the functions p_n we assume $x_0 = 0$.

Remark 6. The functions p_n are given by

$$p_n(i_1, i_2, \ldots, i_n) = \sum_{j=1}^{n} (i_j - 1)(t_j + d_j). \tag{1}$$

We prove this by induction: let $n \in \mathbb{N}$, we have $f_{i_n}^{(n)}(0) = \frac{1}{t_{n-1}}(i_n - 1)(t_n + d_n)$. Let us suppose that for $k \in \mathbb{N}$ $2 \le k \le n$,

$$f_{i_k i_{k+1} \cdots i_n}^{(k)}(0) = \frac{1}{t_{k-1}} \sum_{j=k}^{n} (i_j - 1)(t_j + d_j), \tag{2}$$

then

$$f_{i_{k-1} i_k \cdots i_n}^{(k-1)}(0) = f_{i_{k-1}}^{(k-1)}(f_{i_k i_{k+1} \cdots i_n}^{(k)}(0)) = \frac{1}{t_{k-2}} \sum_{j=k-1}^{n} (i_j - 1)(t_j + d_j).$$

Then (2) holds for any $k \le n$ and in particular

$$p_n(i_1, i_2, \ldots, i_n) = f_{i_1 i_2 \cdots i_n}(0) = \sum_{j=1}^{n} (i_j - 1)(t_j + d_j).$$

It follows that

$$p(i) = \sum_{j=1}^{+\infty} (i_j - 1)(t_j + d_j) \;\; \forall i \in I \tag{3}$$

and

$$K = \left\{ \sum_{j=1}^{+\infty} (i_j - 1)(t_j + d_j) \mid i_j \in I_j \;\; \forall j \in \mathbb{N} \right\}. \tag{4}$$

Example 1. If $m_n = 2$ $\forall n \in \mathbb{N}$ and $d = \log_3 2$, then K is the classical Cantor set.

Indeed, in this case $t_n = d_n = 3^{-n}$, $f_1^{(n)}(x) = x/3$ and $f_2^{(n)}(x) = (x + 2)/3$ $\forall n \in \mathbb{N}$ and by Remark 4 K is invariant with respect to $\mathfrak{F} = \{f_1^{(1)}, f_2^{(1)}\}$.

It may also be noted that (4) becomes

$$K=\Big\{\sum_{j=1}^{+\infty}\frac{c_j}{3^j}\mid c_j\in\{0,2\}\quad\forall j\in\mathbb{N}\Big\}.$$

Remark 7. It is easy to prove, by induction, that

$$\forall n\in\mathbb{N}\ \ t_n=\Big(\prod_{j=1}^{n}m_j\Big)^{-\frac{1}{d}}\text{ and }d_n=\frac{1-m_n^{1-\frac{1}{d}}}{m_n-1}\Big(\prod_{j=1}^{n-1}m_j\Big)^{-\frac{1}{d}}.$$

Remark 8. From Remark 6 we have $K\subseteq[0,1]$ and $0,1\in K$.

Indeed, if we set $k,h\in I$, $k_j=1$, $h_j=m_j$ for every $j\in\mathbb{N}$, then by (3) , $p(k)=0$, $p(h)=\sum_{j=1}^{+\infty}(t_{j-1}-t_j)=t_0=1$ and for all $i\in I$ $0\le p(i)\le p(h)=1$.

It may also be noted that $0\le f_j^{(n)}(x)\le 1$ $\forall x\in[0,1]$ and for every $n,j\in\mathbb{N}$, $j\le m_n$; then we can set $Q=[0,1]$.

Remark 9. Given $n\in\mathbb{N}$, the $\prod_{j=1}^{n}m_j$ intervals of the form $[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n]$ are pairwise disjoint and $|p_n(i_1,i_2,\dots,i_n)-(p_n(k_1,k_2,\dots,k_n)+t_n)|\ge d_n$ for any different $(i_1,i_2,\dots,i_n),(k_1,k_2,\dots,k_n)\in\prod_{j=1}^{n}I_j$.

Moreover, for all $(i_1,i_2,\dots,i_{n+1})\in\prod_{j=1}^{n+1}I_j$ the interval $[p_n(i_1,i_2,\dots,i_n),$ $p_n(i_1,i_2,\dots,i_n)+t_n]$ contains $[p_n(i_1,i_2,\dots,i_{n+1}),p_n(i_1,i_2,\dots,i_{n+1})+t_{n+1}]$ and

$$K=\bigcap_{n=1}^{+\infty}\Big(\bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_n=1}^{m_n}\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big).$$

Now we are going to prove that K is a d-set if and only if the sequence (m_n) is bounded.

Theorem 3.5. *We have* $\mathcal{H}^d(K)=1$ *and so* $\dim K=d$.

Proof. See [3] Chapter 1, Theorem 1.15. □

Lemma 3.3. *For every* $n\in\mathbb{N}$ *and* $(i_1,i_2,\dots,i_n)\in\prod_{j=1}^{n}I_j$*, we have*

$$\mathcal{H}^d\Big(K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big)=t_n^d.$$

Proof. Let $n\in\mathbb{N}$ and $(i_1,i_2,\dots,i_n)\in\prod_{j=1}^{n}I_j$; we define

$$f:K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\to K\bigcap[0,t_n],$$
$$f(x)=x-p_n(i_1,i_2,\dots,i_n).$$

It is clear that $0\le f(x)\le t_n$; moreover, $f(x)\in K$ by (3) and (1). Then f is well defined.

The function f is one-to-one because it is injective and $\forall y\in K\bigcap[0,t_n]$, $y=f(y+p_n(i_1,i_2,\dots,i_n))$. Moreover, f is an isometry and then, by Lemma 2.1, $\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)\le\mathcal{H}^d\Big(K\bigcap[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n]\Big)$. By applying the same arguments to f^{-1} we obtain the opposite inequality.

Finally,

$$1=\mathcal{H}^d(K)=\mathcal{H}^d\Big(\bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_n=1}^{m_n}\Big(K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big)\Big)$$
$$=\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\mathcal{H}^d\Big(K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big)$$
$$=\Big(\prod_{j=1}^{n}m_j\Big)\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)$$

and then

$$\mathcal{H}^d\Big(K\bigcap\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big]\Big)=\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)$$
$$=\Big(\prod_{j=1}^{n}m_j\Big)^{-1}=t_n^d.\qquad\Box$$

Proposition 3.6. *The set* K *is a d-set if and only if the sequence* (m_n) *is bounded.*

Proof. Let $x_0\in K$ and $r\in]0,1]$; by Remark 9, for every $k\in\mathbb{N}$ there exist $(i_1,i_2,\dots,i_k)\in\prod_{j=1}^k I_j$ such that $x_0\in[p_k(i_1,i_2,\dots,i_k),p_k(i_1,i_2,\dots,i_k)+t_k]$. Let

$$n=\min\Big\{k\in\mathbb{N}\mid\exists(i_1,i_2,\dots,i_k)\in\prod_{j=1}^{k}I_j\text{ so that }$$
$$x_0\in\Big[p_k(i_1,i_2,\dots,i_k),p_k(i_1,i_2,\dots,i_k)+t_k\Big]\subseteq[x_0-r,x_0+r]\Big\}$$

(n is well defined because $\lim_k t_k=0$ by Remark 7). We prove that

$$\frac{1}{m_n}r^d\le\mathcal{H}^d\Big(k\bigcap[x_0-r,x_0+r]\Big)\le 2^{1+d}m_nr^d;\tag{5}$$

it will follow that K is a d-set if the sequence (m_n) is bounded.

By Lemma 3.3 $\mathcal{H}^d\Big(K\bigcap[x_0-r,x_0+r]\Big)\ge t_n^d=t_{n-1}^d/m_n$; if $t_{n-1}\le r$, we would have $x_0\in[p_{n-1}(i_1,i_2,\dots,i_{n-1}),p_{n-1}(i_1,i_2,\dots,i_{n-1})+t_{n-1}]\subseteq[x_0-r,x_0+r]$ and this is absurd; then the first inequality follows.

Let us prove the second inequality: we have $[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n]\subseteq[x_0-r,x_0+r]\bigcap[p_{n-1}(i_1,i_2,\dots,i_{n-1}),p_{n-1}(i_1,i_2,\dots,i_{n-1})+t_{n-1}]$ and for every $j\in\mathbb{N}$ the following implications hold:

- if $1\le j<i_{n-1}-1$ then $[x_0-r,x_0+r]\bigcap[p_{n-1}(i_1,i_2,\dots,i_{n-2},j),p_{n-1}(i_1,i_2,\dots,i_{n-2},j)+t_{n-1}]=\varnothing$ because otherwise we would have $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}-1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}-1)+t_{n-1}]\subseteq[x_0-r,x_0+r]$;

- if $i_{n-1}+1<j\le m_{n-1}$ then $[x_0-r,x_0+r]\bigcap[p_{n-1}(i_1,i_2,\dots,i_{n-2},j),p_{n-1}(i_1,i_2,\dots,i_{n-2},j)+t_{n-1}]=\varnothing$ because otherwise we would have $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1)+t_{n-1}]\subseteq[x_0-r,x_0+r]$.

Moreover, at least one of the intervals $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}-1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}-1)+t_{n-1}]$ and $[p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1),p_{n-1}(i_1,i_2,\dots,i_{n-2},i_{n-1}+1)+t_{n-1}]$ does not intersect $[x_0-r,x_0+r]$ because otherwise we would have $[p_{n-1}(i_1,i_2,\dots,i_{n-1}),p_{n-1}(i_1,i_2,\dots,i_{n-1})+t_{n-1}]\subseteq[x_0-r,x_0+r]$.

Then $\mathcal{H}^d\Big(K\bigcap[x_0-r,x_0+r]\Big)<2t_{n-1}^d=2m_nt_n^d\le 2^{1+d}m_nr^d$.

Now we suppose that the sequence (m_n) is not bounded; by taking $x_0=0$ and $r_n=t_n+d_n$ for every $n\in\mathbb{N}$, we have

$$\frac{\mathcal{H}^d\Big(K\bigcap[x_0-r_n,x_0+r_n]\Big)}{r_n^d}=\frac{\mathcal{H}^d\Big(K\bigcap[0,t_n]\Big)}{r_n^d}=\frac{t_n^d}{(t_n+d_n)^d}=\left(\frac{m_n-1}{m_n^{\frac{1}{d}}-1}\right)^d.$$

Let (m_{n_k}) be a subsequence of (m_n) such that $\lim_k m_{n_k}=+\infty$; then

$$\lim_k\frac{\mathcal{H}^d\Big(K\bigcap[x_0-r_{n_k},x_0+r_{n_k}]\Big)}{r_{n_k}^d}=0$$

because $d<1$. □

Remark 10. If in the above proposition we suppose $t_k\le d_k$ for all $k\in\mathbb{N}$, then (5) becomes

$$\frac{1}{m_n}r^d\le\mathcal{H}^d\Big(k\bigcap[x_0-r,x_0+r]\Big)\le 2^dm_nr^d. \tag{6}$$

We recall that by Remark 7 $t_k\le d_k$ if $d\le\log_{(2m_k-1)}m_k$. If $d\le\log_3 2$, then $t_k\le d_k$ independently of m_k.

Example 2. For the classical Cantor set, (6) gives

$$\frac{1}{2}r^d\le\mathcal{H}^d\Big(k\bigcap[x_0-r,x_0+r]\Big)\le 2^{1+d}r^d\quad\forall x_0\in K\quad\forall r\in]0,1].$$

Now we estimate the entropy numbers of K under the assumption that $t_k\le d_k$ $\forall k\in\mathbb{N}$.

To avoid tedious notation, we set for every $k\in\mathbb{N}$,

$$C_k=\bigcup_{i_1=1}^{m_1}\bigcup_{i_2=1}^{m_2}\cdots\bigcup_{i_n=1}^{m_n}\Big[p_n(i_1,i_2,\dots,i_n),p_n(i_1,i_2,\dots,i_n)+t_n\Big].$$

By Remark 9 $K=\bigcap_{k=1}^{+\infty}C_k$.

Let $k\in\mathbb{N}$ and $n_k=\prod_{j=1}^k m_j$; then $\varepsilon_{n_k}(K)\le\varepsilon_{n_k}(C_k)\le t_k/2$.

Since the extreme points of the intervals of C_k are in K, it follows that

$$\varepsilon_{n_k}(K)=\frac{1}{2}t_k=\frac{1}{2}n_k^{-\frac{1}{d}}. \tag{7}$$

Let $h \in \mathbb{N}$ be a divisor of m_{k+1}: we compute $\varepsilon_{hn_k}(K)$.

The set C_{k+1} is a disjoint union of n_{k+1} closed intervals with amplitude t_{k+1} and mutual distance greater than or equal to t_{k+1}. Since hn_k divides $n_{k+1} = m_{k+1}n_k$, we can cover all the m_{k+1} closed intervals of C_{k+1} that are included into a single interval of C_k with h closed intervals of the form

$$\Big[p_{k+1}\Big(i_1, i_2, \dots, i_k, \frac{lm_{k+1}}{h} + 1\Big), p_{k+1}\Big(i_1, i_2, \dots, i_k, \frac{(l+1)m_{k+1}}{h}\Big) + t_{k+1}\Big]$$
$$0 \le l < h$$

and, as before, the extreme points of these intervals are in K; so

$$\varepsilon_{hn_k}(K) = \frac{1}{2}\Big(\frac{m_{k+1}}{h}t_{k+1} + \Big(\frac{m_{k+1}}{h} - 1\Big)d_{k+1}\Big)$$

and by Remark 7 we have

$$\varepsilon_{hn_k}(K) = \frac{(h-1)m_{k+1}^{1-\frac{1}{d}} + m_{k+1} - h}{2h^{1-\frac{1}{d}}(m_{k+1} - 1)}(hn_k)^{-\frac{1}{d}}. \tag{8}$$

If h is not a divisor of m_{k+1}, we have $\varepsilon_{hn_k}(K) \le t_k/(2h)$ and then

$$\varepsilon_{hn_k}(K) \le \frac{1}{2}\, h^{\frac{1}{d}-1}(hn_k)^{-\frac{1}{d}}. \tag{9}$$

Let $\lfloor m_{k+1}/h \rfloor = \max\{n \in \mathbb{N} \mid n \le m_{k+1}/h\}$; then we may not cover K by using hn_k intervals of length $(t_{k+1} + d_{k+1})\lfloor m_{k+1}/h \rfloor$ because at least one of the intervals of C_{k+1} will not be covered; so it must be

$$\varepsilon_{hn_k}(K) > \frac{1}{2}\,(t_{k+1} + d_{k+1})\Big\lfloor \frac{m_{k+1}}{h} \Big\rfloor,$$

i.e.,

$$\varepsilon_{hn_k}(K) > \frac{h^{\frac{1}{d}}(1 - m_{k+1}^{-\frac{1}{d}})}{2(m_{k+1} - 1)}\Big\lfloor \frac{m_{k+1}}{h} \Big\rfloor (hn_k)^{-\frac{1}{d}}. \tag{10}$$

Finally, if $l \in \mathbb{N}$ $l < n_k$, then

$$\varepsilon_{hn_k+l}(K) = \varepsilon_{hn_k}(K) \tag{11}$$

because the additional l intervals can not be equally distributed between the connected components of C_k.

Example 3. Let K be the classical Cantor set; by (7) and (11) we have

$$\varepsilon_{2^k+l}(K) = 2^{-\frac{k}{d}-1} = \frac{1}{2}3^{-k}.$$

It follows that the entropy numbers of the classical Cantor set have the following asymptotic behaviour:

$$\frac{1}{2}n^{-\frac{1}{d}} \le \varepsilon_n(K) < \frac{3}{2}n^{-\frac{1}{d}} \quad \forall n \in \mathbb{N}$$

(see also [6], Example 2.2).

Remark 11. If the sequence (m_n) is bounded, then K is a d-set by Proposition 3.6 and we can apply Proposition 3.1 of [6] to see that there exist $a, b \in]0, +\infty[$ such that

$$an^{-\frac{1}{d}} \leq \varepsilon_n(K) \leq bn^{-\frac{1}{d}} \quad \forall n \in \mathbb{N} .$$

Moreover, by Corollary 2.7 of [6], the box dimension of K is d.

Remark 12. Let us suppose that the sequence (m_n) is not bounded and $t_k \leq d_k \ \ \forall k \in \mathbb{N}$. Let (m_{n_k}) be a subsequence of (m_n) such that $\lim_k m_{n_k} = +\infty$; we set $p_k = \prod_{j=1}^{n_k-1} m_j$ for every $k \in \mathbb{N}$; then, by (7) we have

$$\varepsilon_{p_k}(K) = \frac{1}{2} p_k^{-\frac{1}{d}}.$$

If m_{n_k} is even, then, by setting $2h_k = m_{n_k}$, we have from (8)

$$\varepsilon_{h_k p_k}(K)(h_k p_k)^{\frac{1}{d}} = \frac{2^{1-\frac{1}{d}}(h_k - 1)h_k + h_k^{1+\frac{1}{d}}}{2h_k(2h_k - 1)}. \tag{12}$$

If m_{n_k} is odd, then, we set $2h_k = m_{n_k} + 1$ and by (10) it follows

$$\varepsilon_{h_k p_k}(K)(h_k p_k)^{\frac{1}{d}} > \frac{h_k^{\frac{1}{d}}(1 - m_{n_k}^{-\frac{1}{d}})}{2(m_{n_k} - 1)} \left\lfloor \frac{m_{n_k}}{h_k} \right\rfloor,$$

but $\lfloor m_{n_k}/h_k \rfloor = \lfloor 2 - h_k^{-1} \rfloor = 1$; then

$$\varepsilon_{h_k p_k}(K)(h_k p_k)^{\frac{1}{d}} > \frac{h_k^{\frac{1}{d}}\left(1 - (2h_k - 1)^{-\frac{1}{d}}\right)}{4(h_k - 1)}. \tag{13}$$

Since $\lim_k h_k = +\infty$, it follows from (12) and (13) that

$$\limsup_n \varepsilon_n(K) n^{\frac{1}{d}} = +\infty.$$

3.4.2. *Sierpiński gaskets.* Let X be the plane $\mathbb{R}^2$ with the Euclidean distance; a generalized version of the Sierpiński gasket may be constructed in the following way.

Let (k_n) be a sequence of integer numbers, with $k_n \geq 2$ for all $n \in \mathbb{N}$; the sequence (m_n) is given by

$$m_n = \sum_{j=0}^{k_n} j = \frac{1}{2} k_n(k_n + 1) \quad \forall n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, $i \in I_n$ the contraction $f_i^{(n)}$ is given by

$$f_i^{(n)}(x, y) = \left(\frac{x}{k_n}, \frac{y}{k_n}\right) + (a_i^{(n)}, b_i^{(n)}),$$

where

$$a_i^{(n)} = \frac{k_n - (h_i^{(n)} + 1)^2 + 2(i - 1)}{2k_n}, \ \ b_i^{(n)} = \frac{\sqrt{3}(k_n - h_i^{(n)} - 1)}{2k_n}$$

and $0 \le h_i^{(n)} < k_n$ is so that

$$\sum_{j=0}^{h_i^{(n)}} j < i \le \sum_{j=0}^{h_i^{(n)}+1} j.$$

As before we set $K = p(I)$.

Remark 13. If $k_n = 2$ for all $n \in \mathbb{N}$, then K is the Sierpiński gasket. Indeed, in this case, we have $m_n = 3$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} f_1^{(n)}(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big) + \Big(\frac{1}{4}, \frac{\sqrt{3}}{4}\Big), \\ f_2^{(n)}(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big), \\ f_3^{(n)}(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big) + \Big(\frac{1}{2}, 0\Big). \end{aligned} \tag{14}$$

Remark 14. Even for the points of the Sierpiński gasket we can give a representation by means of series.

For the sake of simplicity we only consider the case in which $k_n = 2 \ \forall n \in \mathbb{N}$. Let us consider the functions $f, g : \{1,2,3\} \to \mathbb{Z}$,

$$f(i) = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ 2 & i = 3 \end{cases}, \quad g(i) = \begin{cases} 1 & i = 1 \\ 0 & i > 1 \end{cases}. \tag{15}$$

Let $x_0 = (0,0)$, then for every $n \in \mathbb{N}$ and $i_1, i_2, \ldots, i_n \in \{1,2,3\}$ we have

$$p_n(i_1, i_2, \ldots, i_n) = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{2^j} \Big(f(i_j), \sqrt{3} g(i_j)\Big). \tag{16}$$

As in Remark 6, even (16) is proven by induction. Let $n \in \mathbb{N}$, by (14) and (15) we have

$$f_{i_n}^{(n)}(0,0) = \frac{1}{4} \Big(f(i_n), \sqrt{3} g(i_n)\Big).$$

Let us suppose that for $k \in \mathbb{N}$ $k \le n$,

$$f_{i_{k+1} i_{k+2} \cdots i_n}^{(k+1)}(0,0) = \frac{1}{2} \sum_{j=k+1}^{n} \frac{1}{2^{j-k}} \Big(f(i_j), \sqrt{3} g(i_j)\Big),$$

then

$$\begin{aligned} f_{i_k i_{k+1} \cdots i_n}^{(k)}(0,0) &= \frac{1}{4} \sum_{j=k+1}^{n} \frac{1}{2^{j-k}} \Big(f(i_j), \sqrt{3} g(i_j)\Big) + \frac{1}{4}\Big(f(i_k), \sqrt{3} g(i_k)\Big) \\ &= \frac{1}{2} \sum_{j=k}^{n} \frac{1}{2^{j-k+1}} \Big(f(i_j), \sqrt{3} g(i_j)\Big). \end{aligned}$$

In particular, for $k = 1$ we have (16).

It follows that

$$p(i) = \frac{1}{2}\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big(f(i_j), \sqrt{3}g(i_j)\Big) \quad \forall i \in I$$

and

$$K = \left\{\frac{1}{2}\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big(f(i_j), \sqrt{3}g(i_j)\Big) \mid i_j \in \{1,2,3\} \quad \forall j \in \mathbb{N}\right\}.$$

4. A measure on K

In this section X is a complete separable metric space.

We recall (§ 2.4) that for every $n \in \mathbb{N}$, $l_1^{(n)}, l_2^{(n)}, \dots, l_{m_n}^{(n)}$ are real numbers in $]0,1[$ so that $\sum_{j=1}^{m_n} l_j^{(n)} = 1$, τ_n is a measure on I_n defined by $\tau_n(A) = \sum_{j\in A} l_j^{(n)}$ $\forall A \subseteq I_n$ and τ is a unique Radon measure on I such that $\tau(I) = 1$ and $\tau_{i_1} \times \tau_{i_2} \times \cdots \times \tau_{i_n} = \pi_{i_1,i_2,\dots,i_n\sharp}\tau$ for any distinct $i_1, i_2, \dots, i_n \in \mathbb{N}$.

Definition. We set $\mu_K = p_\sharp \tau$.

Remark 1. K is the support of μ_K and $\mu_K(K) = 1$.

Definition. Let (ν_n) be a sequence of Radon measures on X. We say that the sequence (ν_n) converges weakly to a Radon measure ν if

$$\lim_n \int_X f d\nu_n = \int_X f d\nu \quad \forall f \in C_c(X).$$

We denote this fact by writing $\nu_n \rightharpoonup \nu$.

Remark 2. We recall that if ν is a Borel regular measure on X and for every $x \in X$ there is $r > 0$ such that $\nu(B_X(x,r)) < +\infty$, then ν is a Radon measure on X. For the proof see [8], chapter 5, theorem V.5.3.

Proposition 4.1. *Let ν be a Borel regular measure on X, with bounded support and such that $\nu(X) = 1$. For every $n \in \mathbb{N}$ we set*

$$\nu_n = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\Big(\prod_{k=1}^{n} l_{i_k}^{(k)}\Big) f_{i_1 i_2 \cdots i_n \sharp}\nu.$$

Then $\nu_n \rightharpoonup \mu_K$.

Proof. Let ν be a Borel regular measure on X with bounded support and such that $\nu(X) = 1$; let C be the support of ν and let $f \in C_c(X)$. We prove that

$$\lim_n \int_X f d\nu_n = \int_X f d\mu_K.$$

Let $\varepsilon > 0$ and $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ $\forall x, y \in X$ with $d_X(x,y) < \delta$; we consider $n'_\varepsilon \in \mathbb{N}$ such that

$$\Big(\prod_{k=1}^{n'_\varepsilon}\rho^{(k)}\Big)\sup_{x\in C} d_X(x, x_0) < \delta. \tag{1}$$

Moreover, let $n''_\varepsilon \in \mathbb{N}$ be such that $|Q| \prod_{j=1}^{n''_\varepsilon} \rho^{(j)} < \delta$; for every $n \in \mathbb{N}$ with $n > n''_\varepsilon$ we have

$$d_X(p_n(\pi_{1,2,\ldots,n}(i)), p(i)) = \lim_k d_X\Big(f_{i_1 i_2 \cdots i_n}(x_0), f_{i_1 i_2 \cdots i_n}(f^{(n+1)}_{i_{n+1} i_{n+2} \cdots i_k}(x_0))\Big)$$
$$\leq |Q| \prod_{j=1}^{n} \rho^{(j)} < \delta \quad \forall i \in I \quad \forall n \in \mathbb{N} \quad n > n''_\varepsilon$$

and then

$$\Big| f(p_n(\pi_{1,2,\ldots,n}(i))) - f(p(i)) \Big| < \varepsilon \quad \forall i \in I \quad \forall n \in \mathbb{N} \quad n > n''_\varepsilon. \tag{2}$$

Then, if $n > \max\{n'_\varepsilon, n''_\varepsilon\}$, we have

$$\Big| \int_X f d\nu_n - \int_X f d\mu_K \Big|$$
$$= \Big| \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l^{(k)}_{i_k} \Big) \int_X f d f_{i_1 i_2 \cdots i_n \sharp} \nu - \int_X f dp_\sharp \tau \Big|$$
$$\leq \Big| \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l^{(k)}_{i_k} \Big) \Big(\int_X f(f_{i_1 i_2 \cdots i_n}(x)) d\nu(x) - f(f_{i_1 i_2 \cdots i_n}(x_0)) \Big) \Big|$$
$$+ \Big| \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l^{(k)}_{i_k} \Big) f(f_{i_1 i_2 \cdots i_n}(x_0)) - \int_I f(p(i)) d\tau(i) \Big|$$
$$\leq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l^{(k)}_{i_k} \Big) \Big| \int_X f(f_{i_1 i_2 \cdots i_n}(x)) d\nu(x) - \int_X f(f_{i_1 i_2 \cdots i_n}(x_0)) d\nu(x) \Big|$$
$$+ \Big| \int_{\prod_{j=1}^{n} I_j} f(p_n(i_1, i_2, \ldots, i_n)) d(\tau_1 \times \tau_2 \times \cdots \times \tau_n) - \int_I f(p(i)) d\tau(i) \Big|$$
$$\leq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l^{(k)}_{i_k} \Big) \int_X \Big| f(f_{i_1 i_2 \cdots i_n}(x)) - f(f_{i_1 i_2 \cdots i_n}(x_0)) \Big| d\nu(x)$$
$$+ \Big| \int_I f(p_n(\pi_{1,2,\ldots,n}(i))) d\tau(i) - \int_I f(p(i)) d\tau(i) \Big|$$
$$\leq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \Big(\prod_{k=1}^{n} l^{(k)}_{i_k} \Big) \int_C \Big| f(f_{i_1 i_2 \cdots i_n}(x)) - f(f_{i_1 i_2 \cdots i_n}(x_0)) \Big| d\nu(x)$$
$$+ \int_I \Big| f(p_n(\pi_{1,2,\ldots,n}(i))) - f(p(i)) \Big| d\tau(i).$$

By (1)

$$d_X\Big(f_{i_1 i_2 \cdots i_n}(x), f_{i_1 i_2 \cdots i_n}(x_0)\Big) < \delta \quad \forall x \in C$$

and then

$$\Big| f(f_{i_1 i_2 \cdots i_n}(x)) - f(f_{i_1 i_2 \cdots i_n}(x_0)) \Big| < \varepsilon \quad \forall x \in C. \tag{3}$$

Then, by (3) and (2) it follows

$$\left|\int_X f d\nu_n - \int_X f d\mu_K\right| < \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\left(\prod_{k=1}^{n} l_{i_k}^{(k)}\right)\varepsilon\nu(C) + \varepsilon\tau(I) = 2\varepsilon$$

because

$$\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\left(\prod_{k=1}^{n} l_{i_k}^{(k)}\right) = 1. \qquad \square$$

Acknowledgement

I wish to thank Prof. H. Triebel who invited me to give a lecture on these topics in Jena and M. Bricchi and Profs. H. Triebel and M. Zähle for the helpful discussion.

References

1. B. Carl and I. Stephani, Entropy, compactness and the approximation of operators. *Cambridge University Press, Cambridge*, 1990.
2. V. Checcucci, A. Tognoli, and E. Vesentini, Lezioni di topologia generale. *Feltrinelli*, 1977.
3. K. J. Falconer, The geometry of fractal sets. *Cambridge University Press, Cambridge*, 1985.
4. K. J. Falconer, Fractal geometry. *John Wiley & Sons*, 1990.
5. G. B. Folland, Real analysis. *John Wiley & Sons*, 1990.
6. G. Follo, Some remarks on entropy numbers of sets with fractional dimension. *Jenaer Schriften zur Mathematik und Informatik*, 2000.
7. J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30**(1981), 713–747.
8. K. Jacobs, Measure and integral. *Academic Press, New York*, 1978.
9. A. Jonsson and H. Wallin, Function spaces on subsets of $\mathbb{R}^N$. *Harwood Academic Publisher, London*, 1984.
10. P. Mattila, Geometry of sets and measures in Euclidean spaces. *Cambridge University Press, Cambridge,* 1995.
11. H. Triebel, Fractals and spectra: related to Fourier analysis and function spaces. *Birkhauser*, 1997.

(Received 11.04.2001)

Author's address:
University of Milano
Dipartimento di matematica
Via Saldini n. 50
Milano 20133
Italy
E-mail: follo@socrates.mat.unimi.it