**Georgian Mathematical Journal**
Volume 8 (2001), Number 4, 727–732

# THE REARRANGEMENT INEQUALITY FOR THE ERGODIC MAXIMAL FUNCTION

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**Abstract**. The equivalence of the decreasing rearrangement of the ergodic maximal function and the maximal function of the decreasing rearrangement is proved. Exact constants are obtained in the corresponding inequalities.

**2000 Mathematics Subject Classification:** 28D05, 26D15.
**Key words and phrases:** Measure-preserving ergodic transformation, ergodic maximal function, decreasing rearrangement.

Let $(X, \mathbb{S}, \mu)$ be a $\sigma$-finite measure space and $T : X \to X$ be a measure-preserving ergodic transformation. For a measurable function $f$ the ergodic maximal function is defined as

$$Mf(x) = \sup_N \frac{1}{N} \sum_{k=0}^{N-1} |f(T^k x)|, \quad x \in X.$$

The decreasing rearrangement of $f$ is the function $f^*$ defined on $[0, \infty)$ by

$$f^*(t) = \inf \Big\{ \lambda : \mu(|f| > \lambda) \leq t \Big\} \tag{1}$$

and its maximal function is denoted by $f^{**}$:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) d\tau, \quad t > 0.$$

The equivalence of $(Mf)^*$ and $f^{**}$, i.e., the validity of inequalities

$$cf^{**}(t) \leq (Mf)^*(t) \leq Cf^{**}(t)$$

with constants $c$ and $C$ independent of $f$ and $t$ (these inequalities sometimes are called rearrangement inequalities) was proved by several authors when $M$ stands for Hardy–Littlewood maximal operator (see [8], [5] for the one-dimensional case and [1] for higher dimensions). This fact is very useful in the proofs of many theorems on the related topics (see [2]).

In the present paper, we prove analogous inequalities for the ergodic maximal operator (see (2) below). The constants $\frac{1}{2}$ and 1 in these inequalities are exact and the corresponding examples are constructed.

ISSN 1072-947X / \$8.00 / 

**Theorem.** *Let $f \in L(X)$. Then*

$$\frac{1}{2} f^{**}(t) \le (Mf)^*(t) \le f^{**}(t) \tag{2}$$

*when* $0 < t < \mu(X)$.

*Remark.* If $\mu(X) < \infty$ and $t \ge \mu(X)$, then $(Mf)^*(t) = 0$. Thus the second inequality in (2) is valid for each $t > 0$, while the first inequality fails to hold whenever $t \ge \mu(X)$ unless $f$ is identically zero.

In the proof of the theorem we can take function $f$ nonnegative since all functions considered depend only on the modulus of $f$. We shall also assume that the measure space $(X, \mathbb{S}, \mu)$ is nonatomic. The case when the space has atoms can easily be reduced to the nonatomic case by "putting" suitable measurable sets into the atoms, keeping the values of $f$ inside the atoms unchanged and defining $T$ correspondingly. This process does not change the distribution functions $\lambda \longmapsto \mu(f > \lambda)$ and $\lambda \longmapsto \mu(Mf > \lambda)$, $\lambda > 0$. Consequently $f^*(t)$ and $(Mf)^*(t)$ keep the same values for each $t > 0$.

The following notation will be used: $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$. $S_n(f)(x) = \sum_{k=0}^n f(T^k x)$ and $A_n(f)(x) = \frac{1}{n+1} S_n(f)(x)$. $\mathbf{1}_E$ stands for the characteristic function of $E$. $\{f > 0\}$ or $(f > 0)$ means $\{x \in X : f(x) > 0\}$.

Since a weak-type estimate for the ergodic maximal operator has a simple form

$$\mu(Mf > \lambda) \le \frac{1}{\lambda} \int\limits_{(Mf > \lambda)} f \, d\mu, \tag{3}$$

where $f \in L(X)$, $\lambda > 0$ (see, e.g., [7]), the second inequality in (2) can be proved easily and it is given below for the sake of completeness.

*Proof of the inequality* $(Mf)^*(t) \le f^{**}(t)$, $t > 0$. Since $\frac{1}{\mu(E)} \int_E f \, d\mu \le \frac{1}{t} \int_0^t f^*(\tau) d\tau$ for each measurable $E$ with $\mu(E) = t$ and $f^{**}(t)$ is a decreasing function (see, e.g., [2]), we have

$$f^{**}(t) \ge \sup_{\mu(E) \ge t} \frac{1}{\mu(E)} \int\limits_E f \, d\mu. \tag{4}$$

Consider the nontrivial case when $(Mf)^*(t) > 0$. It follows from definition (1) that

$$0 < \lambda < (Mf)^*(t) \Longrightarrow \mu(Mf > \lambda) > t. \tag{5}$$

Because of (3) we have

$$\lambda \le \frac{1}{\mu(Mf > \lambda)} \int\limits_{(Mf > \lambda)} f \, d\mu, \quad \lambda > 0. \tag{6}$$

It follows from (5) and (4) that

$$\sup_{0<\lambda<(Mf)^*(t)} \frac{1}{\mu(Mf>\lambda)} \int\limits_{(Mf>\lambda)} f\,d\mu \le f^{**}(t).$$

Consequently, if we let $\lambda$ in (6) tend to $(Mf)^*(t)$ from the left, we get the second inequality in (2). $\square$

For the proof of the first inequality in (2) we need

**Lemma.** *Let $g : X \to \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $g \in L(X)$. Then*

$$\mu(Mg \ge 1) = \min\Big(\int\limits_X g\,d\mu,\ \mu(X)\Big).$$

*Proof.* That $\mu(Mg \ge 1) = \mu(X)$ whenever $\int_X g\,d\mu \ge \mu(X)$ follows from the Individual Ergodic Theorem:

$$\lim_{n\to\infty} A_n(g)(x) = \frac{1}{\mu(X)} \int\limits_X g\,d\mu \tag{7}$$

for a.a. $x \in X$ (see, e.g., [7]). Thus it is sufficient to consider the case where

$$\int\limits_X g\,d\mu < \mu(X). \tag{8}$$

We shall use the filling scheme method (see [6], [7] or [3]) truncating the function $g$ at level 1. Let

$$g_0 = g \quad \text{and} \quad g_{n+1} = \mathbf{1}_{(g_n \ge 1)} + (g_n - 1)^+ \circ T. \tag{9}$$

Observe that $g_n$ takes only nonnegative integer values and

$$g_n = \mathbf{1}_{(g_n \ge 1)} + (g_n - 1)^+, \quad n = 0, 1, \dots . \tag{10}$$

If we consider another sequence

$$h_0 = g - 1 \quad \text{and} \quad h_{n+1} = -h_n^- + h_n^+ \circ T,$$

then, as it can easily be checked by induction,

$$h_n = g_n - 1, \quad n = 0, 1, \dots . \tag{11}$$

That

$$\lim_{n\to\infty} \int\limits_X h_n^+\,d\mu = \lim_{n\to\infty} \int\limits_X (g_n - 1)^+\,d\mu = 0 \tag{12}$$

is proved in [3] (see (19) therein). At the same time, since $T$ is measure-preserving and (10) holds, we obtain

$$\int\limits_X g_{n+1} d\,\mu = \int\limits_X \mathbf{1}_{\{g_n \geq 1\}}\, d\mu + \int\limits_X (g_n - 1)^+ \circ T\, d\mu =$$
$$= \int\limits_X \mathbf{1}_{\{g_n \geq 1\}}\, d\mu + \int\limits_X (g_n - 1)^+\, d\mu = \int\limits_X g_n\, d\mu,$$

$n = 0, 1, \ldots$. Thus, for each $n \geq 0$, we have

$$\int\limits_X g_n\, d\mu = \int\limits_X g\, d\mu. \tag{13}$$

We also use the equality of sets

$$\Big\{x : \max_{0 \leq m \leq n} S_m(h_0)(x) \geq 0\Big\} = (h_n \geq 0), \tag{14}$$

$n = 0, 1, \ldots$, which is proved in [4] (see Lemma 2; see also Lemma 1.1 in [3], where the basic idea of the proof is given). Since

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} g(T^k x) = \lim_{n\to\infty} A_n(g)(x) < 1$$

for a.a. $x$ (see (7), (8)), we have

$$\begin{aligned}(Mg \geq 1) &= \Big\{x : A_n(g)(x) \geq 1 \ \text{ for some } \ n \geq 0\Big\}\\ &= \bigcup_{n=0}^{\infty} \Big\{x : \max_{0\leq m\leq n} A_m(g)(x) \geq 1\Big\} = \bigcup_{n=0}^{\infty} \Big\{x : \max_{0\leq m\leq n} S_m(h_0)(x) \geq 0\Big\}\\ &= \bigcup_{n=0}^{\infty} (h_n \geq 0) = \bigcup_{n=0}^{\infty} (g_n \geq 1)\end{aligned}$$

(the first equality holds if we neglect the sets of measure 0 and all other equalities are exact; (see (11), (14)). Thus

$$\mu(Mg \geq 1) = \lim_{n\to\infty} \mu(g_n \geq 1) \tag{15}$$

(that $(g_n \geq 1) = (h_n \geq 0)$, $n = 0, 1, \ldots$, is an increasing sequence of sets follows from definition (9) and also from (14)).

It follows from (13) and (10) that

$$\int\limits_X g\, d\mu = \int\limits_X g_n\, d\mu = \int\limits_X (\mathbf{1}_{\{g_n\geq 1\}} + (g_n - 1)^+)\, d\mu = \mu(g_n \geq 1) + \int\limits_X (g_n - 1)^+\, d\mu.$$

Hence, taking into account (15) and (12), we get

$$\mu(Mg \geq 1) = \int\limits_X g\, d\mu.$$

*Proof of the inequality* $\frac{1}{2}f^{**}(t) \le (Mf)^*(t)$, $0 < t < \mu(X)$. Fix $t \in (0, \mu(X))$ and assume $f^{**}(t) = \lambda_0$. We shall show that

$$\mu\Big(Mf \ge \frac{1}{2}\,\lambda_0\Big) > t. \tag{16}$$

The first inequality in (2) follows from (16) by virtue of definition (1).

Let $E \in \mathbb{S}$ be a measurable set with

$$\mu(E) = t \tag{17}$$

such that

$$\frac{1}{\mu(E)} \int\limits_E f\,d\mu = \frac{1}{t}\int\limits_0^t f^*(\tau)d\tau = \lambda_0. \tag{18}$$

Since we assume that the space is nonatomic, such $E$ exists (see, e.g., [2], Lemma 2.2.5). Define the function $g$ as follows

$$g = \sum_{m=0}^{\infty} \frac{\lambda_0}{2}\, m\mathbf{1}_{(\{\frac{\lambda_0}{2} m \le f < \frac{\lambda_0}{2}(m+1)\} \cap E)}.$$

Observe that $g \le f$, $\frac{2}{\lambda_0}g$ takes only nonnegative integer values and $f(x) - g(x) < \frac{\lambda_0}{2}$ for each $x \in E$. We have

$$\int\limits_E g\,d\mu > \int\limits_E f\,d\mu - \frac{\lambda_0}{2}\,\mu(E) = \frac{\lambda_0}{2}\,\mu(E)$$

(see (18)). Thus

$$\int\limits_X \frac{2}{\lambda_0}\,g\,d\mu > \mu(E)$$

and because of Lemma we have

$$\begin{aligned}\mu\Big(Mg \ge \frac{\lambda_0}{2}\Big) &= \mu\Big(M\Big(\frac{2}{\lambda_0}\,g\Big) \ge 1\Big) = \min\Big(\frac{2}{\lambda_0}\int\limits_X g\,d\mu, \mu(X)\Big)\\ &> \min(\mu(E), \mu(X)) = t\end{aligned}$$

(see (17)). Since $Mf \ge Mg$, we have proved (16). $\square$

At the end of the paper we shall show that the constants $\frac{1}{2}$ and 1 are exact in the inequalities in (2) and cannot be improved. This is clear for 1 since it may happen that $(Mf)^*(t)$ and $f^{**}(t)$ are equal (e.g., for constant functions). A simple example below shows that the equality

$$\frac{1}{2}\,f^{**}(t) = (Mf)^*(t)$$

can hold for $t$ such that $f^{**}(t)$ does not vanish.

**Example.** Let $\widetilde{T}$ be a (Lebesgue) measure-preserving ergodic transformation of $[0; \frac{1}{2})$ and define $T$ by the equalities $T(x) = x + \frac{1}{2}$ when $x \in [0; \frac{1}{2})$ and

$T(x) = \widetilde{T}(x - \frac{1}{2})$ when $x \in [\frac{1}{2}; 1)$. Then $T$ is a measure-preserving ergidic transformation of $[0; 1)$. If $f = \mathbf{1}_{[\frac{1}{2};1)}$, then $Mf(x) = \frac{1}{2}$ when $x \in [0; \frac{1}{2})$ and $Mf(x) = 1$ when $x \in [\frac{1}{2}; 1)$. Thus $(Mf)^*(\frac{1}{2}) = \frac{1}{2}$, while $f^{**}(\frac{1}{2}) = 1$.

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(Received 17.08.2001)

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